UPPER BOUNDS FOR CONSTANT WEIGHT AND LEE CODES SLIGHTLY OUTSIDE THE PLOTKIN RANGE

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The analogues of the McEliece and Tietäväinen bounds (see [10, p. 565] and [16]) are derived for constant weight and Lee codes.

1. Introduction

Recently McEliece [10, p. 565] and Tietäväinen [16] have derived powerful upper bounds for block codes whose minimum distance is just outside the Plotkin range. In Section 3 the analogues of these bounds are obtained for certain positive semidefinite matrices. The proofs are based on the power sum inequalities (cf. Section 2) which generalize the inequalities of Welch et al. [17] and the mean distance bound of McEliece and Rumsey [11]. Section 4 gives applications to association schemes. The Hamming, Johnson, nonbinary Johnson and Lee schemes are discussed.

2. Power sum inequalities

For a positive integer n, let (Ω, R) be an *n-class colouring structure* (cf. [5]). Hence Ω is a finite set with cardinality $|\Omega| = \upsilon \geq 2$, R is a collection of $n+1$ symmetric relations $R_0, \dots, R_n$ on Ω forming a partition of the Cartesian power Ω^2 and R_0 is the diagonal relation $\{(x,x) : x \in \Omega\}$ of Ω. We call (Ω, R) *regular* if, for $i = 0, \dots, n$, the number $\upsilon_i(x) = |\{y \in \Omega : (x,y) \in R_i\}|$ is independent of the choice of $x \in \Omega$. In this case $\upsilon_i(x)$ is said to be the *valence* of R_i and is denoted by υ_i. The *inner distribution* of a nonempty subset X of Ω is defined to be the $(n+1)$-tuple $(a_0, \dots, a_n)$ of nonnegative rational numbers $a_i = |X|^{-1} |X^2 \cap R_i|$. Clearly, $a_0 = 1$ and $a_0 + \cdots + a_n = |X|$.

Denote by $\mathbb{R}(\Omega, \Omega)$ the algebra of all real $\upsilon \times \upsilon$-matrices S, where the entries are numbered by the elements of Ω^2, the (x,y)-entry of S being written as $S(x,y)$. Consider a matrix $S \in \mathbb{R}(\Omega, \Omega)$ which is *R-invariant* in the sense that in each R_k S is a constant which we denote by $S(k) : S(x,y) = S(k)$ if $(x,y) \in R_k$ $(k = 0, \dots, n)$. So, S

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is symmetric, its diagonal entries $S(x, x)$ $(x \in \Omega)$ are equal to $S(0)$ and

$$S = \sum_{i=0}^{n} S(i) D_i, \tag{1}$$

where $D_i \in \mathbb{R}(\Omega, \Omega)$ is the *adjacency matrix* of R_i for which $D_i(x, y) = 1$ if $(x, y) \in R_i$ and $D_i(x, y) = 0$ if $(x, y) \notin R_i$. For given nonempty subsets X and Y of Ω and for a positive integer r, define $S_r(X, Y)$ to be the mean of the rth power of the function S over the set $X \times Y$, that is,

$$S_r(X, Y) = |X|^{-1} |Y|^{-1} \sum_{x \in X} \sum_{y \in Y} S(x, y)^r. \tag{2}$$

Further, let $S_r(X) = S_r(X, X)$ and $S_r = S_r(\Omega)$. Thus we have

$$S_r(X) = |X|^{-1} \sum_{i=0}^{n} S(i)^r a_i, \tag{3}$$

where $(a_0, \ldots, a_n)$ is the inner distribution of X. Also, if (Ω, R) is regular, then

$$S_r(X, \Omega) = S_r = \frac{1}{v} \sum_{i=0}^{n} S(i)^r v_i \tag{4}$$

holds for all X and r.

Theorem 1. *Suppose $S \in \mathbb{R}(\Omega, \Omega)$ is R-invariant and positive semidefinite. If X and Y are nonempty subsets of Ω, y is an element of Ω and r is a positive integer, then*

$$S_r(X) \geq 0, \tag{5}$$

$$S_r(X) S_r(Y) \geq S_r(X, Y)^2, \tag{6}$$

$$S(0)^r S_r(X) \geq S_r(X, \{y\})^2 \quad \textit{and} \tag{7}$$

$$S_r(X) \geq S_r \quad \textit{if } (\Omega, R) \textit{ is regular.} \tag{8}$$

Proof. Let $S^{(r)}$ be the rth Hadamard power of S, i.e., a $v \times v$-matrix with entries $S^{(r)}(x, y) = S(x, y)^r$ and let ϕ_X be the characteristic vector of X, that is, a v-dimensional column vector whose entries are labelled by the elements of Ω, the x-entry being $\phi_X(x) = 1$ if $x \in X$ and $\phi_X(x) = 0$ if $x \notin X$. Then (2) can be written in the form

$$S_r(X, Y) = |X|^{-1} |Y|^{-1} \phi_X^{\mathrm{t}} S^{(r)} \phi_Y, \tag{9}$$

where t denotes the transpose operation. Since $S^{(r)}$ is a principal sub-matrix of the rth Kronecker power of S, $S^{(r)}$ is positive semidefinite (cf. [8, p. 260]). Hence (5) holds and (6) follows from (9) by using the Schwarz inequality. If in (6) we choose $Y = \{y\}$, we obtain (7). Inequality (8) follows from (4) and (5) by applying (6) to $Y = \Omega$. □

3. Upper bounds for D-sets

Throughout this section we assume that (Ω, R) is a regular n-class colouring structure and $S \in \mathbb{R}(\Omega, \Omega)$ is a nonzero R-invariant and positive semidefinite matrix. Given a subset D of real numbers, call a nonempty subset X of Ω a *D-set* if $S(x, y) \in D$ whenever x and y are distinct elements of X. Denote by $m(S, D)$ the maximal cardinality of such a D-set.

Theorem 2. *Let ϱ be a real number and r a positive integer. If D is the closed interval $[-\varrho, \varrho]$ or if D is the unbounded closed interval $(-\infty, \varrho]$ and r is odd, then*

$$m(S, D) \leq \frac{S(0)^r - \varrho^r}{S_r - \varrho^r} \quad \text{for } \varrho^r < S_r. \tag{10}$$

Proof. Let X be a D-set with cardinality $m = m(S, D)$ and inner distribution $(a_0, \ldots, a_n)$. Since $a_i = 0$ if $i \neq 0$ and $S(i)^r > \varrho^r$, then from (8) and (3) we obtain

$$mS_r \leq mS_r(X) = \sum_{i=0}^{n} S(i)^r a_i \leq S(0)^r + (m-1)\varrho^r.$$

This proves the assertion. □

Theorem 3. *Let ϱ be a nonnegative real number and r a positive even integer. Then*

$$m(S, (-\infty, \varrho]) \leq \frac{2S(0)^r S_r + (S(0)\varrho)^{r-1}(S(0) - \varrho)^2}{S_r(S_r - 2\varrho^r)} \tag{11}$$

provided the denominator of the right hand side of (11) *is positive.*

Proof. Let X be a $(-\infty, \varrho]$-set with cardinality $m = m(S, (-\infty, \varrho])$ and inner distribution $(a_0, \ldots, a_n)$ and let

$$K_t = (-1)^t \sum_{\substack{i=0 \\ S(i)<0}}^{n} S(i)^t a_i \qquad (t = 1, 2, \ldots). \tag{12}$$

Since $S(0) \geq 0$ by semidefiniteness, then

$$\begin{aligned} 0 \leq mS_t \leq mS_t(X) &= \sum_{i=0}^{n} S(i)^t a_i \\ &= S(0)^t a_0 + \sum_{\substack{i=0 \\ S(i)<0}}^{n} S(i)^t a_i + \sum_{\substack{i=1 \\ S(i)\geq 0}}^{n} S(i)^t a_i \\ &\leq S(0)^t + (-1)^t K_t + m\varrho^t. \end{aligned}$$

Hence we have

$$K_r \geq mS_r - S(0)^r - m\varrho^r \quad \text{and} \tag{13}$$

$$K_t \leq S(0)^t + m\varrho^t \quad \text{if } t \text{ is odd.} \tag{14}$$

By the Schwarz inequality,

$$K_r^2 = \left\{ \sum_{S(i)<0} ((-S(i))^{r-1} a_i)^{1/2} ((-S(i))^{r+1} a_i)^{1/2} \right\}^2 \leq K_{r-1} K_{r+1}.$$

So, according to (13) and (14), $mS_r - S(0)^r - m\varrho^r \leq 0$ or $(mS_r - S(0)^r - m\varrho^r)^2 \leq (S(0)^{r-1} + m\varrho^{r-1})(S(0)^{r+1} + m\varrho^{r+1})$. From these two inequalities the latter gives a weaker bound for m implying (11). □

Given two sequences (x_n) and (y_n) of real numbers, write $x_n = \mathrm{o}(y_n)$ if $\lim x_n/y_n = 0$ and $x_n \lesssim y_n$ if $x_n \leq y_n(1 + \mathrm{o}(1))$.

Since S is nonzero, S_2 is positive. We denote $\gamma = S(0)/S_2$.

Corollary 4. *Suppose n tends to infinity and $\varrho = \varrho_n$ are nonnegative real numbers satisfying $\varrho^2 = \mathrm{o}(S_2)$ as $n \to \infty$. Then*

$$m(S, (-\infty, \varrho]) \lesssim \gamma S(0)(2 + \gamma\varrho) \quad \text{as } n \to \infty. \tag{15}$$

Proof. If in (7) we choose $r = 2$ and $X = \Omega$, then (4) and (5) yield $S(0)^2 \geq S_2$. Hence $\varrho = \mathrm{o}(S(0))$ and (15) follows from (11). □

Denote by $\log x$ the Naperian logarithm of x.

Theorem 5. *Suppose n tends to infinity and $\varrho = \varrho_n$ are nonnegative real numbers satisfying $\varrho^2 = \mathrm{o}(S_2)$ and $\gamma\varrho^3 = \mathrm{o}(S_2)$ as $n \to \infty$. Then*

$$m(S, (-\infty, \varrho]) \lesssim \gamma S(0)(2.2 + \log(1 + \gamma\varrho)) \quad \text{as } n \to \infty. \tag{16}$$

Furthermore, if $\gamma\varrho \to \infty$ as $n \to \infty$, then

$$m(S, (-\infty, \varrho]) \lesssim \tfrac{1}{2}\gamma S(0) \log(\gamma\varrho) \quad \text{as } n \to \infty. \tag{17}$$

Proof. Since $\varrho = \mathrm{o}(S(0))$, we can assume that $\varrho < S(0)$. Let X be a $(-\infty, \varrho]$-set with cardinality $m = m(S, (-\infty, \varrho])$. Fix an element c of X and number the other elements $c_1, \ldots, c_{m-1}$ of X in such an order that the numbers $y_i(c) = -S(c, c_i)$ $(i = 1, \ldots, m-1)$ form a decreasing sequence. Denote by $K(c)$ the number of positive $y_i(c)$'s and by $K_t(c)$ the sum of the tth powers of positive $y_i(c)$'s:

$$K_t(c) = \sum_{i=1}^{K(c)} y_i(c)^t \qquad (t = 1, 2, \ldots). \tag{18}$$

Then the numbers K_t defined in (12) can be expressed as the averages

$$K_t = \frac{1}{m} \sum_{c \in X} K_t(c). \tag{19}$$

Applying (7) to $X_s=\{c_1,\dots,c_s\}$, yields

$$S_1(X_s,\{c\})^2\leq S(0)S_1(X_s)\leq \frac{S(0)}{s}(S(0)+(s-1)\varrho).$$

Hence we have

$$\sum_{i=1}^{s} y_i(c)=-sS_1(X_s,\{c\})\leq \varphi(s) \qquad (s=1,\dots,m-1), \tag{20}$$

where $\varphi(z)=\sqrt{S(0)z(S(0)+(z-1)\varrho)}$. Define a map Φ as follows: if z is a nonnegative real number and s an integer with $\varphi(s)\leq z<\varphi(s+1)$, then

$$\Phi(z)=(\varphi(s+1)-\varphi(s))(z-\varphi(s))+\sum_{i=1}^{s}(\varphi(i)-\varphi(i-1))^2.$$

Since φ is concave and increasing for nonnegative values of the argument, so is Φ.

We prove that

$$K_2(c)\leq \Phi(K_1(c)). \tag{21}$$

Let $\varphi(r)\leq K_1(c)<\varphi(r+1)$ where r is a nonnegative integer. Since $K_1(c)\leq\varphi(K(c))$ by (20), then $r\leq K(c)$ and in the case $r=K(c)$ we have $K_1(c)=\varphi(r)$. Because $\varphi(z)$ $(z\geq 0)$ is concave, the numbers

$$x_i=\begin{cases}\varphi(i)-\varphi(i-1) & \text{if } 1\leq i\leq r,\\ K_1(c)-\varphi(r) & \text{if } i=r+1,\\ 0 & \text{if } i>r+1\end{cases}$$

form a decreasing sequence. Since $x_1+\cdots+x_k=\varphi(k)$ if $1\leq k\leq r$ and $x_1+\cdots+x_k=K_1(c)$ if $k\geq r+1$, then from (18) and (20) it follows that $y_1(c)+\cdots+y_k(c)\leq x_1+\cdots+x_k$ $(k=1,\dots,K(c))$ and $y_1(c)+\cdots+y_{K(c)}(c)=K_1(c)=x_1+\cdots+x_{K(c)}$. Hence, by a result of Karamata (see [2, p. 30]),

$$K_2(c)=\sum_{i=1}^{K(c)} y_i(c)^2\leq\sum_{i=1}^{K(c)} x_i^2=\sum_{i=1}^{r+1} x_i^2\leq\Phi(K_1(c)).$$

This proves (21).

Next we derive an estimate for Φ: if $z\geq\varphi(1)=S(0)$, then

$$\Phi(z)\leq\left(1.2+\tfrac{1}{2}\log\frac{z}{S(0)}\right)S(0)^2+2z\sqrt{\varrho S(0)}. \tag{22}$$

Assume $\varphi(s)\leq z<\varphi(s+1)$, where s is a positive integer, and let $i\geq 2$ be an integer. From the inequality $2\sqrt{(i-1)(i-2)}\leq(i-1)+(i-2)$ it follows that

$$(S(0)+\varrho\sqrt{(i-1)(i-2)})^2\leq(S(0)+(i-1)\varrho)(S(0)+(i-2)\varrho)$$

and hence we have

$$(\varphi(i)-\varphi(i-1))^2\leq(\sqrt{i}-\sqrt{i-1})^2S(0)^2+2\varrho S(0)((i-1)^2-(i-1)\sqrt{i(i-2)}).$$

Using the inequality $(i-1)\sqrt{i(i-2)} \geq i(i-2)$, we obtain

$$(\varphi(i)-\varphi(i-1))^2 \leq (\sqrt{i}-\sqrt{i-1})^2 S(0)^2 + 2\varrho S(0) \qquad (i=2,3,\dots). \tag{23}$$

Further, since

$$\begin{aligned}\sum_{i=3}^{s+1} (\sqrt{i}-\sqrt{i-1})^2 &= \sum_{i=3}^{s+1} \frac{1}{(\sqrt{i}+\sqrt{i-1})^2} \\ &\leq \tfrac{1}{4} \sum_{i=3}^{s+1} \frac{1}{i-1} \leq \tfrac{1}{4} \int_1^s \frac{\mathrm{d}x}{x} = \tfrac{1}{4} \log s,\end{aligned}$$

then (23) implies that

$$\begin{aligned}\Phi(z) &\leq \sum_{i=1}^{s+1} (\varphi(i)-\varphi(i-1))^2 \\ &\leq 1.2 S(0)^2 + 2\varrho s S(0) + S(0)^2 \sum_{i=3}^{s+1} (\sqrt{i}-\sqrt{i-1})^2 \\ &\leq (1.2 + \tfrac{1}{4}\log s) S(0)^2 + 2\varrho s S(0).\end{aligned}$$

So, (22) follows from the inequalities $s \leq z^2 S(0)^{-2}$ and $s\sqrt{\varrho S(0)} \leq z$ which are easily obtained from $\varphi(s) \leq z$ and $\varrho \leq S(0)$.

By (19), (21) and Jensen's inequality (see [2, pp. 17 and 18]), we have

$$K_2 = \frac{1}{m} \sum_{c \in X} K_2(c) \leq \frac{1}{m} \sum_{c \in X} \Phi(K_1(c)) \leq \Phi\left(\frac{1}{m} \sum_{c \in X} K_1(c)\right) = \Phi(K_1).$$

Thus, by (13), (14) and (22),

$$\begin{aligned}& m\left\{1 - \frac{\varrho^2}{S_2} - 2\left(\frac{\gamma\varrho^3}{S_2}\right)^{1/2}\right\} \\ &\quad \leq \gamma S(0)\left\{2.2 + \tfrac{1}{2}\log\frac{S(0)+m\varrho}{S(0)} + 2\left(\frac{\varrho}{S(0)}\right)^{1/2}\right\}.\end{aligned} \tag{24}$$

Corollary 4 shows that $m \leq \gamma S(0)(2+\gamma\varrho)(1+\mathrm{o}(1))$. When we substitute this upper bound of m into the right hand side of (24), we get (16). If we replace m by $\gamma S(0)(2.2 + \log(1+\gamma\varrho))(1+\mathrm{o}(1))$ on the right hand side of (24), we obtain (17). □

4. Applications to association schemes

Let (Ω, R) be a *symmetric association scheme with n classes* (see [4, p. 8]). Hence (Ω, R) is an n-class colouring structure with relations $R_0, \dots, R_n$ and, for $i, j, k = 0, \dots, n$, the number

$$p_{ijk} = |\{z \in \Omega \colon (x,z) \in R_i,\ (z,y) \in R_j\}|$$

is independent of the choice of $(x,y)\in R_k$. Then (Ω, R) is regular and the valence v_i of R_i equals $\mathrm{p}_{\mathrm{ii0}}$. The real linear space $\mathscr{A}$ generated by the adjacency matrices D_i of R_i $(i=0,\ldots,n)$ is a commutative $(n+1)$-dimensional subalgebra of $\mathbb{R}(\Omega,\Omega)$ and is composed of symmetric matrices (cf. [10, p. 653]). This algebra is called the *Bose–Mesner algebra* of the scheme. It has a unique basis of primitive idempotents $J_0=v^{-1}J, J_1,\ldots,J_n$ (J is the all-one matrix) which are nonzero matrices of $\mathscr{A}$ satisfying $J_iJ_j=\delta_{ij}J_i$ (see [10, pp. 653 and 654]). Their ranks $\mu_i=\operatorname{rank} J_i$ are called the *multiplicities* of the scheme. Given the two bases $\{D_i\}$ and $\{J_i\}$ of $\mathscr{A}$, we have the basis transformations

$$D_k=\sum_{i=0}^{n} p_k(i)J_i \quad \text{and} \quad J_k=\frac{1}{v}\sum_{i=0}^{n} q_k(i)D_i \qquad (k=0,\ldots,n). \tag{25}$$

The coefficients $p_k(i)$ and $q_k(i)$ are called the *p-numbers* and *q-numbers* of the scheme. These parameters have the following properties (cf. [10, pp. 654 and 655]):

$$\sum_{i=0}^{n} p_k(i)q_i(r)=\sum_{i=0}^{n} q_k(i)p_i(r)=v\delta_{kr}, \tag{26}$$

$$p_0(i)=q_0(i)=1, \qquad p_i(0)=v_i, \qquad q_i(0)=\mu_i, \tag{27}$$

$$\sum_{i=0}^{n} v_i q_k(i)q_r(i)=v\mu_k\delta_{kr}. \tag{28}$$

Given an R-invariant matrix $S\in\mathbb{R}(\Omega,\Omega)$, define the parameters $\lambda_0,\ldots,\lambda_n$ as a solution of the linear equations

$$\sum_{k=0}^{n} \lambda_k q_k(i)=S(i) \qquad (i=0,\ldots,n). \tag{29}$$

Since the coefficient matrix of (29) is nonsingular by (26), the numbers $\lambda_0,\ldots,\lambda_n$ are uniquely defined. Further, from (29) and (26) we obtain

$$\lambda_k=\frac{1}{v}\sum_{i=0}^{n} S(i)p_i(k) \qquad (k=0,\ldots,n). \tag{30}$$

Also, from (4) and (27)–(30) it follows that

$$S_1=\lambda_0 \quad \text{and} \quad S_2=\sum_{k=0}^{n} \lambda_k^2\mu_k. \tag{31}$$

Theorem 6. *An R-invariant matrix $S\in\mathbb{R}(\Omega,\Omega)$ is positive semidefinite if and only if the parameters $\lambda_0,\ldots,\lambda_n$ in (29) are nonnegative.*

Proof. According to (1), (25) and (30),

$$S=v\sum_{i=0}^{n} \lambda_i J_i. \tag{32}$$

Hence $SJ_k = \upsilon\lambda_k J_k$ and the numbers $\upsilon\lambda_k$ are eigenvalues of S. Thus $\lambda_0, \dots, \lambda_n$ are nonnegative if S is positive semidefinite. Conversely, suppose $\lambda_k \geq 0$ for $k = 0, \dots, n$. Since $J_k = J_k^2 = J_k^{\mathrm{t}} J_k$, then $P^{\mathrm{t}} J_k P = (J_k P)^{\mathrm{t}} (J_k P) \geq 0$ for each υ-dimensional column vector P and S is positive semidefinite by (32). □

Example 1. Let F be an *alphabet*, i.e., a finite set of cardinality $q \geq 2$. The *Hamming distance* $d_{\mathrm{H}}(x, y)$ between two vectors x and y of F^n is the number of the coordinate places in which x and y differ. Let $R_0, \dots, R_n$ be the distance relations of F^n induced by $d_{\mathrm{H}} : R_k = \{(x, y) \in F^n \times F^n : d_{\mathrm{H}}(x, y) = k\}$. These relations make F^n a symmetric association scheme called the *Hamming scheme* $H(n, q)$. For this scheme we have

$$\upsilon_i = \mu_i = \binom{n}{i}(q-1)^i \quad \text{and} \quad q_1(i) = n(q-1) - qi$$

(see [4, Section 4.1]). Hence the matrix $S \in \mathbb{R}(F^n, F^n)$ with entries $S(x, y) = (q-1)n/q - d_{\mathrm{H}}(x, y)$ satisfies $S(i) = (q-1)n/q - i = q_1(i)/q$. So, the only nonzero parameter λ_i is $\lambda_1 = 1/q$ and S is positive semidefinite by Theorem 6. According to (31), $S_1 = 0$, $S_2 = (q-1)n/q^2$ and $\gamma = q$.

Let X be a nonempty subset of F^n with inner distribution $(a_0, \dots, a_n)$. From (3)–(5) and (8) we obtain the power sum inequalities

$$\begin{aligned} &\sum_{i=0}^{n} ((q-1)n - qi)^r a_i \\ &\qquad \geq \frac{|X|}{q^n} \sum_{i=0}^{n} \binom{n}{i} (q-1)^i ((q-1)n - qi)^r \geq 0 \qquad (r = 1, 2, \dots) \end{aligned}$$

(cf. [17] and [12]). For $r = 1$, (7) reduces to the mean distance bound

$$d_{\mathrm{H}}(X, X) \leq d_{\mathrm{H}}(X, \{y\}) \left(2 - \frac{q\, d_{\mathrm{H}}(X, \{y\})}{(q-1)n} \right) \qquad (y \in F^{n'}),$$

where $d_{\mathrm{H}}(X, Y)$ is the mean of d_{H} over the set $X \times Y$ (cf. [11] and [12]).

Denote by $m_q(n, d)$ the maximal number of q-ary vectors of length n and Hamming distance at least d apart. Then $m(S, (-\infty, \varrho]) = m_q(n, (q-1)n/q - \varrho)$ and Theorem 2 with $r = 1$ and $\varrho = (q-1)n/q - d$ gives the bound

$$m_q(n, d) \leq \frac{qd}{qd - (q-1)n} \quad \text{for } d > (q-1)n/q$$

which is essentially due to Plotkin [13]. From (15)–(17) we obtain the following asymptotic results: as n tends to infinity, then

$$m_q\left(n, \frac{(q-1)n}{q} - \varrho\right) \lesssim \begin{cases} (q-1)n(2 + q\varrho) & \text{if } 0 \leq \varrho = \mathrm{o}(n^{1/2}), \\ (q-1)n(2.2 + \log(1 + q\varrho)) & \text{if } 0 \leq \varrho = \mathrm{o}(n^{1/3}), \\ \frac{1}{2}(q-1)n \log \varrho & \text{if } \varrho = \mathrm{o}(n^{1/3}) \text{ and } \varrho \to \infty. \end{cases}$$

In the binary case $q=2$ the first of these bounds has been proved by McEliece (see [10, p. 565]) and the other two by Tietäväinen [16]. The generalizations to nonbinary alphabets are due to Perttula [12].

Example 2. Let F be an alphabet of cardinality q and fix some element of F which will be denoted by 0 (zero). The *Hamming weight* $w_H(x)$ of a vector x in F^n is the number of nonzero components of x. The set $W_w=W_w(n,q)$ of all vectors of F^n with Hamming weight w is called the *Hamming surface of radius* w. We denote by $m_q(n,d,w)$ the maximal number of q-ary vectors of length n, Hamming weight w and Hamming distance at least d apart.

Suppose $q=2$. Then the nonempty distance relations of W_w induced by d_H are $R_k=\{(x,y)\in W_w(n,2)^2 : d_H(x,y)=2k\}$ $(k=0,\dots,m)$, where $m=\min(w,n-w)$, and in the case $0<w<n$ they make $W_w(n,2)$ a symmetric association scheme called the *Johnson scheme* $J(w,n)$. Some parameters of this scheme are

$$v_i=\binom{w}{i}\binom{n-w}{i} \quad \text{and} \quad q_1(i)=(n-1)\left(1-\frac{ni}{w(n-w)}\right)$$

(see [4, Section 4.2]). So, for the matrix S with entries

$$S(x,y)=n-n^2d_H(x,y)/(2w(n-w)) \qquad (x,y\in W_w),$$

we have $S(i)=n-n^2i/(w(n-w))$, $\lambda_1=n/(n-1)$, $\lambda_i=0$ for $i\neq 1$, $S_1=0$, $S_2=n^2/(n-1)$ and $\gamma=(n-1)/n$. Hence (3)–(5) and (8) imply that

$$\sum_{i=0}^{m}\left(1-\frac{ni}{w(n-w)}\right)^r a_i\geq\frac{|X|}{\binom{n}{w}}\sum_{i=0}^{m}\binom{w}{i}\binom{n-w}{i}\left(1-\frac{ni}{w(n-w)}\right)^r$$
$$\geq 0 \qquad (r=1,2,\dots)$$

for each nonempty subset X of $W_w(n,2)$ with inner distribution $(a_0,\dots,a_m)$. These inequalities have been proved by Sidelnikov [14]. The upper bounds of Section 3 for $m(S,(-\infty,\varrho])=m_2(n,2w(n-w)(n-\varrho)/n^2,w)$ will be given in the next example.

Example 3. Let F be an alphabet with cardinality $q\geq 3$. Then the Hamming distance does not make $W_w(n,q)$ an association scheme. For $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ in $W_w(n,q)$, denote $e(x,y)=|\{i : x_i=y_i\neq 0\}|$ and $n(x,y)=|\{i : x_i\neq 0$ and $y_i\neq 0\}|$ and define the relations R_{ij} of the surface as follows: $R_{ij}=\{(x,y)\in W_w(n,q)^2 : e(x,y)=w-i,\ n(x,y)=w-j\}$. This classification makes $W_w(n,q)$ $(0<w<n)$ a symmetric association scheme called the *q-ary Johnson scheme* $J_q(w,n)$. Note that R_{00} is the diagonal relation of $W_w(n,q)$ and $d_H(x,y)=i+j$ if $(x,y)\in R_{ij}$. It is known that the numbers $q_{00}(i,j)=1$, $q_{10}(i,j)=(n-1)(1-nj/(w(n-w)))$ and $q_{11}(i,j)=\frac{n}{w}((q-2)w-(q-1)i+j)$ are q-numbers of $J_q(w,n)$ (see [15]).

Consider the matrix $S\in\mathbb{R}(W_w,W_w)$ with entries $S(x,y)=n-d_H(x,y)/\alpha$ where

$$\alpha=\frac{qw(n-w)+(q-2)wn}{(q-1)n^2}.$$

It is easily seen that the only nonzero λ-parameters are $\lambda_{11}=w/((q-1)n\alpha)$ and $\lambda_{10}=qw(n-w)/((q-1)n(n-1)\alpha)$. Consequently, $S(0)=n$, $S_1=0$, $S_2=\lambda_{10}^2q_{10}(0,0)+\lambda_{11}^2q_{11}(0,0)=n^2/(\beta(n-1))$ and $\gamma=\beta(n-1)/n$ where

$$\beta=\frac{(q(n-w)+(q-2)n)^2}{q^2(n-w)^2+(q-2)n(n-1)}.$$

Theorem 2 with $r=1$ and $\varrho=n-d/\alpha$ gives the Johnson bound [7] for the function $m(S,(-\infty,\varrho])=m_q(n,\alpha(n-\varrho),w):m_q(n,d,w)\le d/(d-\alpha n)$ if $d>\alpha n$. Theorem 3 with $r=2$ implies that

$$m_q(n,\alpha(n-\varrho),w)\le\frac{\beta(n-1)(2n^3+\beta\varrho(n-1)(n-\varrho)^2)}{n^3-2\beta\varrho^2n(n-1)}$$

if $0\le\varrho<n/\sqrt{2\beta(n-1)}$. Also, from (15)–(17) we obtain the following asymptotic results: as n tends to infinity, then

$$m_q(n,\alpha(n-\varrho),w)\lesssim\begin{cases}\beta n(2+\beta\varrho), & \text{if } 0\le\varrho=o(n^{1/2}),\\ \beta n(2.2+\log(1+\beta\varrho)) & \text{if } 0\le\varrho=o(n^{1/3}),\\ \tfrac{1}{2}\beta n\log\varrho & \text{if } \varrho=o(n^{1/3}) \text{ and } \varrho\to\infty.\end{cases}$$

By using Example 2, it is easily seen that the above five bounds for $m_q(n,d,w)$ also hold if $q=2$. In this case $\alpha=2w(n-w)/n^2$ and $\beta=1$.

Example 4. Consider the ternary Johnson scheme $J_3(w,n)$ over the alphabet $\{0,1,-1\}$. For $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)\in W_w(n,3)$, let $S(x,y)=x_1y_1+\cdots+x_ny_n$ be the usual inner product of x and y. Then S has a constant value $w-2i+j=(w/n)q_{11}(i,j)$ in R_{ij}. Thus $S(0)=w$, $S_1=0$, $S_2=w^2/n$ and $\gamma=n/w$. Section 3 gives the bounds

$$m(S,[-\varrho,\varrho])\le\frac{n(w^2-\varrho^2)}{w^2-n\varrho^2}\quad\text{if } 0\le\varrho<w/\sqrt{n},$$

$$m(S,(-\infty,\varrho])\le 1-w/\varrho\quad\text{if } \varrho<0,$$

$$m(S,(-\infty,\varrho])\le\frac{n(2w^3+\varrho n(w-\varrho)^2)}{w(w^2-2n\varrho^2)}\quad\text{if } 0\le\varrho<w/\sqrt{2n}$$

and the following asymptotic bounds: as n tends to infinity, then

$$m(S,(-\infty,\varrho])\lesssim\begin{cases}n(2+n\varrho/w) & \text{if } 0\le\varrho=o(wn^{-1/2}),\\ n(2.2+\log(1+n\varrho/w)) & \text{if } 0\le\varrho=o(wn^{-2/3}),\\ \tfrac{1}{2}n\log(n\varrho/w) & \text{if } \varrho=o(wn^{-2/3}) \text{ and } n\varrho/w\to\infty.\end{cases}$$

In some cases these results improve the bounds of Deza and Frankl [6].

Example 5. Let $\mathbb{Z}_q$ be the additive group of integers $0, 1, \ldots, q-1$ modulo q where $q \geq 2$ and let $s=[q/2]$ where $[x]$ is the greatest integer not exceeding x. The *Lee distance* $d_{\mathrm{L}}(x,y)$ (see [9]) between two vectors $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$ of $\mathbb{Z}_q^n$ is defined by $d_{\mathrm{L}}(x,y)=\omega(x_1-y_1)+\cdots+\omega(x_n-y_n)$ where $\omega(i)=\min(i,q-i)$ $(i=0,\ldots,q-1)$. The *Lee composition* $c(x)$ of a vector $x=(x_1,\ldots,x_n)$ in $\mathbb{Z}_q^n$ is a $(s+1)$-tuple $(c_0,\ldots,c_s)$ where $c_k=|\{i:\omega(x_i)=k\}|$. Let Γ be the set of all Lee compositions $c(x)$ where x varies over $\mathbb{Z}_q^n$. Then the relations $R_c=\{(x,y)\in\mathbb{Z}_q^n\times\mathbb{Z}_q^n : c(x-y)=c\}$ $(c\in\Gamma)$ make $\mathbb{Z}_q^n$ a symmetric association scheme called the *Lee scheme* $L(n,q)$ (see [1] and [4, p. 18]). Note that $d_{\mathrm{L}}(x,y)=1z_1+2z_2+\cdots+sz_s$ if $(x,y)\in R_z$ where $z=(z_0,\ldots,z_s)$.

Let $\Omega_0, \Omega_1, \ldots, \Omega_s$ be a partition of $\mathbb{Z}_q$ whose sets are of the form $\Omega_i=\{i, q-i\}$ and let $Q_k(i)=\sum_{h\in\Omega_k}\varepsilon^{hi}$ $(k, i=0,\ldots,s)$ where $\varepsilon=\exp(2\pi\sqrt{-1}/q)$ is a primitive qth root of unity. Astola [1] has shown that the numbers $q_0(z)=1$ and $q_r(z)=\sum_{i=0}^{s} z_i Q_r(i)$ $(z=(z_0,\ldots,z_s)\in\Gamma;\ r=1,\ldots,s)$ are q-numbers of $L(n,q)$. Wyner and Graham [18] (see also [3, pp. 313 and 314]) have proved that the numbers

$$\lambda_r=-\frac{1}{q}\sum_{i=0}^{q-1}\omega(i)\varepsilon^{ri} \qquad (r=1,\ldots,q-1)$$

are nonnegative. Since $\lambda_r=\lambda_{q-r}$ $(r=1,\ldots,q-1)$, then

$$\sum_{r=1}^{s}\lambda_r Q_r(i)=\sum_{r=1}^{q-1}\lambda_{q-r}\varepsilon^{ri}=\frac{1}{q}\sum_{j=1}^{q-1}\omega(j)\left\{1-\sum_{r=0}^{q-1}\varepsilon^{r(i-j)}\right\}=D-\omega(i)$$

where

$$D=\begin{cases}(q^2-1)/(4q) & \text{if } q \text{ is odd,}\\ q/4 & \text{if } q \text{ is even.}\end{cases}$$

Hence

$$\sum_{r=1}^{s}\lambda_r q_r(z)=\sum_{i=0}^{s}z_i\sum_{r=1}^{s}\lambda_r Q_r(i)=Dn-\sum_{i=0}^{s}iz_i$$

for $z=(z_0,\ldots,z_s)\in\Gamma$. So, the matrix S with entries $S(x,y)=Dn-d_{\mathrm{L}}(x,y)$ $(x,y\in\mathbb{Z}_q^n)$ is positive semidefinite, $S(0)=Dn$ and $S_1=0$. From (31) and (27) it follows that

$$\begin{aligned}S_2&=\sum_{r=1}^{s}\lambda_r^2 q_r(n,0,\ldots,0)=n\sum_{r=1}^{s}\lambda_r^2|\Omega_r|=n\sum_{r=1}^{q-1}\lambda_r^2\\ &=\frac{n}{q^2}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\omega(i)\omega(j)\sum_{r=1}^{q-1}\varepsilon^{r(i-j)}=\frac{n}{q}\sum_{i=0}^{q-1}\omega(i)^2-D^2n\\ &=\begin{cases}(q^2-1)(q^2+3)n/(48q^2) & \text{if } q \text{ is odd,}\\ (q^2+8)n/48 & \text{if } q \text{ is even.}\end{cases}\end{aligned}$$

Denote by $M_q(n,d)$ the maximal number of vectors in $\mathbb{Z}_q^n$ with Lee distance at least d apart. Hence $m(S,(-\infty,\varrho])=M_q(n, Dn-\varrho)$ and (10) gives the Plotkin bound (see [18]): $M_q(n, d)\le d/(d-Dn)$ if $d>Dn$. When we use the notation

$$\mu=D^2n/S_2=\begin{cases}3(q^2-1)/(q^2+3) & \text{if } q \text{ is odd,}\\ 3q^2/(q^2+8) & \text{if } q \text{ is even}\end{cases}$$

and replace ϱ by $D\varrho$ in (11), we obtain

$$M_q(n, D(n-\varrho))\le\frac{\mu(2n^2+\mu\varrho(n-\varrho)^2)}{n-2\mu\varrho^2}$$

if $0\le\varrho<\sqrt{n/(2\mu)}$. In this case the asymptotic bounds (15)–(17) are: as n tends to infinity, then

$$M_q(n, D(n-\varrho))\lesssim\begin{cases}\mu n(2+\mu\varrho) & \text{if } 0\le\varrho=\mathrm{o}(n^{1/2}),\\ \mu n(2.2+\log(1+\mu\varrho)) & \text{if } 0\le\varrho=\mathrm{o}(n^{1/3}),\\ \tfrac{1}{2}\mu n\log\varrho & \text{if } \varrho=\mathrm{o}(n^{1/3}) \text{ and } \varrho\to\infty.\end{cases}$$

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