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REPRESENTATION OF LUKASIEWICZ ALGEBRAS BY MEANS OF ORDERED STONE SPACES

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In this paper the author defines the notion of a θ-valued Ordered Lukasiewicz Space and a study of the dual of the category of θ-valued Lukasiewicz algebras, using ordered topological spaces, is given.

The paper is devoted to the study of the dual of the category of θ-valued Lukasiewicz algebras (θ.v.L.a), using ordered topological spaces (see [4]). For the category of distributive lattices see [4, 5].

1. Preliminaries

The notions and notations for the category of θ-valued Lukasiewicz algebras, Luk (θ), are given in [3] and for the category of ordered topological spaces are given in [4].

Definition 1. (a) Let $(X, \leqslant)$ be a quasi-ordered set. A subset E of X will be called *increasing* if $x \in E$, $y \in X$, $x \leqslant y$ imply $y \in E$ (a decreasing set is defined dually).

(b) An *ordered space* is a triple $(X, \mathcal{T}, \leqslant)$, where X is a set, $\mathcal{T}$ a topology on X and $\leqslant$ a quasi-order on X.

(c) An ordered space $(X, \mathcal{T}, \leqslant)$ will be called *totally order disconnected* if, given $x, y \in X$, with $x \not\leqslant y$ there exist an increasing $\mathcal{T}$-clopen set U and a decreasing $\mathcal{T}$-clopen set L such that $U \cap L = \emptyset$, $x \in L$ and $y \in U$.

(d) An ordered space $(X, \mathcal{T}, \leqslant)$ will be called an *ordered Stone space* if it is compact and totally order disconnected.

Throughout this paper all distributive lattices are assumed to have zero and one and the homomorphisms preserve these elements. If A is a distributive lattice, then its dual space is defined to be $T(A) = (X, \mathcal{T}, \leqslant)$, where X is the set of homomorphisms from A onto $\{0, 1\}$, $\mathcal{T}$ is the product topology induced from $\{0, 1\}^A$, and $\leqslant$ is the partial order: $f \leqslant g$ in X iff $f(a) \leqslant g(a)$ for all $a \in A$.

If $\mathcal{X} = (X, \mathcal{T}, \leqslant)$ is an ordered Stone space, then the set

$$L(\mathcal{X}) = \{Y \subseteq X / Y \text{ increasing and } \mathcal{T}\text{-clopen}\}$$

is a distributive lattice with respect to union and intersection of subsets in X.

Lemma [4]. (a) *If A is a distributive lattice, then $T(A)=(X,\mathcal{T},\leqslant)$ is an ordered Stone space.*

(b) *If $\mathcal{X}=(X,\mathcal{T},\leqslant)$ is an ordered Stone space, then $L(\mathcal{X})$ is a distributive lattice with zero and one.*

Theorem 1 [4]. (a) *Let A be a distributive lattice. The map $F_A: A\to L(T(A))$ defined by $F_A(a)=\{f\in X \mid f(a)=1\}$ is a lattice isomorphism.*

(b) *Let $\mathcal{X}=(X,\mathcal{T},\leqslant)$ be an ordered Stone space. The map $G_{\mathcal{X}}:\mathcal{X}\to T(L(\mathcal{X}))$ defined by*

$$G_{\mathcal{X}}(x)(Y)=\begin{cases}1 & \text{if } x\in Y,\\ 0 & \text{if } x\notin Y,\end{cases}$$

for every $Y\in L(\mathcal{X})$ is an isomorphism of ordered Stone spaces.

(c) *If $f: A_1\to A_2$ is a lattice homomorphism, then the map $T(f): T(A_2)\to T(A_1)$ defined by $T(f)(g)=g\circ f$ is a homomorphism of ordered Stone spaces (a continuous and increasing map).*

(d) *If $H:\mathcal{X}_1\to\mathcal{X}_2$ is a homomorphism of ordered Stone spaces, then the map $L(h): L(\mathcal{X}_2)\to L(\mathcal{X}_1)$ defined by $L(h)(Y)=h^{-1}(Y)$ for every $Y\in L(\mathcal{X}_2)$ is a lattice homomorphism.*

(e) *If f and h are in* (c) *and* (d) *the diagrams*

$$\begin{array}{ccc} A_1 & \xrightarrow{\ f\ } & A_2 \\ {\scriptstyle F_{A_1}}\downarrow\wr & & \wr\downarrow{\scriptstyle F_{A_2}} \\ L(T(A_1)) & \xrightarrow[L(T(f))]{} & L(T(A_2)) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{\ h\ } & \mathcal{X}_2 \\ {\scriptstyle G_{\mathcal{X}_1}}\downarrow\wr & & \wr\downarrow{\scriptstyle G_{\mathcal{X}_2}} \\ T(L(\mathcal{X}_1)) & \xrightarrow[T(L(h))]{} & T(L(\mathcal{X}_2)) \end{array}$$

are commutative.

Theorem 1 shows that the category of ordered Stone spaces is equivalent with the dual of the category of distributive lattices.

2. Representation of θ-valued Lukasiewicz algebras

Let A be a θ.v.L.a. and $T(A)$ the ordered Stone space of the distributive lattice A. For every $\alpha\in\mathcal{T}$, $\varphi_\alpha: A\to A$ is a lattice homomorphism· let $\phi_\alpha: T(A)\to T(A)$ defined by $\phi_\alpha=T(\varphi_\alpha)$.

Lemma 1. *The family of maps $(\phi_\alpha \mid \alpha\in J)$ satisfies the following conditions:*

(a) *ϕ_α is a continuous and increasing map, for every $\alpha\in J$;*

(b) *$\phi_\alpha\circ\phi_\beta=\phi_\alpha$ for every $\alpha_1\beta\in J$;*

(c) *if $\alpha,\beta\in J$, $\alpha\leqslant\beta$, then $\phi_\alpha^{-1}(Y)\supseteq\phi_\beta^{-1}(Y)$ for every $Y\in L(T(A))$;*

(d) *if $Y\in L(T(A))$, then $\phi_\alpha^{-1}(Y)$ and $T(A)\backslash\phi_\alpha^{-1}(Y)$ are in $L(T(A))$, for every $\alpha\in J$;*

(e) *if $Y,Z\in L(T(A))$ and for every $\alpha\in J$ $\phi_\alpha^{-1}(Y)=\phi_\alpha^{-1}(Z)$, then $Y=Z$.*

Proof. (a) from Theorem 1 (c);

(b) $(\phi_\alpha \circ \phi_\beta)(f) = \phi_\alpha(\phi_\beta(f)) = \phi_\alpha(f \circ \varphi_\beta) = (f \circ \varphi_\beta) \circ \varphi_\alpha = f \circ (\varphi_\beta \circ \varphi_\alpha) = f \circ \varphi_\alpha = \phi_\alpha(f)$ for every $f \in T(A)$;

(c) In the conditions of Lemma 1, if $f \in \phi_\beta^{-1}(Y)$, then $\phi_\beta(f) \in Y$, or $f \circ \varphi_\beta \in Y$; but for $\alpha \leqslant \beta$ we have $\varphi_\alpha \geqslant \varphi_\beta$. It follows that $f \circ \varphi_\alpha \geqslant f \circ \varphi_\beta$ and $f \circ \varphi_\alpha \in Y$, or $f \in \phi_\alpha^{-1}(Y)$ (only the fact that Y is increasing has been used);

(d) using Theorem 1a), there exist $a \in A$ such that $Y = F_A(a) = \{f \in T(A) \mid f(a) = 1\}$; then $\phi_\alpha^{-1}(Y)$ and $T(A)\backslash\phi_\alpha^{-1}(Y)$ are $\mathcal{T}$-clopen because ϕ_α is continuous; $\phi_\alpha^{-1}(T)$ is increasing and we have:

$$f \in T(A)\backslash\phi_\alpha^{-1}(Y) \Leftrightarrow f \notin \phi_\alpha^{-1}(Y) \Leftrightarrow \phi_\alpha(f) \notin Y = F(a) \Leftrightarrow \phi_\alpha(f)(a) = 0$$
$$\Leftrightarrow f(\varphi_\alpha(a)) = 0 \Leftrightarrow f(\bar{\varphi}_\alpha(a)) = 1 \Leftrightarrow f \circ F_A(\bar{\varphi}_\alpha(a)),$$

then $T(A)\backslash\phi_\alpha^{-1}(Y) = F_A(\bar{\varphi}_\alpha(a))$ and it follows that $T(A)\backslash\phi_\alpha^{-1}(Y)$ is increasing.

(e) Using Theorem 1(a) there exists $a, b \in A$ such that $Y = FA(a)$ and $Z = F_A(b)$; we have:

$$\phi_\alpha^{-1}(Y) = \{f \mid \phi_\alpha(f) \in Y\} = \{f \mid f \circ \varphi_\alpha \in F_A(a)\} = \{f \mid f(\varphi_\alpha(a)) = 1\} = F_A(\varphi_\alpha(a))$$

and $\phi_\alpha^{-1}(Z) = F_A(\varphi_\alpha(b))$. If $\phi_\alpha^{-1}(Y) = \phi_\alpha^{-1}(Z)$ for every $\alpha \in J$, then $F_A(\varphi_\alpha(a)) = F_A(\varphi_\alpha(b))$ for every $\alpha \in J$; it follows that $\varphi_\alpha(a) = \varphi_\alpha(b)$ for every $\alpha \in J$, therefore $a = b$ or $Y = Z$.

Definition 2. (a) An ordered Stone space $\mathcal{X} = (X, \mathcal{T}, \leqslant)$ will be called an *ordered θ-valued Lukasiewicz space* (o.L.s.) if a family of maps $(\phi_\alpha : X \to X \mid \alpha \in J)$ is given which satisfies the following conditions:

(1) for every $\alpha \in J$, ϕ_α is a continuous and increasing map;

(2) $\phi_\alpha \circ \phi_\beta = \phi_\alpha$ for every $\alpha, \beta \in J$;

(3) if $\alpha, \beta \in J$, $\alpha \leqslant \beta$, then $\phi_\alpha^{-1}(Y) \supseteq \phi_\beta^{-1}(Y)$ for every $Y \subseteq X$ increasing and $\mathcal{T}$-clopen;

(4) for every $\alpha \in J$ and $Y \subseteq X$ increasing and $\mathcal{T}$-clopen $\phi_\alpha^{-1}(Y)$ and $T(A)\backslash\phi_\alpha^{-1}(Y)$ are increasing;

(5) if $\phi_\alpha^{-1}(Y) = \phi_\alpha^{-1}(Z)$ for every $\alpha \in J$, where Y and Z are increasing and $\mathcal{T}$-clopen in $\mathcal{X}$, then $Y = Z$.

(b) If $\mathcal{X} = (X, \mathcal{T}, \leqslant, \{\phi_\alpha\}_{\alpha \in J})$ and $\mathcal{X}' = (X', \mathcal{T}', \leqslant', \{\phi'_\alpha\}_{\alpha \in J})$ are o.L.s., a continuous and increasing map $f : X \to X'$ will be called a *θ-application* if for every $\alpha \in J$, the diagrams:

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow{\scriptstyle \phi_\alpha} & & \downarrow{\scriptstyle \phi'_\alpha} \\ X & \xrightarrow[f]{} & X' \end{array}$$

are commutative.

We shall denote by $L(\theta)$ the category of o.L.s. and θ-applications.

Proposition 1. (a) *If A is a θ.v.L.a., then the ordered Stone space $T(A)$ with the family of maps $(\phi_\alpha \mid \alpha \in J)$ given in Lemma 1 is an ordered θ-valued Lukasiewiecz space.*

(b) *If $f: A \to A'$ is a homomorphism of θ.v.L.a., then the map $T(f): T(A') \to T(A)$ (given in Theorem 1(c)) is a θ-application.*

Proof. (a) Follows easily from Theorem 1 and Lemma 1.

(b) We shall prove that for every $\alpha \in J$ the diagram

$$\begin{array}{ccc} T(A') & \xrightarrow{T(f)} & T(A) \\ {\scriptstyle \phi'_\alpha}\downarrow & & \downarrow{\scriptstyle \phi_\alpha} \\ T(A') & \xrightarrow[T(f)]{} & T(A) \end{array}$$

is commutative; we have:

$$\begin{aligned}(\phi_\alpha \circ T(f))(g) &= \phi_\alpha(T(f)(g)) = \phi_\alpha(g \circ f) = (g \circ f) \circ \varphi_\alpha \\ &= g \circ (f \circ \varphi_\alpha) = g \circ (\varphi_\alpha \circ f) = (g \circ \varphi_\alpha) \circ f \\ &= T(f)(\phi'_\alpha(g)) = (T(f) \circ \phi'_\alpha)(g).\end{aligned}$$

Proposition 2. (a) *If $\mathscr{X} = (X, \mathscr{T}, \leqslant, \{\phi_\alpha\}_{\alpha \in J})$ is an o.L.s., then the lattice $L(\mathscr{X})$ associated with the ordered Stone space $(X, \mathscr{T}, \leqslant)$ with the family of maps $(\varphi_\alpha, \bar{\varphi}_\alpha \mid \alpha \in J)$ given by $\varphi_\alpha(Y) = \phi_\alpha^{-1}(Y)$, $\bar{\varphi}_\alpha(Y) = X \backslash \phi_\alpha^{-1}(Y)$ for every $\alpha \in J$ is a θ.v.L.a.*

(b) *If $\mathscr{X} = (X, \mathscr{T}, \leqslant, \{\phi_\alpha\}_{\alpha \in J})$, $\mathscr{X}' = (X', \mathscr{T}', \leqslant', \{\phi'_\alpha\}_{\alpha \in J})$ are o.L.s. and $f: X \to X'$ is a θ-application, then $L(f): L(\mathscr{X}') \to L(\mathscr{X})$ (given in Theorem 1(d)) is a homomorphism of θ.v.L.a.*

Proof. (a) The maps φ_α, $\bar{\varphi}_\alpha$ are well defined (conditions 2 and 3, Definition 2). We shall prove that the conditions of the definition of θ.v.L.a. are verified:

$$\varphi_\alpha(Y \cup Z) = \phi_\alpha^{-1}(Y \cup Z) = \phi_\alpha^{-1}(Y) \cup \phi_\alpha^{-1}(Z) = \varphi_\alpha(Y) \cup \varphi_\alpha(Z),$$

$$\varphi_\alpha(Y \cap Z) = \phi_\alpha^{-1}(Y \cap Z) = \phi_\alpha^{-1}(Y) \cap \phi_\alpha^{-1}(Z) = \varphi_\alpha(Y) \cap \varphi_\alpha(Z),$$

for every $\alpha \in J$ and $Y, Z \in L(\mathscr{X})$.

$$\varphi_\alpha(\emptyset) = \phi_\alpha^{-1}(\emptyset) = \emptyset, \qquad \varphi_\alpha(X) = \phi_\alpha^{-1}(X) = X$$

for every $\alpha \in J$.

$$\varphi_\alpha(Y) \cup \bar{\varphi}_\alpha(Y) = \phi_\alpha^{-1}(Y) \cup [X \backslash \phi_\alpha^{-1}(Y)] = X,$$

$$\varphi_\alpha(Y) \cap \bar{\varphi}_\alpha(Y) = \phi_\alpha^{-1}(Y) \cap [X \backslash \phi_\alpha^{-1}(Y)] = \emptyset,$$

for every $\alpha \in J$ and $Y \in L(\mathscr{X})$.

If $\alpha \leqslant \beta$, then $\phi_\alpha^{-1}(Y) \supseteq \phi_\beta^{-1}(Y)$ or $\varphi_\alpha(Y) \supseteq \varphi_\beta(Y)$ for every $Y \in L(\mathscr{X})$.

$$(\varphi_\alpha \circ \varphi_\beta)(Y) = \varphi_\alpha(\phi_\beta^{-1}(Y)) = \phi_\alpha^{-1}(\phi_\beta^{-1}(Y)) = (\phi_\beta \circ \phi_\alpha)^{-1}(Y) = \phi_\beta^{-1}(Y) = \varphi_\beta(Y)$$

for every $\alpha, \beta \in J$ and $Y \in L(\mathscr{X})$.

If $\varphi_\alpha(Y) = \varphi_\alpha(Z)$ for every $\alpha \in J$, then $\phi_\alpha^{-1}(Y) = \phi_\alpha^{-1}(Z)$ for every $\alpha \in J$, or $Y = Z$.

(b) $L(f)$ is a homomorphism of lattices; we prove that for every $\alpha \in J$ the diagram

$$\begin{array}{ccc} L(\mathscr{X}') & \xrightarrow{L(f)} & L(\mathscr{X}) \\ {\scriptstyle \varphi_\alpha}\downarrow & & \downarrow{\scriptstyle \varphi'_\alpha} \\ L(\mathscr{X}') & \xrightarrow[L(f)]{} & L(\mathscr{X}) \end{array}$$

is commutative:

$$\begin{aligned} (\varphi_\alpha \circ L(f))(Y) &= \varphi_\alpha(L(f)(Y)) = \varphi_\alpha(f^{-1}(Y)) = \phi_\alpha^{-1}(f^{-1}(Y)) \\ &= (f \circ \phi_\alpha)^{-1}(Y) = (\phi'_\alpha \circ f)^{-1}(Y) = f^{-1}(\phi_\alpha^{-1}(Y)) \\ &= f^{-1}(\varphi'_\alpha(Y)) = (L(f) \circ \varphi'_\alpha)(Y). \end{aligned}$$

Lemma 2. (a) *For every θ.v.L.a. A, the isomorphism of distributive lattices given by Theorem* 1(a),

$$F_A : A \to L(T(A))$$

is an isomorphism of θ.v.L.a.

(b) *For every o.L.s.* $\mathscr{X} = (X, \mathscr{T}, \leqslant, \{\phi_\alpha\}_{\alpha \in J})$, *the isomorphism of ordered Stone spaces given by Theorem* 1(b),

$$G_{\mathscr{X}} : \mathscr{X} \to T(L(\mathscr{X}))$$

is an isomorphism of o.L.s.

Proof. (a) We prove that for every $\alpha \in J$, the diagram

$$\begin{array}{ccc} A & \xrightarrow{F_A} & L(T(A)) \\ {\scriptstyle \varphi_\alpha}\downarrow & & \downarrow{\scriptstyle L(T(\varphi_\alpha)) = \varphi_\alpha^*} \\ A & \xrightarrow[F_A]{} & L(T(A)) \end{array}$$

is commutative; we have:

$$(\varphi_\alpha^* \circ F_A)(a) = \varphi_\alpha^*(F_A(a)) = \phi_\alpha^{-1}(F_A(a)) = \phi_\alpha^{-1}\{f \in T(A) \mid f(a) = 1\},$$

$$(F_A \circ \varphi_\alpha)(a) = F_A(\varphi_\alpha(a)) = \{f \in T(A) \mid f(\varphi_\alpha(a)) = 1\},$$

$$\begin{aligned} f \in \phi_\alpha^{-1}\{f \in T(A)/f(a) = 1\} &\Leftrightarrow \phi_\alpha(f)(a) = 1 \Leftrightarrow (f \circ \varphi_\alpha)(a) = 1 \\ &\Leftrightarrow f(\varphi_\alpha(a)) = 1 \Leftrightarrow f \in (F_A \circ \varphi_\alpha)(a). \end{aligned}$$

(b) We prove that for every $\alpha \in J$, the diagram

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{G_{\mathscr{X}}} & T(L(\mathscr{X})) \\ {\scriptstyle \phi_\alpha}\downarrow & & \downarrow {\scriptstyle T(L(\phi_\alpha)) = \phi_\alpha^*} \\ \mathscr{X} & \xrightarrow[G_{\mathscr{X}'}]{} & T(L(\mathscr{X})) \end{array}$$

is commutative; we have:

$$(\phi_\alpha^* \circ G_{\mathscr{X}})(x), \qquad (G_{\mathscr{X}} \circ \phi_\alpha)(x) : L(\mathscr{X}) \to \{0, 1\},$$

$$\begin{aligned} (\phi_\alpha^* \circ G_{\mathscr{X}})(x)(Y) &= \phi_\alpha^*(G_{\mathscr{X}}(x))(Y) = (G_{\mathscr{X}}(x) \cdot \varphi_\alpha)(Y) = G_{\mathscr{X}}(x)(\varphi_\alpha(Y)) \\ &= G_{\mathscr{X}}(x)(\phi_\alpha^{-1}(Y)) \\ &= \begin{cases} 1 & \text{if } x \in \phi_\alpha^{-1}(Y) \\ 0 & \text{if } x \notin \phi_\alpha^{-1}(Y) \end{cases} \\ &= \begin{cases} 1 & \text{if } \phi_\alpha(x) \in Y, \\ 0 & \text{if } \phi_\alpha(x) \notin Y, \end{cases} \end{aligned}$$

$$(G_{\mathscr{X}} \circ \phi_\alpha)(x)(Y) = (G_{\mathscr{X}}(\phi_\alpha(x)))(Y) = \begin{cases} 1 & \text{if } \phi_\alpha(x) \in Y, \\ 0 & \text{if } \phi_\alpha(x) \notin Y. \end{cases}$$

Theorem 2. *The dual of the category* Luk (θ) *is equivalent with the category* $L(\theta)$.

Proof. If we consider the functors

$$T : \text{Luk}\,(\theta) \to L(\theta), \qquad L : L(\theta) \to \text{Luk}\,(\theta)$$

it follows from Lemma 2 and Theorem 1(e) that we have the functorial isomorphisms

$$F : \text{Id}_{\text{Luk}(\theta)} \to L \circ T, \qquad G : \text{Id}_{L(\theta)} \to T \circ L.$$

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