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Connectedness of Opposite-flag Geometries in Moufang Polygons

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We show that the geometry of the elements opposite a certain flag in a Moufang polygon is always connected, up to some small cases. This completes the determination of all Moufang polygons for which this geometry is disconnected.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

A *generalized n-gon*, $n \geq 2$, is a rank 2 geometry whose incidence graph has diameter n and girth $2n$, and each vertex has valency ≥ 3. If the latter condition is not satisfied, then we have a *weak generalized n-gon*. In this paper, we will always consider generalized n-gons with $n \geq 3$ (generalized 2-gons are trivial geometries). They are the irreducible spherical buildings of rank 2. A *generalized polygon* is a generalized n-gon, for some $n \geq 2$. We will view generalized polygons as geometries of rank 2 whose elements are points and lines. The *dual* is obtained by interchanging these names. A *flag* is an incident point–line pair and hence a chamber in the corresponding spherical rank 2 building. Generalized polygons were introduced by Tits [10] and are the basic rank 2 incidence geometries.

Let Γ be a generalized n-gon, $n \geq 3$. Given a fixed flag F in Γ, we define $\Gamma^{(F)}$ to be the set of all flags opposite F in Γ together with all points and lines occurring in these flags. So $\Gamma^{(F)}$ is a rank 2 sub geometry of Γ which we call *opposite-flag geometry*. The question arises: what does an opposite-flag geometry look like? In particular, is it connected? Applying an appropriate modification of the 'free construction' of generalized polygons given in Tits [14], Abramenko [1, Chapter II, Section 2, Proposition 9] outlines a construction of infinite generalized n-gons with opposite-flag geometries having an infinite number of connected components, for arbitrary $n \geq 5$. However, if Γ satisfies the Moufang condition, then Abramenko [1, Chapter II, Section 2, Proposition 7] asserts that with a finite number of finite exceptions, every opposite-flag geometry of any Moufang polygon is connected. It is this result which is proved in the present paper.

MAIN RESULT. *Let Γ be a Moufang polygon, and let F be any flag of Γ. Then the geometry $\Gamma^{(F)}$ is connected, except in the following cases:*

(i) *Γ is the generalized quadrangle associated to the symplectic group* $\mathbf{Sp}_4(2)$. *In this case, $\Gamma^{(F)}$ has two connected components.*
(ii) *Γ is the generalized hexagon (or its dual) associated to the group* $\mathbf{G}_2(2)$. *In this case, $\Gamma^{(F)}$ has four connected components.*
(iii) *Γ is the generalized hexagon associated to the group* $\mathbf{G}_2(3)$. *In this case, $\Gamma^{(F)}$ has three connected components.*
(iv) *Γ is the generalized octagon (or its dual) associated to the group* ${}^2\mathbf{F}_4(2)$. *In this case, $\Gamma^{(F)}$ has two connected components.*

We want to comment briefly on the significance of this theorem in the theory of twin buildings and on already published proofs of parts of our Main Result.

Fundamental results about two-spherical twin buildings are proved under the 'standard assumption' that these twin buildings do not contain any rank 2 residues of type $\mathbf{Sp}_4(2)$, $\mathbf{G}_2(2)$,

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$\mathbf{G}_2(3)$ or ${}^2\mathbf{F}_4(2)$, see in particular Condition (co) in Muelherr and Ronan [6] and Condition $(*)$ in Abramenko and Muelherr [2]. The reason why these exceptions are made is precisely our Main Result implying that in all other cases the rank 2 residues enjoy the described connectedness property which has important consequences for the global structure of the twin buildings. In this context the Main Result was already applied several times (e.g., in the two papers just mentioned), though only sketches or proofs of parts of its statement have been published up to now (see the remarks below). The lack of a complete proof in the literature is one main motivation for the present paper. Another reason for writing it is the new geometric approach to the Main Result in the case of Moufang hexagons, which is due to the second author and presented in Section 2. We expect that this geometric proof, apart from bringing beautiful geometric arguments into the play, is more flexible in view of generalizations than the first author's (unpublished) earlier group theoretic proof.

The statement of our Main Result was first mentioned, but without proof, in Tits [17, (16.7)]. However, the Moufang octagons are not considered in this paper, and the counterexample (iii) is overlooked. The Main Result in its present form was stated as Proposition 7 in Abramenko [1]. However, since this proposition was not applied in [1], a group theoretic proof following Tits' ideas was only sketched there. In his lectures at the Collège de France, January 1998, Tits [19] also gave an alternative group theoretic proof, again based on the observation derived as Corollary 4 below. We comment on this proof at the end of Subsection 3.1.

We also remark that, by using a matrix-technique, Brouwer [3] shows that, if any finite polygon (Moufang or not), has a opposite-flag geometry which is not connected, then it has the same parameters as the counterexamples mentioned in the Main Result.

For $n = 3$, i.e., for projective planes, the Main Result is immediate (it is true for all projective planes, Moufang or not). Likewise, for generalized quadrangles, the result is true without the Moufang condition, see Brouwer [3] (cf. Van Maldeghem [21, (1.7.15)]). By a result of Tits [12, 15] and Weiss [22], Moufang n-gons, $n \geq 3$, only exist for $n = 3, 4, 6, 8$. Hence, in order to prove the Main Result, we may restrict our attention to Moufang hexagons and octagons.

2. Moufang Hexagons

In this geometric approach, our aim is to prove that all Moufang hexagons have connected opposite-flag geometries. In view of Brouwer's result [3], we could restrict ourselves to infinite Moufang hexagons but we give an independent proof here which also works for 'almost all' finite Moufang hexagons (see the remark at the end of this section).

We first recall some geometric definitions and facts concerning Moufang hexagons.

Let Γ be a generalized hexagon. For $i \in \{1, 2, 3, 4, 5, 6\}$, let $\Gamma_i(x)$ be the set of all elements of Γ at distance i (measured in the incidence graph) from the element x (which is a point or a line). Also, elements at distance 6 from each other are called *opposite*. If two elements x, y are not opposite, then there exists a unique element incident with x and at minimal distance from y, and we denote that element by $\mathrm{proj}_x y$ (it is directly related to the usual projection mapping in buildings, see Tits [11, Subsection 3.19]. If two elements x, y are opposite in Γ, then the set $\Gamma_i(x) \cap \Gamma_{6-i}(y)$, $i = 2, 3$, is non-empty (it has the same cardinality as $\Gamma_1(x)$ and $\Gamma_1(y)$) and is denoted for short by $x^y_{[i]}$. For $i = 2$, we sometimes write $x^y_{[2]} = x^y$, see e.g., Ronan [8]. The distance between two elements x and y is denoted by $\delta(x, y)$.

Let L be a line of Γ. Then we say that L is *distance-i-regular*, $i = 2, 3$, if for all lines $M, N \in \Gamma_6(L)$, the condition $|L^M_{[i]} \cap L^N_{[i]}| \geq 2$ implies $L^M_{[i]} = L^N_{[i]}$. Ronan [8, (3.7),(5.9)] showed that all lines of any Moufang hexagon are distance-3-regular and that, up to duality, all lines of any Moufang hexagon are distance-2-regular (but he used another terminology; we follow Van Maldeghem [21, Section 1.9]). The set L^M is called a *trace*.

A *path* is a sequence of consecutively incident elements. *Confluent* lines are lines which are incident with a common point. The number of points on a line of any generalized polygon is a constant, which we call the *length* of any line.

Now let Γ be a Moufang hexagon. Without loss of generality (replacing Γ by its dual if necessary), we may assume that Γ has distance-2-regular and distance-3-regular lines. We show a lemma.

LEMMA 1. *Let L, M be two opposite lines in Γ and let p be any point of Γ. Suppose that $|\Gamma_3(L) \cap \Gamma_3(M) \cap \Gamma_4(p)| \geq 3$. Then there is a unique point x of $\Gamma_3(L) \cap \Gamma_3(M)$ collinear with p, and all other points of $\Gamma_3(L) \cap \Gamma_3(M)$ belong to $\Gamma_4(p)$.*

PROOF. Let $x_1, x_2, x_3 \in \Gamma_3(L) \cap \Gamma_3(M) \cap \Gamma_4(p)$. Note that x_1, x_2, x_3 are mutually opposite points. The path (p, L_i, p_i, M_i, x_i), $i = 1, 2, 3$, defines the elements L_i, p_i and M_i. Let L' be the unique element of $\Gamma_3(x_2) \cap \Gamma_2(M_1)$. If $L' \neq M$, then by the distance-3-regularity we have $x_1, x_2, x_3 \in \Gamma_3(L') \cap \Gamma_3(M)$, hence replacing L' by L, we may assume without loss of generality that M_1 is confluent with L. If $L' = M$, then by interchanging the names of L and M, we also obtain that M_1 meets L. Let $\{q_1\} = \Gamma_1(L) \cap \Gamma_1(M_1)$, and let (x_2, M_2', q_2, L) be a path connecting x_2 with L. Note that we may assume that $L_1 \neq L$, otherwise the assertion follows. So it is clear that $q_1 \neq p_1$, since otherwise all points of L_1 different from p_1 are opposite x_2. Consequently we have the ordinary hexagon $(p, L_1, p_1, M_1, q_1, L, q_2, M_2', x_2, M_2, p_2, L_2, p)$, showing that L_2 is opposite L. We can also see from this that $\{M_1, M_2'\} \subseteq L^M \cap L^{L_2}$. Hence, by the distance-2-regularity of lines, every element of L^M is at distance 4 from L_2. Consider the line $M_3' := \mathrm{proj}_{x_3}(L)$. By the foregoing, M_3' is at distance 4 from L_2, Since p is clearly opposite $\mathrm{proj}_L(x_3)$ (indeed, there is a path $(p, L_1, p_1, M_1, q_1, L, \mathrm{proj}_L(x_3))$), the unique line N meeting both L_2 and M_3' is not incident with p. But if it were not incident with x_3 either, then $\delta(x_3, p) = 6$, a contradiction. Hence $L_2 \in \Gamma_3(x_2) \cap \Gamma_3(x_3)$, and hence by the distance-3-regularity, $L_2 \in \Gamma_3(x)$, for all $x \in \Gamma_3(L) \cap \Gamma_3(M)$. So we may now take $L_2 = M$, without loss of generality and the assertion follows easily. □

We can now show the following.

PROPOSITION 2. *Let Γ be a Moufang hexagon with distance-2-regular lines of length ≥ 7, and let F be some flag of Γ. Then the opposite-flag geometry $\Gamma^{(F)}$ is connected.*

PROOF. We put $F = \{p, L\}$, with p a point of Γ and L a line. Let M and N be two distinct lines opposite L.

We note that $|\Gamma_1(x)| \geq 7$, for all points x of Γ by Van Maldeghem [21, (1.9.5)].

First we assume that $M_p := \mathrm{proj}_p M \neq \mathrm{proj}_p N =: N_p$, and that M and N are opposite. Let S be some indexing set with the same cardinality as $\Gamma_1(M)$ (and hence $\ell := |S|$ is the length of any line of Γ). Then we can put $\Gamma_1(M) = \{x_i(M) | i \in S\}$. Let $(x_i(M), L_i(M, N), p_i(\{M, N\}), L_i(N, M), x_i(N), N)$ be a path connecting $x_i(M)$ and N, with $i \in S$. If, for some $i \in S$, $\delta(p_i(\{M, N\}), p) = 2$, then clearly $M_p = \mathrm{proj}_p p_i(\{M, N\}) = N_p$, contradicting our hypothesis. Hence, by the previous lemma, there are at least $\ell - 2 \geq 5$ elements i of S such that $p_i(\{M, N\})$ is opposite p. At least $\ell - 4 \geq 3$ amongst these are such that both $x_i(M)$ and $x_i(N)$ are opposite p. We gather such i in $S' \subseteq S$. By the distance-2-regularity of lines, we have either that for at most one element $j \in S$, the line $L_j(N, M)$ is not opposite L, or that for all elements $j \in S$, the line $L_j(N, M)$ is not opposite L. Similarly for the $L_j(M, N)$'s. Without loss of generality, we may assume that no line $L_j(N, M)$ is opposite L, for all $j \in S$ (because in the other case, there exist at least $|S'| - 2 \geq \ell - 6 \geq 1$ elements $i \in S'$ such that the path $(M, x_i(M), L_i(M, N), p_i(\{M, N\}), L_i(N, M), x_i(N), N)$ is contained in $\Gamma^{(F)}$). Now fix $i \in S'$. Let N', $N' \neq N$, be any line through $x_i(N)$, opposite both L and M (N' exists since $\Gamma_1(x_i(N))| \geq 7$). We let N' play the same role as N and we use similar notation (note that we do not require $\mathrm{proj}_p N' \neq M_p$).

Suppose that $L_j(N', M)$ is not opposite L, for all $j \in S$. Then we consider any line M' through $x_i(M)$ opposite both L and N, $M' \neq M$. Note that M' exists and is automatically opposite N'. Due to the distance-2-regularity of N and N', we have $N^{M'} \cap N^L = N'^{M'} \cap N'^L = \{L_i(N, M)\}$ (because the confluent lines M and M' define different traces $N^M \neq N^{M'}$ and $N'^M \neq N'^{M'}$). For a similar reason at most one of the traces M'^N or $M'^{N'}$ coincides with M'^L. Noticing that $p_i(\{M, N\}) \in \Gamma_3(M) \cap \Gamma_3(N) \cap \Gamma_3(M') \cap \Gamma_3(N')$ implies that at most two elements of $\Gamma_3(M') \cap \Gamma_3(N)$ (respectively $\Gamma_3(M') \cap \Gamma_3(N')$) are not opposite p, we now see that M' and either N or N' belong to the same connected component of $\Gamma^{(F)}$. Hence, since both $x_i(M)$ and $x_i(N)$ are opposite p, the lines M and N are in the same connected component of $\Gamma^{(F)}$.

So we may assume that $L_j(N', M)$ is opposite L, for all $j \in S \setminus \{i\}$, and for all lines $N' \in \Gamma_6(L) \cap \Gamma_6(M) \cap \Gamma_1(x_i(N))$ with $N' \neq N$. For at most one such N' we have $M^{N'} = M^L$. So there exists at least one choice for N' such that $|M^{N'} \cap M^L| \leq 1$. As in the previous paragraph, it follows that N' (and hence N) and M are in the same connected component of $\Gamma^{(F)}$.

If M and N are not opposite, then we claim that there is always a line N' opposite both M and L in the same connected component of $\Gamma^{(F)}$ as N. Moreover, N' can be chosen such that $\mathrm{proj}_p N' \neq M_p$. Indeed, consider any point $x \in \Gamma_1(N) \cap \Gamma_6(p)$, $x \neq \mathrm{proj}_N M$. Any line $N_1 \in \Gamma_1(x) \cap \Gamma_6(L) \cap \Gamma_6(M_p)$, $N_1 \neq N$, satisfies $\delta(M, N_1) = \delta(M, N) + 2$ and $\mathrm{proj}_p N_1 \neq M_p$. There are at least $|\Gamma_1(x)| - 3$ choices for N_1. If $\delta(M, N) = 4$, then we can take $N_1 = N'$, if $\delta(M, N) = 2$, then repeating the argument with M and N_1 proves the claim.

Finally, if $M_p = N_p$, then we may pick any line $P \in \Gamma_6(L)$ with $\mathrm{proj}_p P \neq M_p$ (this is always possible). By the previous part, both M and N belong to the same connected component of $\Gamma^{(F)}$ as P. $\square$

REMARK. The Main Result for generalized hexagons now follows from the previous proposition, from Brouwer [3] (only needed when lines have length at most 6), and from the fact that no infinite Moufang hexagon has lines with finite length $l \leq 6$. Indeed, this follows from Tits' unpublished classification of Moufang hexagons in Tits [13] (see also Tits and Weiss [20]). However, we do not need the full strength of this classification here but only the following ingredients.

First of all, the root groups of a Moufang hexagon Γ constitute a root datum of type $\mathbf{G}_2$. This is shown in Tits [18]. Secondly, the root groups corresponding to the long roots of a root datum of type $\mathbf{G}_2$ can be coordinatized by the additive group of a (commutative) field $\mathbb{K}$, and those corresponding to the short roots by the additive group of a Jordan division algebra J over $\mathbb{K}$ (cf. Faulkner [5], Theorem 3.55). Recall that J is in particular a $\mathbb{K}$-vector space endowed with a cubic form $N : J \to \mathbb{K}$ such that $N(a) \neq 0$, for all $a \in J \setminus \{0\}$. In order to show that no infinite Moufang hexagon Γ has lines of finite length, it suffices to verify that J has to be finite dimensional if $\mathbb{K}$ is finite. However, this follows directly from the theorem of Chevalley-Warning which implies that every cubic form on a vector space of dimension at least 4 over a finite field necessarily has a non-trivial zero.

3. MOUFANG OCTAGONS

3.1. A general lemma. The following discussion is based on ideas developed in Tits [17], Section 16 (see also Abramenko [1, Chapter II, Section 2]). We have to introduce some notation.

Let Δ be a spherical Moufang building of rank r (see Ronan [9], Chapter 6 for the basic properties of Moufang buildings). Fix an apartment Σ of Δ and a chamber $c \in \Sigma$. Denote by Φ the set of all roots (half apartments) of Σ, and set $\Phi^+ := \{\alpha \in \Phi | c \in \alpha\}$. For any $\alpha \in \Phi$, U_α will be the root group associated to α. Let $c_1, \ldots, c_r$ be the chambers of Σ which are adjacent but not equal to c. For any i, $1 \leq i \leq r$, we denote by α_i the unique element of Φ^+ not containing c_i. We set

$$U := \langle U_\alpha | \alpha \in \Phi^+ \rangle \qquad \text{and} \qquad U' := \langle U_{\alpha_i} | 1 \leq i \leq r \rangle,$$

which are subgroups of $\mathrm{Aut}(\Delta)$. Finally, we define (slightly modifying the notation of the remark just preceding Section 2)

$$\Delta^o(c) := \{x \in \Delta | x \text{ and } c \text{ are opposite in } \Delta\}.$$

A connected component of $\Delta^o(c)$ is by definition a maximal subset M of $\Delta^o(c)$ such that any two chambers of M can be connected by a gallery in M.

LEMMA 3. *The index $[U : U']$ is equal to the number of connected components of $\Delta^o(c)$. In particular, $\Delta^o(c)$ is (gallery-)connected if and only if $U = U'$.*

PROOF. Let $c^o, c_1^o, \ldots, c_r^o$ be the chambers of Σ which are opposite $c, c_1, \ldots, c_r$, respectively, and let $M = M(c, c^o)$ be the connected component of $\Delta^o(c)$ containing c^o. Since U acts simple-transitively on $\Delta^o(c)$ (cf. Ronan [9], Theorem 6.15) and hence transitively on the set of its connected components, it suffices to show that $\mathrm{Stab}_U(M) = U'$.

First we show that $\mathrm{Stab}_U(M) \leq U'$. Assume that $u \in U$ stabilizes M. Then there is a gallery $\gamma = (c^o = x_0, x_1, \ldots, x_\ell = u(c^o))$ in $\Delta^o(c)$ connecting c^o and $u(c^o)$. We prove $u \in U'$ by induction on the length ℓ of γ. If $\ell = 0$, then $u(c^o) = c^o$ and hence $u = 1$, again by Ronan [9, Theorem 6.15]. For $\ell > 0$, there is an i such that the panels $x_1 \cap c^o$ and $c_i^o \cap c^o$ coincide. Since $c_i^o \in \alpha_i$ and $x_1, c^o \notin \alpha_i$, there exists a $u_i \in U_{\alpha_i}$ satisfying $u_i(x_1) = c^o$. Applying the induction hypothesis to the gallery $(c^o = u_i(x_1), u_i(x_2), \ldots, u_i(x_\ell) = u_i u(c^o))$, we obtain $u_i u \in U'$ and hence $u \in U'$.

Now we show that $U' \leq \mathrm{Stab}_U(M)$. Clearly, it suffices to verify that $U_{\alpha_j} \subseteq \mathrm{Stab}_U(M)$, for all $j \in \{1, \ldots, r\}$. Let $u \in U_{\alpha_j}$ be arbitrary. Since $u(c^o)$ contains the panel $c^o \cap c_j^o$, c^o and $u(c^o)$ are adjacent chambers. Hence $u(c^o) \in M$ and $u(M) = M$. □

Now let Γ be a Moufang polygon and let F be a flag in Γ. Then Γ can be considered as a spherical Moufang building of rank $r = 2$, F as a chamber of Γ and $\Gamma^o(F)$ as the set of chambers of $\Gamma^{(F)}$. Choosing an apartment Σ of Γ which contains F, setting $c = F$ and defining U, U' as above, we obtain the following specialization of Lemma 3.

COROLLARY 4. *The number of connected components (in the usual graph theoretic sense) of $\Gamma^{(F)}$ is equal to $[U : U']$.*

REMARK 1. By Corollary 4, our problem is reduced to prove that $U' = U$ in the generic case. This can also be carried out in the case of Moufang hexagons as was sketched in [1]. However, we preferred to give the new geometric proof in the present paper. On the other hand, we did not find an analogous geometric argument for Moufang octagons so we shall have to apply the group theoretic approach to that case in the following.

REMARK 2. A way to establish the equality $U' = U$ without carrying out explicit calculations with commutation relations (as below) was given by Tits [19] in his course at Collège de

France, January 1998. His idea is the following. By considering the action of a (suitable) torus T, one turns the group

$$U'' := \langle U_\alpha | \alpha \in \Phi^+ \text{ and } \alpha \text{ contains every chamber adjacent to } c\rangle$$

into a group *with operators*. This allows one to show that, in the generic case, every subgroup (with operators) of U'' is a product of subgroups of the U_α, with $\alpha \in \Phi^+$ containing every chamber adjacent to c. It is then easy to deduce $U'' \le U'$ and hence $U' = U$. The statement concerning U'' is achieved by showing that the U_α, α as above, have no isomorphic subquotients with respect to the operators. This method needs an explicit calculation in T, and it also relies on the classification of Moufang polygons. On the one hand, it is somewhat more involved than the method below, because it needs some additional lemmas. On the other hand, the calculations to perform afterwards are shorter.

3.2. Moufang octagons. Our discussion will be based on Tits' classification of Moufang octagons developed in [16], and we shall also use the notations introduced there.

Given a Moufang octagon Γ, there exist a field $\mathbb{K}$ of characteristic 2 and an endomorphism σ of $\mathbb{K}$ satisfying $\sigma^2(a) = a^2$, for all $a \in \mathbb{K}$, such that Γ is (isomorphic to) the building associated to the group ${}^2\mathbf{F}_4(\mathbb{K}, \sigma) =: G$. We can identify the root groups U_α ($\alpha \in \Phi$) with the subgroups U_i ($1 \le i \le 16$) of G introduced in Tits [16], Section 1, and U with $\langle U_i | 1 \le i \le 8\rangle$, as well as U' with $\langle U_1 \cup U_8\rangle$. Recall that there are parametrizations $x_{2j+1} : \mathbb{K} \to U_{2j+1}$ and $x_{(2j)} : \mathbb{K}^2 \to U_{2j}$ satisfying $x_{2j+1}(a)x_{2j+1}(b) = x_{2j+1}(a + b)$ and

$$x_{(2j)}(a, b)x_{(2j)}(\tilde{a}, \tilde{b}) = x_{(2j)}(a + \tilde{a}, b + \tilde{b} + \sigma(a)\tilde{a}), \text{ for all } a, b, \tilde{a}, \tilde{b} \in \mathbb{K}.$$

Furthermore, $x_{2j}(a) = x_{(2j)}(a, 0)$, $x'_{2j}(b) = x_{(2j)}(0, b)$ and $U'_{2j} = \{x'_{2j}(b) | b \in \mathbb{K}\}$. As Tits [16], we shall often abbreviate $x_i(a)$ by a_i and $x'_{2i}(b)$ by $b_{2i'}$. If there is no danger of ambiguity, we shall also use the notation $i := 1_i = x_i(1)$ and $2i' := 1_{2i'} = x'_{2i}(1)$.

Note that $U(\mathbf{GF}(2), \mathrm{id})$ can always be considered as a subgroup of U. It is therefore useful to determine $U'(\mathbf{GF}(2), \mathrm{id})$ first.

LEMMA 5. *For* $\mathbb{K} = \mathbf{GF}(2)$ *and* $\sigma = \mathrm{id}$ *the subgroup* U' *of* U *coincides with*

$$\widetilde{U} := \{(a_1)_1(a_2)_2(b_2)_{2'} \dots (a_8)_8(b_8)_{8'} | a_i, b_{2j} \in \mathbf{GF}(2); a_2 + a_4 + a_6 = 0\}.$$

PROOF. First we observe that $\widetilde{U}$ is in fact a subgroup of U (and hence contains U'). This follows mainly from an inspection of formulae (1)–(15) given in Tits [16, Subsection 1.7] for commutators $[u_i, u_j]$ with $u_i \in U_i$, $u_j \in U_j$ and $1 \le i < j \le 8$. One notes that $[u_i, u_j]$ can always be expressed as a product with factors from $U_{i+1}, U_{i+2}, \dots, U_{j-1}$, where the number of factors from $U_\ell \setminus U'_\ell$ with $\ell \in \{2, 4, 6\}$ is even. From this and the fact that $(U_\ell \setminus U'_\ell) \cdot (U_\ell \setminus U'_\ell) \subseteq U'_\ell$, for all $\ell \in \{2, 4, 6\}$ in our case ($\mathbb{K} = \mathbf{GF}(2)$), one easily deduces that $\widetilde{U}$ is closed under multiplication and hence a subgroup of U. This subgroup is obviously generated by (and we use the notation introduced above) $1, 3, 5, 7, 8, 2', 4', 6', 8'$, 24 and 46. Using again formulae (1)–(15) in Tits [16, (1.7.1)], we shall show that all of these elements are contained in U', thus completing the proof of the lemma. Indeed, we obtain successively:

$$\begin{aligned}[][8', [1, 8']] &= [8', 2'34566'7] = [8', 2'3] = [8', 2'][2', [8', 3]][8', 3] = \\ &= (6'54')[2', 6']6' = 54' \in U',\end{aligned}$$

$$\begin{aligned}[][8', [1, 8]] &= [8', 2344'56'7] = [8', 23] = [8', 2][2, [8', 3]][8', 3] = \\ &= (76'4')[2, 6']6' = 74' \in U',\end{aligned}$$

$$
\begin{aligned}
[1, 74'] &= [1, 7] = 35 \in U', \\
[8', 35] &= [8', 3] = 6' \in U', \\
[1, 6'] &= 4' \in U', \\
(54')4' &= 5 \in U', \\
(74')4' &= 7 \in U', \\
(35)5 &= 3 \in U', \\
3[1, 8]76'54' &= 3(2344'56'7)76'54' = 24 \in U', \\
[1, 24] &= [2, [1, 4]][1, 4] = [2, 2']2' = 2' \in U', \\
32'[1, 8']76'5 &= 32'(2'34566'7)76'5 = 46 \in U'.
\end{aligned}
$$

The lemma is proved. □

LEMMA 6. *If* $|\mathbb{K}| > 2$*, then* $U' = U$.

PROOF. We have to show $U_i \subseteq U'$, for all i, $2 \leq i \leq 7$. Combining Lemma 5 with formulae (1)–(15) in Tits [16, (1.7.1)], we first obtain

$$
\begin{aligned}
&[U_1, 42] = [U_1, 4] = U_2' \subseteq U', \\
&[U_1, 6'] = U_4' \subseteq U', \\
&[3, U_8'] = U_6' \subseteq U', \\
&[64, U_8] = [6, U_8] = U_7 \subseteq U'.
\end{aligned}
$$

From the identities $[2', a_8] = a_3(\sigma(a)a)_{4'}(\sigma(a)a^2)_{6'}$ and $[2', a_{8'}] = a_{4'}a_5\sigma(a)_{6'}$, for all $a \in \mathbb{K}$, we then deduce $U_3 \subseteq U'$ and $U_5 \subseteq U'$, respectively. Formula (7) of Tits [16, (1.7.1)] shows that

$$[t_1, u_8] \in U_3(tu)_2(\sigma(t)\sigma(u)u)_4 U_4' U_5 U_6' U_7, \quad \forall t, u \in \mathbb{K}. \tag{$*$}$$

Replacing t by $a\sigma(a)b^{-1}$ and u by $b\sigma(a)^{-1}$ in $(*)$, we deduce that $a_2b_4 \in U'$, for all $a, b \in \mathbb{K}^\times$ $(= \mathbb{K} \setminus \{0\})$. Since $(a_2b_4)(\tilde{a}_2b_4) \in (a + \tilde{a})_2 U_2' U_3 U_4'$, this implies $(a + \tilde{a})_2 \in U'$, for all $a, \tilde{a} \in \mathbb{K}^\times$. Since $|\mathbb{K}| > 2$, we obtain $x_2 \in U'$, for all $x \in \mathbb{K}$ and consequently $U_2 \subseteq U'$. Similarly, $U_4 \subseteq U'$. Finally, formula (6) of Tits [16, (1.7.1)] implies $[a_1, 8'] \in U_2' U_3 U_4 U_5 U_6' U_7 a_6$, for all $a \in \mathbb{K}$. Hence we have $U_6 \subseteq U'$ as well. □

Combining Lemma 5 and Lemma 6 with Corollary 4, we obtain:

COROLLARY 7. *The opposite-flag geometry* $\Gamma^{(F)}$ *is connected for any Moufang octagon with lines of length* > 5*. In the (up to duality, unique) excluded case,* $\Gamma^{(F)}$ *has exactly two connected components.*

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