The Stability of Certain Partial Difference Equations

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Abstract—This paper is concerned with the linear delay partial difference equation

\[ u(i, j + 1) = a(i, j)u(i + 1, j) + b(i, j)u(i, j) + p(i, j)u(i - \sigma, j - \tau), \]

where \( \sigma \) and \( \tau \) are positive integers, \( a(i, j), b(i, j), p(i, j) \) are real sequences defined on \( i \geq \sigma, j \geq \tau \). Sufficient conditions for this equation to be stable are derived. Stability of certain nonlinear partial difference equations is studied also. (C) 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Partial difference equations have been posed from various practical problems [1,2] and from the approximation of solutions of partial differential equations by the finite difference methods [3,4]. Recently, the qualitative analysis of partial difference equations has received much attention; see [5-12].

In this paper, we first consider the linear partial difference equation

\[ u(i, j + 1) = a(i, j)u(i + 1, j) + b(i, j)u(i, j) + p(i, j)u(i - \sigma, j - \tau), \tag{1.1} \]

where \( \sigma \) and \( \tau \) are positive integers, \( a(i, j), b(i, j), p(i, j) \) are real sequences defined on \( i \geq \sigma, j \geq \tau \).

Let \( N_t = t, t + 1, \ldots \) and \( \Omega = \mathbb{N}_\sigma \times \mathbb{N}_\tau \setminus \mathbb{N}_0 \times \mathbb{N}_1 \). It is easy to construct by induction a double sequence \( u(i, j) \) which equals \( \varphi_{i,j} \) on \( \Omega \) and satisfies (1.1) on \( \mathbb{N}_0 \times \mathbb{N}_1 \). The solution of the initial value problem of (1.1) is unique.

Stability of (1.1) has been investigated in [11,12]. Based on [12] and applying Liapunov functions, we obtain some new and extensive stability criteria. Let

\[ ||\varphi|| = \sup_{(i,j) \in \Omega} |\varphi(i,j)|. \]

For any \( H > 0 \), let \( S_H = \{ \varphi \mid ||\varphi|| < H \} \).

DEFINITION 1.1. (See [12].) Equation (1.1) is said to be stable if for every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that for every \( \varphi \in S_\delta \), the corresponding solution \( u = \{ u(i, j) \} \) of (1.1) satisfies

\[ |u(i, j)| < \epsilon, \quad i, j \in \mathbb{N}_0. \]

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Definition 1.2. (See [12].) Equation (1.1) is said to be exponentially asymptotically stable if for any \( \delta > 0 \), there exists a real number \( \xi \in (0,1) \) such that \( \varphi \in S_{\delta} \) implies that

\[
|u(i,j)| \leq \delta^j \xi^j, \quad i,j \in N_0.
\]

Definition 1.3. Equation (1.1) is said to be linearly stable if there exists a constant \( M \geq 0 \) such that the solutions of (1.1) satisfy

\[
|u(i,j)| \leq M \| \varphi \|, \quad (i,j) \in N_0^2.
\]

Remark 1.1. It is easy to see that equation (1.1) is stable if equation (1.1) is linearly stable.

Definition 1.4. For a real function \( h : N^2_0 \rightarrow (0, \infty) \), equation (1.1) is said to be \( h \)-stable, if

\[
|h(i,j)u(i,j)| \leq M \| \varphi \|, \quad (i,j) \in N_0^2,
\]

where \( M \) is a positive constant.

2. MAIN RESULTS

Let \( V(u,i,j) : R \times N_0^2 \rightarrow R^+ = [0, \infty) \). If for any solution \( \{u(i,j)\} \) of equation (1.1), there exists a constant \( c > 0 \) such that

\[
V(u,i,j) > c|u(i,j)|, \quad (i,j) \in N_0^2,
\]

then we call \( V(u,i,j) \) as a positive Liapunov function.

The following lemmas are obvious.

Lemma 2.1. If there exists a positive Liapunov function \( V(u,i,j) \) and a constant \( M > 0 \) such that

\[
V(u,i,j) \leq M \| \varphi \|, \quad (i,j) \in N_0^2,
\]

where \( \{u(i,j)\} \) is a solution of (1.1) with the initial function \( \{\varphi(i,j)\} \), then (1.1) is linearly stable.

Lemma 2.2. For a given real function \( h(i,j) > 0 \), if the following equation:

\[
h(i,j+1)u(i,j+1) - a(i,j)h(i+1,j)u(i+1,j) + b(i,j)h(i,j)u(i,j) + p(i,j)h(i-\sigma,j-\tau)u(i-\sigma,j-\tau)
\]

is linearly stable, then (1.1) is \( h \)-stable, where \( \{u(i,j)\} \) is a solution of (1.1) with the initial function \( \{\varphi(i,j)\} \).

In the following, we will show two kinds of results according to the different order in the space \( N_0^2 \). At first, we define an order in \( N_0^2 \) as follows. For any points \( (i,j) \) and \( (p,q) \) in \( N_0^2 \), if \( j < q \), then claim that \( (i,j) < (p,q) \).

Lemma 2.3. (See [12].) Assume that

\[
|a(i,j)| + |b(i,j)| + |p(i,j)| \leq 1, \quad (i,j) \in N_0^2.
\]  \hspace{1cm} (2.1)

Then equation (1.1) is linearly stable.

Proof. For a given solution \( \{u(i,j)\} \) of equation (1.1), let

\[
V(u,i,j) = \max_{j \geq 0} |u(i,j)|, \quad j \geq 0,
\]

where
and denote

\[ w_u(j) = V(u, i, j). \]

For \( j = 0 \), we have

\[ w_u(0) = V(u, i, 0) = V(\varphi, i, 0) \leq \| \varphi \|. \]

From (1.1), we have

\[ w_u(0) = v_{\kappa}(i, 0) = v_{\kappa}(i, 0) \leq \| \varphi \|. \]

Hence, \( w_u(1) < \| \varphi \| \). Assume that for some fixed integer \( n \geq 0 \)

\[ w_u(j) \leq \| \varphi \|, \quad j < n. \]

Then, we have

\[ |u(i, n)| \leq (|a(i, n - 1)| + |b(i, n - 1)| + |p(i, n - 1)|) \| \varphi \| \leq \| \varphi \|. \]

By induction, we have

\[ w_u(n) < \| \varphi \|, \quad n \geq 0. \]

That is, \( |u(i, j)| \leq \| \varphi \|, \quad i, j \geq 0. \) The proof is complete.

**Theorem 2.1.** Let

\[ c_j' = \sup_{i \geq 0}(|a(i, j)| + |b(i, j)| + |p(i, j)|), \quad c_j = \max (1, c_j'), \quad (2.2) \]

and \( c_i = 1 + r_j, \quad j \geq 1. \) If

\[ \sum_{j=1}^{\infty} r_j < \infty, \quad (2.3) \]

then equation (1.1) is linearly stable.

**Proof.** Similar to the proof of Lemma 2.2, by induction we can prove the following estimation:

\[ w_u(n) \leq \left( \prod_{j=1}^{n-1} c_j \right) \| \varphi \|, \quad n \geq 1. \]

Hence,

\[ \ln w_u(n) \leq \ln \| \varphi \| + \sum_{j=1}^{n-1} \ln c_j = \ln \| \varphi \| + \sum_{j=1}^{n-1} \ln (1 + r_j) \leq \ln \| \varphi \| + \sum_{j=1}^{n-1} r_j \leq \sum_{j=1}^{\infty} r_j, \]

and hence,

\[ w_u(n) \leq \| \varphi \| \exp \left( \sum_{j=1}^{\infty} r_j \right) = M \| \varphi \|, \]

where \( M = \exp(\sum_{j=1}^{\infty} r_j) \). The proof is complete.

**Remark 2.1.** Theorem 2.1 improves Lemma 2.3.
EXAMPLE 2.1. Consider the partial difference equation
\[ u(i, j + 1) = \frac{1}{2} u(i + 1, j) + \frac{1}{2} u(i, j) + 2^{-j} u(i - \sigma, j - \tau), \] (2.4)
where \( \sigma, \tau \) are integers. Thus,
\[
|a(i, j)| = \frac{1}{2}, \quad |b(i, j)| = \frac{1}{2}, \quad |p(i, j)| = 2^{-j},
\]
\[
c_j = 1 + 2^{-j}, \quad r_j = 2^{-j},
\]
since
\[
\sum_{j=1}^{\infty} r_j = 1 < \infty.
\]
Thus, from Theorem 2.1, equation (2.4) is linearly stable.

Obviously, the condition (2.1) does not hold. So Lemma 2.3 cannot apply to (2.4).

From Theorem 2.1, it is easy to see the following result.

COROLLARY 2.1. If \( a(i, j), b(i, j), p(i, j) \) converge uniformly to zero in \( j \), then equation (1.1) is linearly stable.

In fact, in this case, there exists an integer \( t > 0 \) such that
\[
|a(i, j)| + |b(i, j)| + |p(i, j)| < 1, \quad j > t.
\]
Hence, the conditions of Theorem 2.1 are satisfied.

LEMMA 2.4. Assume that if \( \tau = 0 \), there exists \( \xi \in (0, 1) \) such that
\[
|a(i, j)| + |b(i, j)| + |p(i, j)| \xi^{-\tau} \leq \xi, \quad (i, j) \in N^2_0. \quad (2.5)
\]
Then equation (1.1) is \( \xi^{-\tau} \)-stable.

PROOF. Let \( h(i, j) = \xi^{-j}, \quad j \geq 0 \). Because
\[
h(i, j + 1) u(i, j + 1) = a(i, j) h(i + 1, j) u(i + 1, j) + b(i, j) h(i, j) u(i, j) + p(i, j) h(i, j) u(i, j)
\]
\[+ p(i, j) h(i - \sigma, j - \tau) u(i - \sigma, j - \tau), \]
implies that
\[
u(i, j + 1) = \left[ a(i, j) u(i + 1, j) + b(i, j) u(i, j) + p(i, j) \xi^{-\tau} u(i, j - \tau) \right] \xi,
\]
from (2.5), we have
\[
\left[ |a(i, j)| + |b(i, j)| + |p(i, j)| \xi^{-\tau} \right] \xi \leq 1.
\]
Hence, from Theorem 2.1, equation (2.6) is linearly stable.

Then, from Lemma 2.2, equation (1.1) is \( \xi^{-\tau} \)-stable. The proof is complete.

If \( a(i, j) \neq 0 \), for any \( i, j \geq 0 \), then equation (1.1) can be written in the form
\[
u(i + 1, j) = \frac{1}{a(i, j)} \left[ u(i, j + 1) - b(i, j) u(i, j) - p(i, j) u(i - \sigma, j - \tau) \right]. \quad (2.7)
\]
We define another order in \( N^2_0 \) as follows. For any points \((i, j)\) and \((p, q)\) in \( N^2_0 \), if \( i < p \), then claim that \((i, j) < (p, q)\). By the symmetry, we have following result.
THEOREM 2.2.

(i) If \(|1/a(i,j)| \leq 1\), then equation (1.1) is linearly stable.

(ii) Let

\[ d'_i = \sup_{j \geq 0} \left[ \frac{1}{|a(i,j)|} (1 + |b(i,j)| + |p(i,j)|) \right] \]

\[ d_j = \max(1, d'_i) \text{ and } d_i = 1 + s_i, \ i \geq 1. \]

If \( \sum_{i=1}^{\infty} s_i < \infty \), then equation (1.1) is linearly stable.

(iii) Assume that \( \tau = 0 \) and there exists \( \xi \in (0, 1) \) such that

\[ \frac{1}{|a(i,j)|} (1 + |b(i,j)| + |p(i,j)|) \xi^{-\tau} \leq \xi. \]

Then equation (1.1) is \( \xi^{-1} \)-stable.

COROLLARY 2.2. Assume that there exist constants \( H > 0 \) and \( L > 0 \) such that

\[ |f(u)| \leq L|u|, \quad \text{as } |u| \leq H. \]

(i) If

\[ |a(i,j)| + |b(i,j)| + |p(i,j)| \leq 1, \quad (i,j) \in N_0^2, \]

then equation (2.8) is stable, where \( \varphi \in S_H \). Notice that if \( \varphi \in S_H \), then \( \{u(i,j)\} \) satisfies \( |u| \leq H \).

(ii) If \( \sigma = 0 \) and there exists \( \xi \in (0, 1) \) such that

\[ |a(i,j)| + |b(i,j)| + L|p(i,j)| \xi^{-\tau} \leq \xi, \quad (i,j) \in N_0^2, \]

then the solution of (2.8) satisfies

\[ |u(i,j)| \leq M\|\varphi\|\xi^i, \]

where \( \|\varphi\| < H \).

COROLLARY 2.3. Assume that \( \sigma = 0 \), \( \lim_{|u| \to \infty} |f(u)/u| = 0 \), and there exist \( M > 0, \xi \in (0, 1) \), such that \( |p(i,j)| < M \) and

\[ |a(i,j)| + |b(i,j)| < \xi, \quad (i,j) \in N_0^2. \]

Then there exists a constant \( H > 0 \) such that

\[ |u(i,j)| \leq M\|\varphi\|\xi^i, \quad (i,j) \in N_0^2, \]

where \( \varphi \in S_H \).

Now we define a new order in \( N_0^2 \) as follows. Let \( (i,j), (p,q) \in N_0^2 \). If \( i + j < p + q \) or \( i + j = p + q, j < q \), then \( (i,j) < (p,q) \).
Theorem 2.3. Assume that there exist \( \alpha, \beta \in (0, 1) \) such that

(i) \[ |u(i, 0)| \leq \| \varphi \| \alpha^i, \quad i \geq 0; \]

(ii) \[ \alpha |a(i, j)| + |b(i, j)| + \alpha^{-\sigma} \beta^{-\tau} |p(i, j)| \leq \beta, \quad (i, j) \in N_0^2. \tag{2.9} \]

Then equation (1.1) is \( \alpha^{-i} \beta^{-j} \)-stable; i.e.,

\[ |u(i, j)| < M \| \varphi \| \alpha^i \beta^j, \quad (i, j) \in N_0^2. \]

Proof. Let \( h(i, j) = \alpha^{-i} \beta^{-j}. \)

We consider the following equation:

\[ h(i, j + 1)u(i, j + 1) = a(i, j)h(i + 1, j)u(i + 1, j) + b(i, j)h(i, j)u(i, j) + t(i, j)h(i - \sigma, j - \tau)u(i - \sigma, j - \tau), \tag{2.10} \]

which equals to

\[ u(i, j + 1) = \left[ a(i, j)u(i + 1, j) + b(i, j)u(i, j) + \alpha^{-\sigma} \beta^{-\tau} p(i, j)u(i - \sigma, j - \tau) \right] \beta^{-1}. \]

From (2.9), by Theorem 2.1, we can obtain that equation (2.10) is linearly stable. Using Lemma 2.2, equation (1.1) is \( h \)-stable; i.e.,

\[ |u(i, j)| \leq M \| \varphi \| \alpha^i \beta^j, \quad (i, j) \in N_0^2. \]

Similar to Corollaries 2.2 and 2.3, we have the following corollary.

Corollary 2.4. Assume that there exist \( L > 0, H > 0, \alpha > 0, \beta > 0 \) such that

\[ |f(u)| \leq L |u|, \quad |u| \leq H, \]

and

\[ |a(i, j)| + |b(i, j)| + \alpha^{-\sigma} \beta^{-\tau} |p(i, j)| \leq \beta, \quad (i, j) \in N_0^2. \]

Then the solutions of equation (2.8) satisfy

\[ |u(i, j)| \leq M \| \varphi \| \alpha^i \beta^j, \quad (i, j) \in N_0^2, \quad \alpha, \beta \in (0, 1); \]

i.e., equation (2.8) is \( \alpha^{-i} \beta^{-j} \)-stable.

Corollary 2.5. Assume that there exist \( \alpha > 0, \beta > 0 \) such that

\[ \lim_{|u| \to 0} \left| \frac{f(u)}{u} \right| = 0, \]

and there exist \( M > 0, \xi \in (0, 1) \) such that \( |p(i, j)| < M \) and

\[ |a(i, j)| + |b(i, j)| < \xi, \quad (i, j) \in N_0^2. \]

Then there exists \( H > 0 \) such that the solutions of (2.8) satisfy

\[ |u(i, j)| \leq M \| \varphi \| \alpha^i \beta^j, \quad (i, j) \in N_0^2, \quad \alpha, \beta \in (0, 1), \]

where \( \| \varphi \| < H. \)

Similar to Theorem 2.1, we can obtain the following result.
**Corollary 2.6.** Let

\[ p_j' = \sup_{i \geq 0} \left\{ \alpha |a(i,j)| + b(i,j) + \alpha^{-\gamma} \beta^{-\gamma} |p(i,j)| \right\} \beta^{-1}, \]

\[ p_j = \max(1, p_j'), \text{ and } p_j = 1 + t_j, j \geq 1. \]

If

\[ \sum_{j=1}^{\infty} t_j < \infty, \]

then equation (1.1) is \( \alpha^i \beta^j \)-stable.

**Example 2.2.** Consider the partial difference equation

\[ \frac{1}{16} u(i+1, j) + u(i, j) + \frac{1}{4} \left( \frac{13}{8} - \frac{1}{i+j+1} \right) u(i, j+1) + \frac{-1}{16(i+j+1)} u(i-2, j-2) = 0, \tag{2.11} \]

where \( |a(i,j)| = 1/16, \ |b(i,j)| = (1/4)(13/8 - (i+j+1)^{-1}), \ |p(i,j)| = |16(i+j+1)|^{-1}. \)

Let \( \alpha = 1/2, \ \beta = 1/2. \) Then \( \alpha |a(i,j)| + b(i,j) + \alpha^{-2} \beta^{-2} |p(i,j)| = 1/16 + 13/32 < 1/2 = \beta. \)

By Theorem 2.3, the solutions of (2.11) satisfy

\[ |u(i,j)| \leq M \| \varphi \| 2^{-(i+j)}, \quad (i,j) \in \mathbb{N}_0^2. \]

**References**