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The Jordan Canonical Forms of complex orthogonal and skew-symmetric matrices

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Abstract

We study the Jordan Canonical Forms of complex orthogonal and skew-symmetric matrices, and consider some related results. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction and notation

Every square complex matrix A is similar to its transpose, A^{T} ([2, Section 3.2.3] or [1, Chapter XI, Theorem 5]), and the similarity class of the n-by-n complex symmetric matrices is all of M_n [2, Theorem 4.4.9], the set of n-by-n complex matrices. However, other natural similarity classes of matrices are non-trivial and can be characterized by simple conditions involving the Jordan Canonical Form.

For example, A is similar to its complex conjugate, $\overline{A}$ (and hence also to its adjoint, $A^* = \overline{A}^{\mathrm{T}}$), if and only if A is similar to a real matrix [2, Theorem 4.1.7]; the Jordan Canonical Form of such a matrix can contain only Jordan blocks with real eigenvalues and pairs of Jordan blocks of the form $J_k(\lambda) \oplus J_k(\overline{\lambda})$ for non-real λ. We denote by $J_k(\lambda)$ the standard upper triangular k-by-k Jordan block with eigenvalue

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λ; see [2, Chapter 3] and [3, Chapter 6] for basic facts about the Jordan Canonical Form and how it behaves when acted on by a primary matrix function such as the inverse, exponential, log, or square.

It is easy to see that a nonsingular A is similar to its inverse, A^{-1}, if and only if its Jordan Canonical Form contains only Jordan blocks with eigenvalues ± 1 and pairs of blocks of the form $J_k(\lambda) \oplus J_k(\lambda^{-1})$ for $\lambda \in \mathbb{C}\backslash\{-1, 0, 1\}$. A natural and important class of such matrices is the group of complex orthogonal matrices: $AA^{\mathrm{T}} = I$ and hence $A^{-1} = A^{\mathrm{T}}$, which is similar to A. However, it may not be obvious that a complex orthogonal matrix has an additional constraint on its Jordan structure: its even-dimensional Jordan blocks with eigenvalue ± 1 must occur in pairs.

In the same spirit, it is easy to see that A is similar to $-A$ if and only if its Jordan Canonical Form contains only Jordan blocks with eigenvalue 0 and pairs of blocks of the form $J_k(\lambda) \oplus J_k(-\lambda)$ for $\lambda \in \mathbb{C}\backslash\{0\}$. The complex skew-symmetric matrices ($A^{\mathrm{T}} = -A$) are certainly of this type, but again it may not be obvious that they, too, are constrained: their even-dimensional Jordan blocks with eigenvalue 0 must occur in pairs. This constraint is a consequence of the constraint on a complex orthogonal matrix's even-dimensional Jordan blocks with eigenvalue $+1$; it is also closely related to the fact that a complex skew-symmetric matrix has even rank (see [1, Chapter XI, Theorem 6] or [2, Section 4.4, Problem 26].

2. Results

Our main objective is to present a new approach to the following classical characterizations of the Jordan Canonical Forms of complex orthogonal and skew-symmetric matrices. For a different approach, see [1, Chapter XI].

Theorem 1. *An n-by-n complex matrix is similar to a complex orthogonal matrix if and only if its Jordan Canonical Form can be expressed as a direct sum of matrices of only the following five types*:
(a) $J_k(\lambda) \oplus J_k(\lambda^{-1})$ *for* $\lambda \in \mathbb{C}\backslash\{-1, 0, 1\}$ *and any* k,
(b) $J_k(1) \oplus J_k(1)$ *for any even* k,
(c) $J_k(-1) \oplus J_k(-1)$ *for any even* k,
(d) $J_k(1)$ *for any odd* k, *and*
(e) $J_k(-1)$ *for any odd* k.

Theorem 2. *An n-by-n complex matrix is similar to a complex skew-symmetric matrix if and only if its Jordan Canonical Form can be expressed as a direct sum of matrices of only the following three types*:
(a) $J_k(\lambda) \oplus J_k(-\lambda)$ *for* $\lambda \in \mathbb{C}\backslash\{0\}$ *and any* k,
(b) $J_k(0) \oplus J_k(0)$ *for any even* k, *and*
(c) $J_k(0)$ *for any odd* k.

3. The complex orthogonal case

Our first, and key, observation is that the similarity class of the complex orthogonal group is generated by complex symmetric similarities.

Lemma 1. *Let $A \in M_n$ be nonsingular. The following are equivalent*:
(a) *A is similar to a complex orthogonal matrix*;
(b) *A is similar to a complex orthogonal matrix via a complex symmetric similarity*;
(c) *there exists a nonsingular complex symmetric S such that $A^{\mathrm{T}} = SA^{-1}S^{-1}$; and*
(d) *there exists a nonsingular complex symmetric S such that $A^{\mathrm{T}}SA = S$.*

Proof. Assuming (a), suppose that X is nonsingular and $XAX^{-1} = L$ is complex orthogonal. The algebraic polar decomposition [3, Theorem 6.4.16] ensures that there is a nonsingular complex symmetric G and a complex orthogonal Q such that $X = QG$. Then $L = XAX^{-1} = QGAG^{-1}Q^{\mathrm{T}}$, so $GAG^{-1} = Q^{\mathrm{T}}LQ$ is a product of complex orthogonal matrices and hence is complex orthogonal. Assuming (b), suppose that $A = GQG^{-1}$ for some complex symmetric G and complex orthogonal Q. Then $A^{-1} = GQ^{\mathrm{T}}G^{-1}$ and $A^{\mathrm{T}} = G^{-1}Q^{\mathrm{T}}G = G^{-2}A^{-1}G^2$, which is (c) with $S = G^{-2}$. Now assume (c) and write $S = Y^{\mathrm{T}}Y$ for some $Y \in M_n$ [2, Corollary 4.4.6], so $A^{\mathrm{T}} = SA^{-1}S^{-1} = Y^{\mathrm{T}}YA^{-1}Y^{-1}Y^{-\mathrm{T}}$, or $(YAY^{-1})^{\mathrm{T}} = Y^{-\mathrm{T}}A^{\mathrm{T}}Y^{\mathrm{T}} = YA^{-1}Y^{-1} = (YAY^{-1})^{-1}$; YAY^{-1} is therefore complex orthogonal and so (a) follows. The equivalence of (c) and (d) is clear. □

Since a complex orthogonal matrix A is similar to $A^{\mathrm{T}} = A^{-1}$, to each Jordan block $J_k(\lambda)$ of A there is a corresponding Jordan block $J_k(\lambda^{-1})$. When $\lambda^2 \neq 1$, these two Jordan blocks are different, so they are paired; when $\lambda^2 = 1$, they are the same, so there is no evident pairing—but there is a pairing if k is even. As a first step, we show that each of the pairs of Jordan blocks described in Theorem 1 (a)–(c) is similar to a complex orthogonal matrix.

Lemma 2. *For any positive integer k and any $\lambda \neq 0$, $J_k(\lambda) \oplus J_k(\lambda^{-1})$ is similar to a complex orthogonal matrix.*

Proof. Let B be any symmetric matrix to which $J_k(\lambda)$ is similar, and define the symmetric matrix

$$H \equiv \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \in M_{2k}. \tag{1}$$

Then $J_k(\lambda) \oplus J_k(\lambda^{-1})$ is similar to $J_k(\lambda) \oplus J_k(\lambda)^{-1}$, which is similar to $B \oplus B^{-1}$. Moreover,

$$H\left(B \oplus B^{-1}\right)^{-1} H^{-1} = H\left(B^{-1} \oplus B\right) H = B \oplus B^{-1} = \left(B \oplus B^{-1}\right)^{\mathrm{T}}.$$

Thus, $J_k(\lambda) \oplus J_k(\lambda^{-1})$ is similar to $B \oplus B^{-1}$, which by Lemma 1 is similar to a complex orthogonal matrix. □

Our next step is to show that each of the Jordan blocks described in Theorem 1 (d) and (e) is similar to a complex orthogonal matrix.

Lemma 3. *For any odd positive integer k, each of $J_k(1)$ and $J_k(-1)$ is similar to a complex orthogonal matrix.*

Proof. Since $J_k(-1)$ is similar to $-J_k(1)$, it is sufficient to prove that $J_k(1)$ is similar to a complex orthogonal matrix whenever k is an odd positive integer. We use Lemma 1 and note that the case $k = 1$ is trivial: $J_1(1)^{\mathrm{T}} = S_1 J_1(1)^{-1} S_1^{-1}$with $S_1 \equiv [1]$. For each successive $n = 1, 2, \ldots$ we show how to construct a nonsingular symmetric S_{2n+1} such that

$$J_{2n+1}^{\mathrm{T}}(1) S_{2n+1} J_{2n+1}(1) = S_{2n+1}. \tag{2}$$

For $n = 1$, look for an S_3 of the form

$$S_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -S_1 & x_1 \\ 1 & x_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & x_1 \\ 1 & x_1 & 0 \end{bmatrix},$$

which is automatically symmetric and nonsingular. It satisfies the constraints (6), and hence the identity (2), if and only if $-1 + x_1 + x_1 = 0$; the choice $x_1 = 1/2$ is forced on us. For $n = 2$, look for an S_5 of the form

$$S_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -S_3 & x \\ 1 & x^{\mathrm{T}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & x_1 \\ 0 & 0 & 1 & -\frac{1}{2} & x_2 \\ 0 & -1 & -\frac{1}{2} & 0 & 0 \\ 1 & x_1 & x_2 & 0 & 0 \end{bmatrix},$$

which again is automatically symmetric and nonsingular. The constraints (6) require that $-1 - 1/2 + x_1 = 0$ and $-1/2 + 0 + x_2 = 0$, so the choices $x_1 = 3/2$ and $x_2 = 1/2$ are forced on us. The pattern of the construction is now clear: with $n \geqslant 3$ and a constructed S_{2n-1} in hand (necessarily symmetric and nonsingular), look for an S_{2n+1} of the form

$$S_{2n+1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -S_{2n-1} & x \\ 1 & x^{\mathrm{T}} & 0 \end{bmatrix} \tag{3}$$

for some

$$x = \begin{bmatrix} x_1\ x_2\ \ldots\ x_{2n-2}\ 0 \end{bmatrix}^{\mathrm{T}} \in \mathbb{C}^{2n-1}. \tag{4}$$

Let $e_k \in \mathbb{C}^{2n-1}$ have a 1 in position k and zeroes elsewhere. Then

$$J_{2n+1}(1)^{\mathrm{T}} S_{2n+1} J_{2n+1}(1)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ e_1 & J_{2n-1}(1)^{\mathrm{T}} & 0 \\ 0 & e_{2n-1}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -S_{2n-1} & x \\ 1 & x^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} 1 & e_1^{\mathrm{T}} & 0 \\ 0 & J_{2n-1}(1) & e_{2n-1} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -J_{2n-1}(1)^{\mathrm{T}} S_{2n-1} J_{2n-1}(1) & e_1 + J_{2n-1}(1)^{\mathrm{T}} (x - S_{2n-1} e_{2n-1}) \\ 1 & e_1^{\mathrm{T}} + (x - S_{2n-1} e_{2n-1})^{\mathrm{T}} J_{2n-1}(1) & -e_{2n-1}^{\mathrm{T}} S_{2n-1} e_{2n-1} + x^{\mathrm{T}} e_{2n-1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -S_{2n-1} & e_1 + J_{2n-1}(1)^{\mathrm{T}} (x - S_{2n-1} e_{2n-1}) \\ 1 & e_1^{\mathrm{T}} + (x - S_{2n-1} e_{2n-1})^{\mathrm{T}} J_{2n-1}(1) & 0 \end{bmatrix},$$

where the last equality relies on (2) for $n-1$ and the requirements in (3) and (4) that the lower right entry of S_{2n-1} and the last entry of x are both zero. In order to satisfy (2), x must satisfy

$$e_1 + J_{2n-1}(1)^{\mathrm{T}} (x - S_{2n-1} e_{2n-1}) = x,$$

which is equivalent to

$$J_{2n-1}(0)^{\mathrm{T}} x = J_{2n-1}(1)^{\mathrm{T}} S_{2n-1} e_{2n-1} - e_1;$$

fortunately, this equation has a unique solution of the form (4). Alternatively, we may use the constraints (6) to determine (uniquely) $x_1, x_2, \ldots, x_{2n-2}$ in succession. □

Since a direct sum of matrices that are individually similar to complex orthogonal matrices must be similar to a complex orthogonal matrix, the preceding two results establish the sufficiency of the conditions in Theorem 1. We now turn to establishing their necessity. The only issue is the asserted impossibility of having an odd number of Jordan blocks of a given even dimension associated with either of the eigenvalues $+1$ or -1. The following technical lemma is the key to our explanation of this phenomenon.

Lemma 4. *Suppose* $X = [x_{ij}] \in M_{n,m}$ *satisfies*

$$J_n^{\mathrm{T}}(1) X J_m(1) = X. \tag{5}$$

Then the first column of X *is zero if either* (a) $n < m$, *or* (b) $n = m$, n *is even, and* X *is symmetric.*

Proof. A computation reveals that (5) holds if and only if $x_{i-1,j-1} + x_{i-1,j} + x_{i,j-1} + x_{i,j} = x_{i,j}$ for all $i = 1, \ldots, n$ and all $j = 1, \ldots, m$, where we adopt the convention that $x_{p,q} \equiv 0$ if either $p = 0$ or $q = 0$. Thus, (5) is equivalent to the constraints

$$x_{i-1,j-1} + x_{i-1,j} + x_{i,j-1} = 0 \quad \text{for all } i = 1, \ldots, n \text{ and all } j = 1, \ldots, m. \tag{6}$$

Examining (6) for $i = 1, 2, \ldots$ shows that $x_{1,1} = x_{1,2} = \cdots = x_{1,m-1} = 0$, $x_{2,1} = \cdots = x_{2,m-2} = 0$, and, indeed, that $x_{ij} = 0$ whenever $i + j \leqslant m$. In particular, $x_{i,1} = 0$ for all $i = 1, \ldots, n$ if $n < m$. Now suppose that $m = n$. We already know that the entire triangular portion of X above its counter-diagonal is filled with zeroes; in particular, $x_{i,1} = 0$ for $i = 1, \ldots, n-1$. For $j = n - i + 2$, (6) ensures that $x_{i-1,n-i+2} + x_{i,n-i+1} = 0$ for all $i = 2, \ldots, n$; in particular, $x_{n,1} = (-1)^{n+1} x_{1,n}$. If X is symmetric, then $x_{n,1} = x_{1,n}$ as well, so $x_{n,1} = 0$ if X is symmetric and n is even. □

We now apply this observation to exclude unpaired Jordan blocks of the form $J_k(\pm 1)$ with k even.

Lemma 5. *Let $r, k_1, \ldots, k_r$ be positive integers with k_1 even, and suppose that $k_1 > k_2 \geqslant \cdots \geqslant k_r$ if $r > 1$. Then neither $J_{k_1}(1) \oplus \cdots \oplus J_{k_r}(1)$ nor $J_{k_1}(-1) \oplus \cdots \oplus J_{k_r}(-1)$ is similar to a complex orthogonal matrix.*

Proof. First consider the case $\lambda = 1$ and suppose that $J \equiv J_{k_1}(1) \oplus \cdots \oplus J_{k_r}(1)$ is similar to a complex orthogonal matrix. Lemma 1 guarantees that there is a nonsingular symmetric S such that $J^{\mathrm{T}} S J = S$, so $J_{k_i}(1)^{\mathrm{T}} S_{ij} J_{k_j}(1) = S_{ij}$ for $i, j = 1, \ldots, r$, where we partition $S = [S_{ij}]$ with each $S_{ij} \in M_{k_i,k_j}$. Since S_{11} is symmetric and has even dimension, and since $k_1 > k_i$ for all $i = 2, \ldots, r$ if $r > 1$, Lemma 4 ensures that the first column of $S_{i,1}$ is zero for all $i = 1, \ldots, r$, so S is singular. This contradiction establishes the case $\lambda = 1$. The case $\lambda = -1$ now follows immediately, since $J_k(-1)$ is similar to $-J_k(1)$. □

A simple argument permits us to combine the two exclusions described in the preceding lemma.

Theorem 3. *Let $r, k_1, \ldots, k_r$ and $p, l_1, \ldots, l_p$ be positive integers with k_1 and l_1 even; suppose that $k_1 > k_2 \geqslant \cdots \geqslant k_r$ if $r > 1$ and that $l_1 > l_2 \geqslant \cdots \geqslant l_p$ if $p > 1$. Then $J_{k_1}(1) \oplus \cdots \oplus J_{k_r}(1) \oplus J_{l_1}(-1) \oplus \cdots \oplus J_{l_p}(-1)$ is not similar to a complex orthogonal matrix.*

Proof. Write $J_+ \equiv J_{k_1}(1) \oplus \cdots \oplus J_{k_r}(1)$ and $J_- \equiv J_{l_1}(-1) \oplus \cdots \oplus J_{l_p}(-1)$. If $J_+ \oplus J_-$ is similar to a complex orthogonal matrix, then there is a symmetric nonsingular S such that $(J_+^{\mathrm{T}} \oplus J_-^{\mathrm{T}})S = S(J_+^{-1} \oplus J_-^{-1})$. Partition $S = [S_{ij}]_{i,j=1,2}$ conformally to $J_+ \oplus J_-$. Then $J_+^{\mathrm{T}} S_{12} = S_{12} J_-^{-1}$, so $S_{12} = 0$ since J_+ and J_-^{-1} have no eigenvalues in common ([2, Problem 9 or 13, Section 2.4] or [3, Theorem 4.4.6]). Then $S_{21} = S_{12}^{\mathrm{T}} = 0$ and $S = S_{11} \oplus S_{22}$. Lemma 1 ensures that both J_+ and J_- are similar to complex orthogonal matrices, which contradicts Lemma 5. □

The last ingredient in our proof of necessity is the following cancellation lemma.

Lemma 6. *Let $C \in M_k$ be similar to a complex orthogonal matrix. If $B \oplus C$ is similar to a complex orthogonal matrix for some $B \in M_n$, then B is similar to a complex orthogonal matrix.*

Proof. If SCS^{-1} is complex orthogonal, then $(I_n \oplus S)(B \oplus C)(I_n \oplus S)^{-1} = B \oplus SCS^{-1}$, so there is no loss of generality if we assume that C is complex orthogonal. Suppose C and $A \equiv X\,(B \oplus C)\,X^{-1}$ are complex orthogonal. Then

$$A^{\mathrm{T}} = X^{-\mathrm{T}}(B^{\mathrm{T}} \oplus C^{\mathrm{T}})X^{\mathrm{T}} = X(B^{-1} \oplus C^{\mathrm{T}})X^{-1} = A^{-1},$$

so

$$\left(B^{\mathrm{T}} \oplus C^{\mathrm{T}}\right) X^{\mathrm{T}}X = X^{\mathrm{T}}X\,\left(B^{-1} \oplus C^{\mathrm{T}}\right)$$

and we see that there is an S such that

$$\left(B^{\mathrm{T}} \oplus C^{\mathrm{T}}\right) S = S\left(B^{-1} \oplus C^{\mathrm{T}}\right) \quad \text{and} \quad S = S^{\mathrm{T}} \text{ is nonsingular.} \tag{7}$$

Partition $S = \left[S_{ij}\right]_{i,j=1,2}$ conformally to $B \oplus C$, so $S_{21} = S_{12}^{\mathrm{T}}$ and both S_{11} and S_{22} are symmetric. We now show that there is an S satisfying (7) for which S_{22} is nonsingular. Adding $0 \oplus tC^{\mathrm{T}}$ to both sides of (7) gives

$$\begin{bmatrix} B^{\mathrm{T}} & 0 \\ 0 & C^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathrm{T}} & S_{22} + tI \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathrm{T}} & S_{22} + tI \end{bmatrix}\begin{bmatrix} B^{-1} & 0 \\ 0 & C^{\mathrm{T}} \end{bmatrix}.$$

Let

$$S(t) \equiv \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathrm{T}} & S_{22} + tI \end{bmatrix},$$

and define the polynomial $p(t) \equiv \det S(t)$. Then $p(0) \neq 0$, so p is not the zero polynomial. Since there are only finitely many values of $t \in \mathbb{C}$ for which $p(t) = 0$ and $S_{22} + tI$ is singular, we may assume that S_{22} is nonsingular and S satisfies (7). The identity

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathrm{T}} & S_{22} \end{bmatrix}\begin{bmatrix} I & 0 \\ -S_{22}^{-1}S_{12}^{\mathrm{T}} & I \end{bmatrix} = \begin{bmatrix} S_{11} - S_{12}S_{22}^{-1}S_{12}^{\mathrm{T}} & S_{12} \\ 0 & S_{22} \end{bmatrix}$$

shows that $Y \equiv S_{11} - S_{12}S_{22}^{-1}S_{12}^{\mathrm{T}}$, the Schur complement of S_{22} in S, is nonsingular; it is also symmetric since S_{11} and S_{22} are both symmetric. Comparing the block entries of both sides of (7) gives the four identities (a) $B^{\mathrm{T}}S_{11} = S_{11}B^{-1}$, (b) $B^{\mathrm{T}}S_{12} = S_{12}C^{\mathrm{T}}$, (c) $C^{\mathrm{T}}S_{12}^{\mathrm{T}} = S_{12}^{\mathrm{T}}B^{-1}$, and (d) $C^{\mathrm{T}}S_{22} = S_{22}C^{\mathrm{T}}$. Taking the transpose of (b) gives $S_{12}^{\mathrm{T}}B = CS_{12}^{\mathrm{T}}$, and taking the inverse in (d) gives $S_{22}^{-1}C = CS_{22}^{-1}$. Hence,

$$S_{22}^{-1}(S_{12}^{\mathrm{T}}B) = S_{22}^{-1}(CS_{12}^{\mathrm{T}}) = CS_{22}^{-1}S_{12}^{\mathrm{T}}.$$

Invoking (b) and the orthogonality of C now gives

$$\begin{aligned} B^{\mathrm{T}}(S_{12}S_{22}^{-1}S_{12}^{\mathrm{T}}) &= (B^{\mathrm{T}}S_{12})(S_{22}^{-1}S_{12}^{\mathrm{T}}B)B^{-1} \\ &= (S_{12}C^{\mathrm{T}})(CS_{22}^{-1}S_{12}^{\mathrm{T}})B^{-1} \\ &= (S_{12}S_{22}^{-1}S_{12}^{\mathrm{T}})B^{-1}. \end{aligned}$$

Subtracting this identity from (a) gives $B^{\mathrm{T}}Y = YB^{-1}$. Since Y is nonsingular and symmetric, Lemma 1 ensures that B is similar to a complex orthogonal matrix. □

We now can resolve the last open issue involving the necessity of the conditions in Theorem 1.

Theorem 4. *Let A be a complex orthogonal matrix. Then the even-sized Jordan blocks of A corresponding to each of the eigenvalues $+1$ and -1 are paired.*

Proof. Take the direct sum of all the pairs of Jordan blocks of A of the form $J_k(\lambda)$ and $J_k(\lambda^{-1})$ and denote it by C_1; notice that C_1 contains all pairs of Jordan blocks of A with eigenvalues ± 1. Now take the direct sum of all the remaining (necessarily unpairable) odd-sized Jordan blocks of A with eigenvalues $+1$ or -1, if any, and denote it by C_2. Let $C \equiv C_1 \oplus C_2$. If C contains all the Jordan blocks of A, we are done; if not, let B denote the direct sum of all the remaining Jordan blocks of A that have not been incorporated into C. Notice that B must be a direct sum of only single unpairable even-sized Jordan blocks with eigenvalues either $+1$ or -1. Theorem 3 ensures that B is not similar to a complex orthogonal matrix. According to Lemmata 2 and 3, C is similar to a complex orthogonal matrix. Lemma 6 now guarantees that B is similar to a complex orthogonal matrix, so the direct sum forming B must be empty. □

4. The skew-symmetric case

Let $A \in M_n$ be a given skew-symmetric matrix. Since A is similar to $A^{\mathrm{T}} = -A$, to each Jordan block $J_k(\lambda)$ of A there is a corresponding block $J_k(-\lambda)$. When $\lambda \neq 0$, these two Jordan blocks are different, so they are paired; when $\lambda = 0$, they are the same, so it is not evident that there is any pairing—but again there is in the even-dimensional case. Our approach to Theorem 2 parallels our approach to Theorem 1 and relies on it.

The same argument as in the proof of Lemma 1 shows that complex symmetric similarities play a special role in the similarity class of the complex skew-symmetric matrices.

Lemma 7. *A given $A \in M_n$ is similar to a complex skew-symmetric matrix if and only if there is a nonsingular symmetric S such that $A^{\mathrm{T}} = -SAS^{-1}$.*

Lemma 8. *For any positive integer k and any $\lambda \in \mathbb{C}$, $J_k(\lambda) \oplus J_k(-\lambda)$ is similar to a skew-symmetric matrix.*

Proof. Let B be any symmetric matrix to which $J_k(\lambda)$ is similar, so that $J_k(\lambda) \oplus J_k(-\lambda)$ is similar to the symmetric matrix $B \oplus (-B)$. With the nonsingular symmetric matrix H defined in (1), one checks that $B \oplus (-B) = -H\,(B \oplus (-B))\,H^{-1}$, so Lemma 7 ensures that $B \oplus (-B)$, and hence $J_k(\lambda) \oplus J_k(-\lambda)$, is similar to a skew-symmetric matrix. □

Lemma 9. *For any odd positive integer k, $J_k(0)$ is similar to a skew-symmetric matrix.*

Proof. Using Lemma 7, it suffices to exhibit a nonsingular symmetric S such that $J_k^{\mathrm{T}}(0)S = -SJ_k(0)$. Let $S \in M_k$ have counterdiagonal entries $1, -1, 1, -1, \ldots$ and zero entries elsewhere:

$$S \equiv \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & -1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Then S is nonsingular and, since k is odd, it is symmetric. □

The preceding two results establish the sufficiency of the conditions in Theorem 2. The following lemma relies on Lemma 5 and the observation that e^A is complex orthogonal whenever A is skew-symmetric: $\left(e^A\right)^{\mathrm{T}} = e^{A^{\mathrm{T}}} = e^{-A} = \left(e^A\right)^{-1}$.

Lemma 10. *Let $r, k_1, \ldots k_r$ be positive integers with k_1 even, and suppose that $k_1 > k_2 \geqslant \cdots \geqslant k_r$ if $r > 1$. Then $J_{k_1}(0) \oplus \cdots \oplus J_{k_r}(0)$ is not similar to a skew-symmetric matrix.*

Proof. If $J \equiv J_{k_1}(0) \oplus \cdots \oplus J_{k_r}(0)$ is similar to a skew-symmetric matrix, then e^J is similar to a complex orthogonal matrix, whose Jordan Canonical Form is $J_{k_1}(1) \oplus \cdots \oplus J_{k_r}(1)$, which contradicts Lemma 5 . □

The following cancellation lemma is an analog of Lemma 6.

Lemma 11. *Let C be similar to a skew-symmetric matrix. If $B \oplus C$ is similar to a skew-symmetric matrix, then B is also similar to a skew-symmetric matrix.*

Proof. Using the argument and notation of Lemma 6, one shows that there is a nonsingular symmetric S with S_{22} nonsingular such that

$$\left(B^{\mathrm{T}} \oplus -C\right) S = S\,(-B \oplus -C) \tag{8}$$

The four identities obtained from (8) are (a′) $B^{\mathrm{T}} S_{11} = -S_{11}B$, (b′) $B^{\mathrm{T}} S_{12} = -S_{12}C$, (c′) $CS_{12}^{\mathrm{T}} = S_{12}^{\mathrm{T}}B$, and (d′) $CS_{22} = S_{22}C$. Using (b′), (d′), and (c′), we find that

$$B^{\mathrm{T}} S_{12} S_{22}^{-1} S_{12}^{\mathrm{T}} = -S_{12}^{\mathrm{T}} C S_{22}^{-1} S_{12}^{\mathrm{T}} = -S_{12}^{\mathrm{T}} S_{22}^{-1} C S_{12}^{\mathrm{T}} = -S_{12}^{\mathrm{T}} S_{22}^{-1} S_{12}^{\mathrm{T}} B.$$

Subtracting this identity from (a′) gives $B^{\mathrm{T}}Y = -YB$, so Lemma 7 ensures that B is similar to a skew-symmetric matrix. □

Just as in the preceding section, the following result completes the proof of necessity of the conditions in Theorem 2; its proof is entirely parallel to that of Theorem 4.

Theorem 5. *Let $A \in M_n$ be skew-symmetric. Then the even-sized singular Jordan blocks of A are paired.*

5. Other results

The matrix e^S is complex orthogonal whenever $S \in M_n$ is skew-symmetric, but not every complex orthogonal matrix can be achieved in this way. For example, when $n = 1$, the only skew-symmetric matrix is $S = [0]$, and $Q = [-1] \neq e^0$. More generally, consider a complex orthogonal Q that is similar to $J_k(-1)$ with k odd. If $Q = e^S$ for some skew-symmetric matrix S, then S can have only one block in its Jordan Canonical Form, and that block must be $J_k(im\pi)$ for some odd integer m; this violates Theorem 2. Using Theorem 2, it is not difficult to characterize the complex orthogonal matrices that can be represented as e^S for some complex skew-symmetric S.

Theorem 6. *Let Q be a complex orthogonal matrix. There exists a skew-symmetric S such that $Q = e^S$ if and only if for each odd positive integer k, the number of blocks $J_k(-1)$ in the Jordan Canonical Form of Q is even (possibly zero).*

Proof. Suppose $Q = e^S$, with S skew-symmetric. Theorem 2 guarantees that each Jordan block $J_k(im\pi)$ of S with eigenvalue $im\pi$ and m an odd integer is paired with a corresponding block $J_k(-im\pi)$. Since these are the only Jordan blocks of S that can create Jordan blocks of Q with eigenvalue -1, it follows that all odd-sized Jordan blocks of Q with eigenvalue -1 come in pairs.

Conversely, suppose that the odd-sized Jordan blocks of Q with eigenvalue -1 can be paired. If the Jordan Canonical Form of Q contains $J_k(\lambda) \oplus J_k(\lambda^{-1})$ for some $\lambda \in \mathbb{C}\backslash\{0\}$, let S have a pair of Jordan blocks of the form $J_k(\ln\lambda) \oplus J_k(-\ln\lambda)$, where the same branch of ln is chosen in both blocks. Do the same when $\lambda = \pm 1$ and k is even, and when $\lambda = -1$ and k is odd. If Q has a Jordan block of the form $J_k(1)$ with k odd, let S have a Jordan block $J_k(0)$. With such a choice for S, Q and e^S are similar. Since they are both complex orthogonal matrices, they are complex orthogonally similar [3, Corollary 6.4.19], say, $Q = Q_1 e^S Q_1^{\mathrm{T}} = e^{Q_1 S Q_1^{\mathrm{T}}}$, and $Q_1 S Q_1^{\mathrm{T}}$ is skew-symmetric. □

Theorem 7. *Let A be a complex orthogonal matrix. Then $A = B^2$ for some complex orthogonal $B \in M_n$ if and only if for each positive integer k, the number of blocks $J_k(-1)$ in the Jordan Canonical Form of A, if any, is even.*

Proof. Suppose $A = B^2$ for some complex orthogonal B. Then the only Jordan blocks of A with eigenvalue -1 must come from pairs of Jordan blocks of B of the form $\pm(J_k(i) \oplus J_k(-i))$. This guarantees that the Jordan blocks of A with eigenvalue -1 can be paired.

Conversely, suppose that the Jordan blocks of A corresponding to -1 can be paired. Then for every pair $J_k(\lambda) \oplus J_k(\lambda^{-1})$, take a complex orthogonal matrix that has a Jordan block of the form $J_k(\sqrt{\lambda}) \oplus J_k((\sqrt{\lambda})^{-1})$, where the same branch of the square root is chosen in both blocks. The only blocks that cannot be written this way are the odd-dimensional Jordan blocks with eigenvalue 1. But $J_k(1)$ is similar to $J_k^2(1)$. As in Theorem 6, the proof can be finished by invoking the fact that two similar complex orthogonal matrices are complex orthogonally similar [3, Corollary 6.4.19]. □

References

[1] F.R. Gantmacher, The Theory of Matrices, vol. 2, Chelsea Publishing Company, New York, 1989.
[2] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, New York, 1985.
[3] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, New York, 1991.