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Asymptotics of Sobolev Orthogonal Polynomials for Coherent Pairs of Measures*

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Strong asymptotics for the sequence of monic polynomials $Q_n(z)$, orthogonal with respect to the inner product

$$(f, g)_S = \int f(x)\, g(x)\, d\mu_1(x) + \lambda \int f'(x)\, g'(x)\, d\mu_2(x), \qquad \lambda > 0,$$

with z outside of the support of the measure μ_2, is established under the additional assumption that μ_1 and μ_2 form a so-called coherent pair with compact support. Moreover, the asymptotic behaviour of the (square of) the norm $(Q_n, Q_n)_S$ and of the zeros of Q_n is obtained. © 1998 Academic Press

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1. INTRODUCTION AND MAIN RESULTS

The study of orthogonal polynomials with respect to inner products that involve derivatives (the so-called Sobolev orthogonal polynomials) has been a very active field of research for the past 10 years (see [8] for a wide bibliography on this subject). Although algebraic properties of such polynomials (existence, recurrence relations, etc.) have been widely studied, the definitive asymptotic results were known only for the case when the measure associated with the derivatives is discrete (see [5, 9] and recently [1]). The outer as well as the inner asymptotics for the orthogonal polynomials with both measures having an absolutely continuous component (excluding the trivial cases) is in general an open question. In this paper we study orthogonality with respect to the inner product

$$(f, g)_S = \int f(x)\, g(x)\, d\mu_1(x) + \lambda \int f'(x)\, g'(x)\, d\mu_2(x),$$

with $\lambda > 0$, in the case when both measures are non-discrete compactly supported on $\mathbf{R}$ and satisfy an additional assumption—they form a coherent pair.

The concept of coherence was introduced by Iserles *et al.* in [3], with the goal of obtaining Sobolev orthogonal polynomials that satisfy certain special properties. Later the algebraic properties of the Sobolev orthogonal polynomials with respect to coherent pairs and the relations involving also the orthogonal polynomials corresponding to both measures that form the coherent pair have been widely studied by Marcellán and co-workers [6, 7], Meijer [11, 12], Pérez [16], and others. Let us recall this concept for a pair of positive measures.

DEFINITION 1. Let μ_i, $i = 1, 2$, be two positive measures and let $\{P_n(x)\}_n$ and $\{T_n(x)\}_n$ be the respective monic orthogonal polynomial sequences (MOPS). Then (μ_1, μ_2) is a coherent pair of measures if there exist non-zero constants $\sigma_1, \sigma_2, \ldots$ such that

$$T_n(x) = \frac{P'_{n+1}(x)}{n+1} - \sigma_n \frac{P'_n(x)}{n}, \qquad n \geqslant 1. \tag{1}$$

Recently, Meijer [13] gave the complete classification of coherent pairs. In particular, he proved that if (μ_1, μ_2) form a coherent pair of measures then necessarily one of the two measures μ_i must be classical (Laguerre or Jacobi).

Let w_1, w_2 be two non-negative weights on $(-1, 1)$ related by

$$\frac{w_2(x)}{w_1(x)} = \frac{1-x^2}{|x-\xi|}, \tag{2}$$

for a fixed $\xi \in \mathbf{R} \setminus (-1, 1)$, and let ν_1, ν_2 be the corresponding absolutely continuous measures on $[-1, 1]$:

$$d\nu_i(x) = w_i(x)\, dx, \quad i = 1, 2. \tag{3}$$

Then, following [13], the complete classification of all coherent pairs of measures with compact support (up to a constant factor or a linear change of variable) is given by the following

PROPOSITION 1. *Let μ_1, μ_2 be two measures, and let the support* $\operatorname{supp}(\mu_1) = [-1, 1]$. *Then (μ_1, μ_2) form a coherent pair of measures if and only if one of the three following cases holds*:

Case 1 (*Absolutely Continuous Case*). $\mu_i = \nu_i$, $i = 1, 2$, *where either*

$$w_1(x) = \rho^{(\alpha, \beta)}(x)$$

or

$$w_2(x) = \rho^{(\alpha, \beta)}(x)$$

with $\rho^{(\alpha, \beta)}(x) = (1-x)^\alpha (1+x)^\beta$.

Case 2 (*Mass Points in the First Measure*).

$$\mu_1 = \nu_1 + M\delta_\xi, \qquad \mu_2 = \nu_2, \quad M > 0,$$

where

$$w_2(x) = \rho^{(0, \beta)}(x) \qquad \text{and} \qquad \xi = 1$$

or

$$w_2(x) = \rho^{(\alpha, 0)}(x) \qquad \text{and} \qquad \xi = -1.$$

Case 3 (*Mass Points in the Second Measure*).

$$\mu_1 = \nu_1, \quad \mu_2 = \nu_2 + M\delta_\xi, \qquad M > 0, \quad |\xi| \geqslant 1,$$

where

$$w_1(x) = \rho^{(\alpha, \beta)}(x).$$

In all the cases v_1 *and* v_2 *are related by* (2), (3) *and* $\alpha, \beta \in \mathbf{R}$ *can take any admisible value* (*i.e., such that* $w_1, w_2 \in L_1[-1, 1]$).

Given a coherent pair of measures (μ_1, μ_2) we consider the Sobolev inner product

$$(p, q)_S = \langle p, q \rangle_1 + \lambda \langle p', q' \rangle_2$$

where $\langle p, q \rangle_i = \int p(x)\, q(x)\, d\mu_i(x)$, $i = 1, 2$, and $\lambda > 0$ is fixed. It is easy to see that $(\cdot, \cdot)_S$ is an inner product, so that there exists a monic orthogonal polynomial sequence, $\{Q_n\}_n$, with respect to $(\cdot, \cdot)_S$.

The following theorem allows us to obtain the strong asymptotics of the sequence $\{Q_n(x)\}_n$ and shows that it is actually described in terms of the measure μ_2.

THEOREM 1. *Let* (μ_1, μ_2) *be a coherent pair of measures,* $\operatorname{supp}(\mu_1) = [-1, 1]$, $\{T_n(x)\}_n$ *the MOPS associated to* μ_2, *and* $\{Q_n(x)\}_n$ *the MOPS with respect to* $(\cdot, \cdot)_S$. *Then,*

$$\lim_{n \to \infty} \frac{Q_n(x)}{T_n(x)} = \frac{1}{\Phi'(x)} \tag{4}$$

uniformly on compact subsets of $\bar{\mathbf{C}} \setminus [-1, 1]$, *where* $\Phi(x) = (x + \sqrt{x^2 - 1})/2$ *with* $\sqrt{x^2 - 1} > 0$ *when* $x > 1$.

In the next section this theorem is proved. In Section 3 we study the norm and zero asymptotic behaviour of $Q_n(x)$ and establish that the zero asymptotics, as expected, can be described in terms of the support of the measure μ_2.

A concept, closely connected with coherent pairs, is symmetric coherence (see [3]), which takes place in the case that both μ_1 and μ_2 are symmetric with respect to the origin, and in the right-hand side of (1) the indices $n+2$ and n are involved. In this case, (4) can also be established following the same line of reasoning that we propose. Nevertheless, we believe that actually the asymptotic relation (4) is a general fact that takes place under much milder assumptions on both measures, say when their absolutely continuous parts belong to the Szegő class.

2. PROOF OF THEOREM 1

In order to establish Theorem 1 we need some preliminary results that may be of independent interest. In fact, (4) is a direct consequence of the following algebraic relation, obtained in [3] (see also [6] and [16]), and whose proof we present for the sake of completeness.

PROPOSITION 2. *Let (μ_1, μ_2) be a coherent pair of measures. Then, with the notation introduced in Section* 1, *the following relation holds*:

$$P_{n+1}(x) - \sigma_n \frac{n+1}{n} P_n(x) = Q_{n+1}(x) - \alpha_n(\lambda) Q_n(x), \qquad n \geqslant 1, \tag{5}$$

where

$$\alpha_n(\lambda) = \sigma_n \frac{n+1}{n} \frac{k_n}{\tilde{k}_n} \neq 0, \qquad n \geqslant 1, \tag{6}$$

with $k_n = \langle P_n(x), P_n(x) \rangle_1$ and $\tilde{k}_n = (Q_n, Q_n)_S$.

Proof. The coefficients of $Q_n(x)$ are rational functions of λ with numerator and denominator of the same degree. Hence, for $\lambda \to \infty$ there exists a limit polynomial $R_n(x)$, $n \geqslant 0$, of the polynomial $Q_n(x)$:

$$R_0(x) = Q_0(x) = 1$$
$$R_1(x) = Q_1(x)$$
$$R_n(x) = \lim_{\lambda \to +\infty} Q_n(x), \qquad n \geqslant 2.$$

Clearly, R_n is monic, has degree exactly n, and is independent of λ. Moreover,

$$\langle R_n, 1 \rangle_1 = 0, \qquad n \geqslant 1,$$
$$\langle R_n', x^m \rangle_2 = 0, \qquad n \geqslant 2, \quad 0 \leqslant m \leqslant n-2.$$

From here we can deduce that

$$R_n(x) = P_n(x) - \sigma_{n-1} \frac{n}{n-1} P_{n-1}(x), \qquad n \geqslant 2, \tag{7}$$

$$R_n'(x) = n T_{n-1}(x), \qquad n \geqslant 2. \tag{8}$$

On the other hand, expressing R_{n+1} as a linear combination of $\{Q_i(x)\}$,

$$R_{n+1}(x) = Q_{n+1}(x) + \sum_{i=0}^{n} a_i^{(n+1)} Q_i(x),$$

and using the orthogonality and relations (7) and (8) we obtain:

$$a_i^{(n+1)} = -\sigma_n \frac{n+1}{n} \langle P_n(x), Q_i(x) \rangle_1 \tilde{k}_i^{-1}.$$

Thus $a_i^{(n+1)}=0$ when $0\leqslant i\leqslant n-1$ and

$$a_n^{(n+1)}=-\sigma_n\,\frac{n+1}{n}\,\frac{k_n}{\tilde{k}_n}=-\alpha_n(\lambda). \qquad \blacksquare$$

In this way, in order to obtain from (5) the asymptotics of the $Q_n(x)$ it is essential to study the limit behaviour of the parameters σ_n and $\alpha_n(\lambda)$.

PROPOSITION 3. *The parameters σ_n of the coherence relation* (1) *satisfy*:

(1) *In Case* 3 *of Proposition* 1,

$$\lim_{n\to\infty}\sigma_n=\Phi(\xi). \tag{9}$$

(2) *In the rest of the cases,*

$$\lim_{n\to\infty}\sigma_n=\frac{1}{4\Phi(\xi)}, \tag{10}$$

where we assume $\Phi(\pm1)=\pm1/2$.

Proof. The coherence condition (1) can be rewritten as follows:

$$\sigma_n=\frac{\dfrac{1}{n+1}\,\dfrac{P'_{n+1}(x)}{P_{n+1}(x)}\,\dfrac{P_{n+1}(x)}{P_n(x)}-\dfrac{T_n(x)}{P_n(x)}}{\dfrac{1}{n}\,\dfrac{P'_n(x)}{P_n(x)}}. \tag{11}$$

Note that when (μ_1,μ_2) belong to Case 1 or 2 of Proposition 1, both μ_1, μ_2 satisfy Szegő's condition. Recall that a measure μ on $[-1,1]$ whose absolutely continuous part is given by a weight $w(x)$ satisfies Szegő's condition (and we write $\mu\in S$) if

$$\rho(\theta)=\pi w(\cos\theta)\;|\sin\theta|\in L_1[0,2\pi]$$

and

$$\int_0^{2\pi}\log\rho(\theta)\,d\theta>-\infty.$$

To such a measure we can associate the Szegő function

$$D(z,\mu):=\exp\left\{\frac{1}{4\pi}\int_0^{2\pi}\log\rho(\theta)\,\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\right\},\qquad |z|<1, \tag{12}$$

so that the sequence of monic orthogonal polynomials with respect to μ, $\{P_n(x)\}_n$, has the following strong outer asymptotics:

$$P_n(x)=\frac{D(0,\mu)\,\Phi^n(x)}{D\left(\frac{1}{2\Phi(x)},\mu\right)}\,(1+o(1)), \tag{13}$$

uniformly on compact subsets of $\mathbf{C}\setminus[-1,1]$, with $\Phi(x)$ given in Theorem 1.

In particular, in this case we have that

$$\frac{P_{n+1}(x)}{P_n(x)}\to\Phi(x), \qquad \frac{P_n'(x)}{nP_n(x)}\to\frac{1}{\sqrt{x^2-1}}, \tag{14}$$

uniformly on compact subsets of $\overline{\mathbf{C}}\setminus[-1,1]$. Finally, it follows from (13) that the relative asymptotics $T_n(x)/P_n(x)$ when $\mu_1, \mu_2\in S$ is

$$\frac{T_n(x)}{P_n(x)}\to\frac{D(0,\mu_2)}{D(0,\mu_1)}\,\frac{D\left(\frac{1}{2\Phi(x)},\mu_1\right)}{D\left(\frac{1}{2\Phi(x)},\mu_2\right)}, \tag{15}$$

uniformly on compact subsets of $\mathbf{C}\setminus[-1,1]$, and in consequence only depends on the ratio of the weights $w_2(x)/w_1(x)$, which for a coherent pair is given by (2). These remarks allow us to follow a unifying approach for the proof of Theorem 1 in Cases 1 and 2. In fact, the Jensen–Poisson formula (see e.g. [15, p. 107]) and straightforward computation show that

$$\frac{D(z,\mu_2)}{D(z,\mu_1)}=\begin{cases}\dfrac{(1-z^2)\sqrt{-\Phi(\xi)}}{(z-2\Phi(\xi))}, & \xi\leqslant-1,\\[2ex] -\dfrac{(1-z^2)\sqrt{\Phi(\xi)}}{(z-2\Phi(\xi))}, & \xi\geqslant 1,\end{cases} \tag{16}$$

where we assume $\Phi(\pm1)=\pm1/2$. Hence, in Cases 1 and 2,

$$\frac{T_n(x)}{P_n(x)}\to\frac{\Phi(x)}{\Phi(\xi)}\cdot\frac{1-4\Phi(\xi)\,\Phi(x)}{1-4\Phi^2(x)}=\Phi'(x)\left(1-\frac{1}{4\Phi(\xi)\,\Phi(x)}\right). \tag{17}$$

Then, taking limits in (11) and using (14)–(17) we obtain that

$$\lim_{n\to\infty}\sigma_n=\frac{1}{4\Phi(\xi)}.$$

It remains for us to consider Case 3. Denote by $\{T_n^*(x)\}_n$ the sequence of monic orthogonal polynomials with respect to the absolutely continuous measure $v_2 \in S$. The coherence condition (1) now can be rewritten as

$$\sigma_n = \frac{\frac{1}{n+1}\frac{P'_{n+1}(x)}{P_{n+1}(x)}\frac{P_{n+1}(x)}{P_n(x)} - \frac{T_n(x)}{T_n^*(x)}\frac{T_n^*(x)}{P_n(x)}}{\frac{1}{n}\frac{P'_n(x)}{P_n(x)}}, \tag{18}$$

Since the asymptotic behaviour of $T_n^*(x)/P_n(x)$ is given by (17), it only remains to study the ratio $T_n(x)/T_n^*(x)$. This was done by several authors, see e.g. [14, Lemma 16, p. 132; 2; or 4]. Under our assumptions,

$$\frac{(x-\xi)\,T_n(x)}{T_{n+1}^*(x)} \to \left(1 - \frac{\Phi(\xi)}{\Phi(x)}\right)^2, \tag{19}$$

uniformly on compact subsets of $\overline{\mathbf{C}} \backslash [-1, 1]$. Using (14) and (19) we have

$$\begin{aligned}\lim_{n\to\infty} \frac{T_n(x)}{T_n^*(x)} &= \lim_{n\to\infty} \frac{T_n(x)}{T_{n+1}^*(x)} \lim_{n\to\infty} \frac{T_{n+1}^*(x)}{T_n^*(x)} \\ &= \left(1 - \frac{\Phi(\xi)}{\Phi(x)}\right)^2 \frac{1}{x-\xi}\,\Phi(x) = \frac{(\Phi(x)-\Phi(\xi))^2}{(x-\xi)\,\Phi(x)},\end{aligned} \tag{20}$$

uniformly on compact subsets of $\mathbf{C} \backslash ([-1, 1] \cup \{\xi\})$.

Then, taking limits in (18) and using (14)–(17) and (20), we prove that

$$\lim_{n\to\infty} \sigma_n = \Phi(x) - \frac{(\Phi(x)-\Phi(\xi))^2}{x-\xi}\left(1 - \frac{1}{4\Phi(\xi)\,\Phi(x)}\right) = \Phi(\xi). \quad \blacksquare \tag{21}$$

Proposition 4. *The sequence $\alpha_n(\lambda)$ defined in* (6) *satisfies*

$$\lim_{n\to\infty} \alpha_n(\lambda) = 0. \tag{22}$$

Proof. Recall that

$$\alpha_n(\lambda) = \sigma_n \frac{n+1}{n} \frac{k_n}{\tilde{k}_n} \neq 0, \qquad n \geqslant 1, \tag{23}$$

where $k_n = \langle P_n(x), P_n(x) \rangle_1$ and $\tilde{k}_n = (Q_n, Q_n)_S$. Using the extremal property of $\{T_n\}$, we have

$$\tilde{k}_n = \int_{-1}^{1} Q_n^2(x)\, d\mu_1(x) + \lambda \int_{-1}^{1} (Q_n'(x))^2\, d\mu_2(x) \geqslant \lambda n^2 \int_{-1}^{1} T_{n-1}^2\, dv_2(x)$$
$$\geqslant \lambda n^2 k_{n-1}'(v_2)$$

with $k_n'(v_2) = \inf \int (x^n + a_{n-1} x^{n-1} + \cdots + a_0)^2\, dv_2(x)$.

Therefore,

$$0 < \frac{k_n}{\tilde{k}_n} \leqslant \frac{k_n}{\lambda n^2 k_{n-1}'(v_2)}. \tag{24}$$

Since $\mu_1, v_2 \in S$,

$$\frac{k_n}{k_{n-1}'(v_2)} \to \frac{1}{4} \left(\frac{D(0, \mu_1)}{D(0, v_2)} \right)^2,$$

where as above $D(z, \mu_1)$ and $D(z, v_2)$ are the Szegő functions associated with μ_1 and v_2, respectively. Thus, taking limits in (24) we obtain

$$\lim_{n \to \infty} \frac{k_n}{\tilde{k}_n} = 0,$$

and it remains to use Proposition 3. ∎

Now, we are ready to prove the main result in one step.

Proof of Theorem 1. With the notation

$$Y_n(x) := \frac{Q_n(x)}{P_n(x)}, \qquad \delta_n(x) := \alpha_{n-1}(\lambda) \frac{P_{n-1}(x)}{P_n(x)},$$
$$\beta_n := 1 - \sigma_{n-1} \frac{n}{n-1} \frac{P_{n-1}(x)}{P_n(x)},$$

Eq. (5) can be rewritten as

$$Y_n(x) - \delta_n(x) Y_{n-1}(x) = \beta_n(x), \tag{25}$$

which uniquely defines the sequence $\{Y_n\}$ of analytic functions in $\overline{\mathbf{C}} \setminus [-1, 1]$, with the initial values $Y_0 = Y_1 = 1$. It is clear that

$$|Y_n(x)| \leqslant |\delta_n(x)|\, |Y_{n-1}(x)| + |\beta_n(x)|. \tag{26}$$

Using (14) and (22) we obtain that there exists $n_0 \in \mathbf{N}$ such that

$$|\delta_n(x)| < \tfrac{1}{2}, \qquad n \geqslant n_0. \tag{27}$$

On the other hand,

$$|\beta_n(x)| = \left|1 - \sigma_{n-1} \frac{n}{n-1} \frac{P_{n-1}(x)}{P_n(x)}\right| \leqslant 1 + \frac{n}{n-1} |\sigma_{n-1}| \left|\frac{P_{n-1}(x)}{P_n(x)}\right|.$$

From (9), (10), (14), and the inequality $|\Phi(x)| > 1/2$ for $x \notin [-1, 1]$, we deduce that there exist $B > 0$ and $n_1 \in \mathbf{N}$ such that

$$|\beta_n(x)| < B, \qquad n \geqslant n_1. \tag{28}$$

Then, by (27) and (28) in (26), we have for $n \geqslant n_2 = \max\{n_0, n_1\}$ that

$$|Y_n(x)| < \tfrac{1}{2} |Y_{n-1}(x)| + B. \tag{29}$$

Consider the sequence

$$Z_n(x) = \begin{cases} |Y_n(x)|, & n \leqslant n_2, \\ \tfrac{1}{2} Z_{n-1}(x) + B, & n > n_2. \end{cases}$$

For $m > n_2$,

$$Z_{m+r} = \left(\frac{1}{2}\right)^r Z_m + 2B\left(1 - \frac{1}{2^r}\right), \qquad r = 1, 2, \ldots. \tag{30}$$

Taking limits when $r \to \infty$ in (30), we obtain that $Z_n(x)$ is uniformly bounded for all n sufficiently large. Moreover, $0 < |Y_n(x)| \leqslant Z_n(x)$, for all $n \in \mathbf{N}$. Hence, $Y_n(x)$ is uniformly bounded. Taking limits in (25) and using (9), we have in Case 3 that,

$$Y_n(x) \to 1 - \frac{\Phi(\xi)}{\Phi(x)},$$

and in the other cases, by (10), that

$$Y_n(x) \to 1 - \frac{1}{4\Phi(\xi)\,\Phi(x)},$$

both uniformly on compact subsets of $\overline{\mathbf{C}} \setminus [-1, 1]$.

In this way, we have established the following assertion that gives the asymptotics of $\{Q_n\}_n$ relative to $\{P_n\}_n$:

PROPOSITION 5. *Uniformly on compact subsets of* $\overline{\mathbf{C}}\setminus[-1,1]$,

1. *in Case* 3 *of Proposition* 1,

$$\lim_{n\to\infty}\frac{Q_n(x)}{P_n(x)}=1-\frac{\Phi(\xi)}{\Phi(x)}; \tag{31}$$

2. *in the rest of the cases,*

$$\lim_{n\to\infty}\frac{Q_n(x)}{P_n(x)}=1-\frac{1}{4\Phi(\xi)\,\Phi(x)}. \tag{32}$$

Now we can derive (4).

Cases 1 *and* 2. Combining (17) and (32) we obtain

$$\lim_{n\to\infty}\frac{Q_n(x)}{T_n(x)}=\lim_{n\to\infty}\frac{Q_n(x)}{P_n(x)}\lim_{n\to\infty}\frac{P_n(x)}{T_n(x)}$$
$$=\left(1-\frac{1}{4\Phi(\xi)\,\Phi(x)}\right)\frac{\Phi(\xi)(4\Phi(x)^2-1)}{\Phi(x)(4\Phi(x)\,\Phi(\xi)-1)}=\frac{1}{\Phi'(x)},$$

uniformly on compact subsets of $\overline{\mathbf{C}}\setminus[-1,1]$.

Case 3. Using (17), (20), and (31) we have

$$\lim_{n\to\infty}\frac{Q_n(x)}{T_n(x)}=\lim_{n\to\infty}\frac{Q_n(x)}{P_n(x)}\lim_{n\to\infty}\frac{P_n(x)}{T_n^*(x)}\lim_{n\to\infty}\frac{T_n^*(x)}{T_n(x)}$$
$$=\left(1-\frac{\Phi(\xi)}{\Phi(x)}\right)\frac{\Phi(x)}{\Phi'(x)(\Phi(x)-\Phi(\xi))}=\frac{1}{\Phi'(x)}, \tag{33}$$

with $x\in\overline{\mathbf{C}}\setminus([-1,1]\cup\xi)$; clearly, (33) holds also in a neighborhood of ξ.

Thus, the theorem is proved. ∎

As we pointed out above, Theorem 1 allows us to establish the strong outer asymptotics of the sequence $\{Q_n\}_n$:

COROLLARY 1. *With the hypothesis of Theorem* 1 *and notation introduced above,*

(1) *if* $\operatorname{supp}(\mu_2)=[-1,1]$,

$$Q_n(x)=\frac{1}{\Phi'(x)}\,\frac{D(0,\mu_2)\,\Phi^n(x)}{D\left(\dfrac{1}{2\Phi(x)},\mu_2\right)}\,(1+o(1)), \tag{34}$$

uniformly on compact subsets of $\mathbf{C}\setminus[-1,1]$.

(2) *if* (μ_1, μ_2) *belongs to Case* 3 *of Proposition* 1 *with* $|\xi| > 1$,

$$Q_n(x) = \frac{1}{\Phi'(x)} \frac{(\Phi(x) - \Phi(\xi))^2}{x - \xi} \frac{D(0, v_2)}{D\left(\dfrac{1}{2\Phi(x)}, v_2\right)} \Phi^{n-1}(x)(1 + o(1)), \tag{35}$$

uniformly on compact subsets of $\overline{\mathbf{C}} \setminus ([-1, 1] \cup \{\xi\})$.

3. NORM AND ZERO ASYMPTOTICS

Now we study the (Sobolev) norm behaviour of $Q_n(x)$. With the notation $k_n = \langle P_n(x), P_n(x) \rangle_1$, $k'_n = \langle T_n(x), T_n(x) \rangle_2$, and $\tilde{k}_n = (Q_n, Q_n)_S$, introduced above, the following theorem holds:

THEOREM 2.

$$k_n + \lambda n^2 k'_{n-1} \leqslant \tilde{k}_n \leqslant k_n + \sigma_{n-1}^2 \left(\frac{n}{n-1}\right)^2 k_{n-1} + \lambda n^2 k'_{n-1}, \qquad n \geqslant 2. \tag{36}$$

In particular, in Cases 1 *and* 2,

$$\lim_{n \to \infty} \frac{4^{n-1} \tilde{k}_n}{n^2} = 2\pi D^2(0, \mu_2) \lambda. \tag{37}$$

Proof. Using the extremal property of k_n and k'_n we have

$$\begin{aligned} \tilde{k}_n = (Q_n(x), Q_n(x))_S &= \langle Q_n(x), Q_n(x) \rangle_1 + \lambda \langle Q'_n(x), Q'_n(x) \rangle_2 \\ &\geqslant \langle P_n(x), P_n(x) \rangle_1 + \lambda n^2 \langle T_{n-1}(x), T_{n-1}(x) \rangle_2. \end{aligned} \tag{38}$$

On the other hand, from the extremal property of $\tilde{k}_n$, for the limit polynomials R_n satisfying (8) we have

$$\begin{aligned} \tilde{k}_n \leqslant (R_n(x), R_n(x))_S &= \langle R_n(x), R_n(x) \rangle_1 + \lambda \langle R'_n(x), R'_n(x) \rangle_2 \\ &= \langle R_n(x), R_n(x) \rangle_1 + \lambda n^2 k'_{n-1}. \end{aligned} \tag{39}$$

By (7),

$$\begin{aligned} \langle R_n(x), R_n(x) \rangle_1 &= \left\langle P_n - \sigma_{n-1} \frac{n}{n-1} P_{n-1}, P_n - \sigma_{n-1} \frac{n}{n-1} P_{n-1} \right\rangle_1 \\ &= k_n + \sigma_{n-1}^2 \left(\frac{n}{n-1}\right)^2 k_{n-1}. \end{aligned} \tag{40}$$

Combining (39) and (40) we obtain that

$$\tilde{k}_n \leqslant k_n + \sigma_{n-1}^2 \left(\frac{n}{n-1}\right)^2 k_{n-1} + \lambda n^2 k'_{n-1}. \tag{41}$$

From inequalities (38) and (41) the result (36) follows.
In particular, taking limits in (36) with $n \to \infty$ we obtain (37). ∎

Finally, we make some remarks on the behaviour of the zeros of $Q_n(x)$.
First, the strong asymptotics in (34) implies weak asymptotics. That is, if we associate with each $Q_n(x)$ the discrete unit measure with equal positive masses at its zeros (with account of multiplicity),

$$\omega_n = \frac{1}{n} \sum_{Q_n(\xi)=0} \delta_\xi,$$

then if supp $(\mu_2) = [-1, 1]$,

$$d\omega_n(x) \to \frac{1}{\pi} \frac{dx}{\sqrt{1-x^2}} \tag{42}$$

in the weak-* topology.
Furthermore, Corollary 1 implies the following assertion:

COROLLARY 2. *The zeros of Sobolev monic orthogonal polynomials are, in all the cases, dense in* supp(μ_2), *i.e.*,

$$\bigcap_{n \geqslant 1} \overline{\bigcup_{k=n}^{\infty} \{x : Q_k(x) = 0\}} = \mathrm{supp}(\mu_2). \tag{43}$$

Moreover, if μ_2 *has a mass point* $\xi \in \mathbf{R} \backslash [-1, 1]$, *exactly one zero of* $Q_n(x)$ *is attracted by* ξ *and the rest accumulate at* $[-1, 1]$.

Proof. It is sufficient to observe that $Q_n(x)/T_n(x)$ is a sequence of analytic functions in $\overline{\mathbf{C}} \backslash$supp (μ_2), $\Phi(x)$ is analytic and has no zeros in $\overline{\mathbf{C}} \backslash [-1, 1]$, and the zero asymptotics of $T_n(x)$ is known in all the cases. Hence, the zeros of $Q_n(x)$ cannot accumulate outside supp(μ_2). On the other hand, (42) shows that they must be dense in supp(μ_2). ∎

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