JOURNAL OF FUNCTIONAL ANALYSIS 17, 311–327 (1974)

Factoring Weakly Compact Operators

W. J. DAVIS, T. FIGIEL, W. B. JOHNSON,* AND A. PELCZYNSKI

Ohio State University, Columbus, Ohio 43210 and
The Institute of Mathematics, Polish Academy of Sciences

Communicated by Tosio Kato

Received February 5, 1974

The main result is that every weakly compact operator between Banach spaces factors through a reflexive Banach space. Applications of the result and technique of proof include new results (e.g., separable conjugate spaces embed isomorphically in spaces with boundedly complete bases; convex weakly compact sets are affinely homeomorphic to sets in a reflexive space) and simple proofs of known results (e.g., there is a reflexive space failing the Banach–Saks property; if X is separable, then $X = Z^{**}/Z$ for some Z; there is a separable space which does not contain l_1 whose dual is nonseparable).

1. INTRODUCTION

The main result proved here is that if $T: X \to Y$ is a weakly compact operator between Banach spaces, then T factors through a reflexive Banach space, i.e., there are a reflexive space R and (bounded, linear) operators $S: X \to R$ and $L: R \to Y$ so that $T = LS$. Of course, an immediate application of this result is that every weakly compact subset of a Banach space is (in the respective weak topologies) affinely homeomorphic to a subset of a reflexive Banach space.

The proof of the main result, being very short, simple and self-contained, can be quickly assimilated by anyone familiar with such rudiments of functional analysis as are usually presented in a first year graduate course. The reader who is interested primarily in the factorization theorem needs to read only the part of Lemma 1 that occurs at the beginning of Section 2.

The remainder of Section 2 contains further more or less immediate applications of the Factorization Theorem. For example, Remark 2 gives that every weakly compact operator from a locally convex

* Supported in part by NSF GP-33578.

space into a Fréchet space factors through a reflexive Banach space (a linear operator is said to be weakly compact if it transforms a neighborhood of 0 into a weakly relatively compact set). Corollary 5 yields that no regular summability method will sum some subsequence of an arbitrary weakly null sequence in a reflexive Banach space.

Section 3 contains applications of the technique of factorization to some embedding problems involving bases. The main theorem here is that every Banach space with separable dual is a quotient of a space with shrinking basis. This answers a question in [14], and provides an alternate approach to proving that separable conjugate spaces have the Radon–Nikodym and Krein–Milman properties.

In Section 4 it is shown that if X is a weakly compactly generated Banach space (abbreviated *WCG*) then there is a Banach space Z such that Z^{**}/Z is isomorphic to X. (We regard consistently any Banach space as a subset of its second dual under the canonical embedding.) This is a nonseparable version of the James–Lindenstrauss Theorem [11]. Since, when X is separable, our construction can yield a Z^* with shrinking basis, this gives an alternate proof of the aforementioned theorem. As another illustration of the method a separable space Y is constructed which has nonseparable dual but contains no isomorphic copy of l_1. The first such example was given by James [6] and discussed later in [12].

Our notation is rather standard. X, Y, Z, E, F etc. denote Banach spaces, unless stated otherwise, and R is used for a reflexive Banach space. The linear span (resp. convex hull) of a subset $A \subset X$ is denoted span A (resp. conv A). The bars always refer to the closure with respect to the norm topology. The unit ball of X is denoted B_X. All operators are assumed to be linear and continuous. The weak topology of X generated by a family Γ of linear functionals separating the points of X is denoted $\sigma(X, \Gamma)$.

Given a sequence $(X_n, \|\cdot\|_n)_{n=1}^\infty$ of Banach spaces and a Banach space E of (numerical) sequences with the norm satisfying $\|(t_i)\|_E = \|(|t_i|)\|_E$ for any $(t_i) \in E$, $(\sum_{n=1}^\infty (X_n, \|\cdot\|_n))_E$ will denote the Banach space of all sequences $(x_n) \in \prod_{n=1}^\infty X_n$, so that

$$\|(x_n)\| = \|(\|x_n\|_n)\|_E < \infty.$$

Finally, let us recall that an operator $T\colon X^* \to Y^*$ is w^* continuous iff T satisfies one of the following equivalent conditions: (i) $T = S^*$, where $S\colon Y \to X$, (ii) $T^*(Y) \subseteq X$, (iii) T is continuous with respect to the $\sigma(X^*, X)$ and $\sigma(Y^*, Y)$ topologies.

2. THE FACTORIZATION THEOREM

Let W be a convex, symmetric and bounded subset of a Banach space $(X, \|\cdot\|)$. For each $n = 1, 2, \ldots$, the gauge $\|\cdot\|_n$ of the set $U_n = 2^n W + 2^{-n} B_X$ is a norm equivalent to $\|\cdot\|$. Define, for $x \in X$, $|||x||| = (\sum_{n=1}^{\infty} \|x\|_n^2)^{1/2}$, let $Y = \{x \in X: |||x||| < \infty\}$ and $C = B_Y = \{x \in X: |||x||| \leqslant 1\}$. Finally, let j denote the identity embedding of Y into X.

In the sequel, if X and W are given, Y, C, j and U_n's will always be defined in this way. Here are some properties of this construction.

LEMMA 1. (i) $W \subseteq C$.

(ii) $(Y, |||\cdot|||)$ *is a Banach space and* j *is continuous.*

(iii) $j^{**}: Y^{**} \to X^{**}$ *is one to one and* $(j^{**})^{-1}(X) = Y$.

(iv) Y *is reflexive iff* W *is weakly relatively compact.*

Proof. (i) If $w \in W$, then $\|w\|_n \leqslant 2^{-n}$, $n = 1, 2, \ldots$, hence $|||w||| < 1$; i.e., $w \in C$.

(ii) Let $X_n = (X, \|\ \|_n)$, $Z = (\sum_{n=1}^{\infty} X_n)_{l_2}$. The mapping $\varphi: Y \to Z$ given by $\varphi(y) = (jy, jy, \ldots)$ is a linear isometric embedding, and $\varphi(Y) = \{z = (x_n) \in Z: x_n = x_1,$ for $n = 1, 2, \ldots\}$ is a closed subspace. j may be regarded as the composition of φ and the projection of Z onto the first coordinate.

(iii) Observe that $\varphi^{**}(y^{**}) = (j^{**}y^{**}, j^{**}y^{**}, \ldots)$, for $y^{**} \in Y^{**}$, and, since φ is an isometry, $(\varphi^{**})^{-1}(0) = \{0\}$, $(\varphi^{**})^{-1}(\varphi(Y)) = Y$.

(iv) First let us notice that the $\sigma(X^{**}, X^*)$ closure of C in X^{**} is nothing but $j^{**}(B_{Y^{**}})$. Indeed, $B_{Y^{**}}$ is $\sigma(Y^{**}, Y^*)$ compact (Alaoglu Theorem), $C = B_Y$ is $\sigma(Y^{**}, Y^*)$ dense in $B_{Y^{**}}$ (Goldstine Theorem) and j^{**} is w^* continuous. Therefore, $j^{**}(B_{Y^{**}})$ is $\sigma(X^{**}, X^*)$ closed (being $\sigma(X^{**}, X^*)$ compact) and $j^{**}(C) = C$ is $\sigma(X^{**}, X^*)$ dense in it.

Now, if W is weakly relatively compact, i.e., $\overline{W}$ is $\sigma(X, X^*)$ compact, then the sets $2^n\overline{W} + 2^{-n}B_{X^{**}}$, $n = 1, 2, \ldots$, contain C and are $\sigma(X^{**}, X^*)$ closed, hence they contain $j^{**}(B_{Y^{**}})$. Since

$$\bigcap_n (2^n\overline{W} + 2^{-n}B_{X^{**}}) \subseteq \bigcap_n (X + 2^{-n}B_{X^{**}}) = X,$$

it follows that $j^{**}(B_{Y^{**}}) \subseteq X$, hence, by (iii), $Y^{**} \subseteq Y$; i.e., Y is reflexive.

The other implication (of iv) follows from (i).

Before we proceed let us mention that all the results of the present paper, as well as their proofs, remain valid if, in the definition of $|||\cdot|||$ and Z, the l_2 norm is replaced by the norm of any reflexive space E of sequences such that the unit vectors have norm 1. The reflexivity is critically needed for (iii) to hold, however, there are applications where nonreflexive norms are to be used.

COROLLARY 1. *Weakly compact operators factor through reflexive spaces.*

Proof. Let T: $Z \to X$ be weakly compact and let W of Lemma 1 be $T(B_Z)$. The operators $j^{-1} \circ T$: $Z \to Y$ and j: $Y \to X$ provide the required factorization.

COROLLARY 2. *Every weakly compact subset K of a Banach space X is affinely homeomorphic (in the respective weak topologies) to a subset of a reflexive Banach space.*

Proof. Let in Lemma 1 $W = \mathrm{conv}(K \cup (-K))$. Then, by the Krein–Smulian Theorem, W is weakly relatively compact, hence, by (iv), Y is reflexive. Therefore $K' = j^{-1}(K)$ is weakly compact (being weakly closed and bounded, by (i)). $j\,|_{K'}$ is the homeomorphism we need.

Remark 1. Since there is a one to one operator T^* mapping Y into a reflexive space R^* with unconditional basis (let $X = Y^*$ in Remark 3 below), K is affinely homeomorphic to a weakly compact subset of a reflexive space with unconditional basis, e.g., to $T^*(K')$.

Remark 2. If K is a weakly compact subset of a Fréchet space F, then there are a reflexive Banach space R and an operator T: $R \to F$ so that $K \subseteq T(B_R)$.

For F can be regarded as a closed subspace of the topological product of a sequence (X_n) of Banach spaces. Let K_n be the image of the projection of K onto the n-th coordinate. By Corollary 2, there is a sequence $(T_n)_{n=1}^{\infty}$ of operators T_n: $R_n \to X_n$ such that the R_n's are reflexive Banach spaces and $K_n \subseteq T_n(B_{R_n})$, $n = 1, 2,\ldots,$. Let $Z = (\sum R_n)_2$ and define S: $Z \to \prod X_n$ by $S((r_n)) = (2^n T_n r_n)_{n=1}^{\infty}$. Then Z is reflexive and $S(B_Z) \supseteq \prod K_n \supseteq K$. Clearly, one can take $R = S^{-1}(F)$, $T = S\,|_R$.

Using the observation one gets easily that any weakly compact operator mapping a locally convex space into a Fréchet space factors through a reflexive Banach space, and also a Fréchet version of

Corollary 2. Of course, a construction similar to that of Lemma 1 can be carried on in Fréchet spaces as well.

COROLLARY 3. *A Banach space X is WCG iff there is a reflexive space R and a one to one operator $T\colon R \to X$ with $T(R)$ dense in X.*

Proof. The sufficiency is obvious; the necessity follows from the proof of Corollary 2.

Remark 3. The R of Corollary 3 may be chosen to have unconditional basis (cf. Remark 5 below).

COROLLARY 4 ([1]). *If X is WCG, then B_{Z^*} is $\sigma(Z^*, Z)$ sequentially compact for any subspace $Z \subseteq X$.*

Proof. Let $T\colon R \to X$ be the operator of Corollary 3. Let $(z_n{}^*)_{n=1}^{\infty}$ be a sequence in B_{Z^*}. Extend each $z_n{}^*$ to an $x_n{}^* \in B_{X^*}$. By Eberlein's Theorem (in fact, a separable version thereof) there is a subsequence (n_k) so that $(T^* x_{n_k}^*(x))$ is convergent for $x \in R$. Since $T(R)$ is dense in X, and $(x_{n_k}^*)$ is bounded, it is $\sigma(X^*, X)$ convergent. Therefore $(z_{n_k}^*)$ is $\sigma(Z^*, Z)$ convergent, as required.

COROLLARY 5 (cf. [15, 2]). *For any regular summability method S there exist a reflexive space R and a sequence $(r_n)_{n=1}^{\infty}$ in R so that (r_n) tends weakly to zero, but no subsequence of it is S-summable.*

Proof. It was proved in [15] that for any S the space $C[0, 1]$ contains a sequence (x_n) with such a property. By Corollary 2 there is a reflexive space R and a one to one operator $T\colon R \to C[0, 1]$ so that $(x_n) \subset T(B_R)$. Obviously, one can take $r_n = T^{-1}(x_n)$, $n = 1, 2, \ldots,$.

Remark 4. It follows from Corollary 7 below that one can find R and (r_n) so that, in addition, $(r_n)_{n=1}^{\infty}$ is an unconditional basis.

3. FACTORIZATION THROUGH SPACES WITH BASES

In the sequel, whenever a linear subspace $\Gamma \subseteq X^*$ separating the points of X has been fixed, X will be regarded as a subset of Γ^* (the embedding being given by $x(\gamma) = \gamma(x)$, for $\gamma \in \Gamma$), and, if $A \subseteq X$, then $\tilde{A}$ will denote the $\sigma(\Gamma^*, \Gamma)$ closure of A in Γ^*. Recall that X is norm closed in Γ^*, iff $c = \inf_{\|x\|=1} \sup\{|\gamma(x)| : \|\gamma\| \leqslant 1\} > 0$; B_X is $\sigma(X, \Gamma)$ closed, iff $c = 1$; i.e., if Γ norms X.

By a Schauder decomposition of a Banach space Z we shall mean a sequence $(S_n)_{n=1}^{\infty}$ of projections on Z such that $S_n S_m = S_{\min(n,m)}$, for $n, m = 1, 2, \ldots$, and $S_n z \to z$ for $z \in Z$. The subspaces $(S_n - S_{n-1})(Z)$, where $S_0 = 0$, will be called the summands of the decomposition. $(S_n)_{n=1}^{\infty}$ is called shrinking, iff $(S_n{}^*)_{n=1}^{\infty}$ is a Schauder decomposition of Z^*, equivalently, iff $S_n^{**} z^{**} \overset{w^*}{\to} z^{**}$ for $z^{**} \in Z^{**}$. $(S_n)_{n=1}^{\infty}$ is said to be boundedly complete, iff any bounded sequence $(z_n)_{n=1}^{\infty}$ in Z such that $S_n z_n = S_n z_{n+1} = z_n$, $n = 1, 2, \ldots$, is convergent.

If the S_n's are of finite rank, and rank $S_n \leqslant 1 +$ rank S_{n-1} for $n = 1, 2, \ldots$, then, picking a vector $x_n \neq 0$ from each nontrivial summand $(S_n - S_{n-1})(Z)$, one gets a Schauder basis. If $(x_n)_{n=1}^{\infty}$ is a Schauder basis, the notation $x_n{}^*$ is reserved for the biorthogonal functionals, and S_n denotes the projection $S_n(x) = \sum_{i=1}^{n} x_i{}^*(x) x_i$, for $x \in Z$.

If $(x_i)_{i \in I}$ is an unconditional basis for Z, and $\alpha \subseteq I$, then P_α will denote the projection $P_\alpha(x) = \sum_{i \in \alpha} x_i{}^*(x) x_i$.

Now we can formulate the postponed part of Lemma 1.

LEMMA 1 (cont'd). (v) *If $\Gamma \subset X^*$ separates the points of X, then $\tilde{C} \subseteq \overline{\operatorname{span}}\, \tilde{W}$.*

(vi) *If Γ norms X and W is $\sigma(X, \Gamma)$ compact, then so is C.*

(vii) *If $\Gamma = X^*$, then the topologies $\sigma(Y^{**}, Y^*)$ and $\sigma(X^{**}, X^*)$ coincide on $\tilde{C}$, hence $\sigma(Y, Y^*)$ and $\sigma(X, X^*)$ coincide on C.*

(viii) *If $S: X \to X$ is a linear operator, and $S(W) \subseteq aW$, then $S(j(Y)) \subseteq j(Y)$, and $|||j^{-1} S j||| \leqslant \max(\| S \|, | a |)$,*

(ix) *If $(S_n)_{n=1}^{\infty}$ is a Schauder decomposition for X so that $S_n(W) \subseteq aW$, for some a and all n, then $(j^{-1} S_n j)_{n=1}^{\infty}$ is a Schauder decomposition for Y, which is shrinking (resp. boundedly complete), if (S_n) was so.*

(x) *If $(x_i)_{i \in I}$ is an unconditional basis for X, and $P_\alpha(W) \subseteq aW$, for all finite subsets α of I, then $\{x_i\} \cap Y$ is an unconditional basis for Y.*

Proof. (v) $C \subseteq U_n = 2^n W + 2^{-n} B_X \subseteq 2^n \tilde{W} + 2^{-n} B_{\Gamma^*}$, $n = 1, 2, \ldots,$. The latter sets are $\sigma(\Gamma^*, \Gamma)$ closed, being the algebraic sums of a $\sigma(\Gamma^*, \Gamma)$ closed and $\sigma(\Gamma^*, \Gamma)$ compact set, hence they contain $\tilde{C}$. Therefore

$$\tilde{C} \subseteq \bigcap_{n=1}^{\infty} (2^n \tilde{W} + 2^{-n} B_{\Gamma^*}) \subseteq \bigcap_{n=1}^{\infty} (\operatorname{span} \tilde{W} + 2^{-n} B_{\Gamma^*}) = \overline{\operatorname{span}}\, \tilde{W}.$$

(vi) Now the sets U_n are $\sigma(X, \Gamma)$ closed, so the $\| \cdot \|_n$'s are

$\sigma(X, \Gamma)$ lower semicontinuous, hence so is $|||\cdot|||$; i.e., C is $\sigma(X, \Gamma)$ closed. Since $\tilde{W} = W$, using (v) and the norm closedness of X in Γ^* we get $\tilde{C} \subseteq \overline{\text{span}}\, \tilde{W} = \overline{\text{span}}\, W \subseteq X$. Thus C is $\sigma(\Gamma^*, \Gamma)$ closed in Γ^*, hence, being bounded, it is $\sigma(\Gamma^*, \Gamma)$ compact.

(vii) Since, by (iii), j^{**} is one to one, $B_{Y^{**}}$ can be identified with $\tilde{C}$. The $\sigma(X^{**}, X^*)$ topology of $\tilde{C}$ is Hausdorff and weaker than its $\sigma(Y^{**}, Y^*)$ topology, which is compact. Therefore both topologies coincide on $\tilde{C}$ and, a fortiori, on C.

(viii) Let $b = \max(\| S \|, | a |)$. Then

$$S(U_n) = 2^n S(W) + 2^{-n} S(B_X) \subseteq b2^n W + b2^{-n} B_X = bU_n ,$$

i.e., $\| S \|_n \leqslant b$, $n = 1, 2,...,$. Hence, for any $y \in Y$,

$$||| j^{-1}Sjy ||| = \left(\sum_{n=1}^{\infty} \| Sjy \|_n^2 \right)^{1/2} \leqslant b \left(\sum_{n=1}^{\infty} \| jy \|_n^2 \right)^{1/2} = b \, ||| y |||.$$

(ix) By (viii),

$$S_n(jY) \subseteq jY, \qquad \text{and} \qquad \sup_n ||| j^{-1}S_n j ||| \leqslant \max(| a |, \sup_n \| S_n \|) < \infty.$$

Obviously, $(j^{-1}S_n j)$ is an increasing sequence of commuting projections. If $y \in Y$, then $(j^{-1}S_n jy)_{n=1}^{\infty}$ is bounded in Y and $S_n jy \to jy$. By (vii), $(j^{-1}S_n jy)$ is $\sigma(Y, Y^*)$ convergent to y. Therefore $\bigcup_n (j^{-1}S_n j)(Y)$ is a linear subspace $\sigma(Y, Y^*)$-dense in Y, hence it is strongly dense. This proves that $(j^{-1}S_n j)$ is a Schauder decomposition.

If (S_n) is a shrinking Schauder decomposition then $(S_n^{**}x^{**})$ converges $\sigma(X^{**}, X^*)$ to x^{**}. Using (vii) we get easily that $(j^{-1}S_n j)^{**}y^{**}$ is $\sigma(Y^{**}, Y^*)$ convergent to y^{**} which is equivalent to shrinkingness of $(j^{-1}S_n j)_{n=1}^{\infty}$.

To get the boundedly complete case observe first that C is $\sigma(X, X^*)$ closed. (Just read the first sentence of the proof of (vi) replacing U_n by $\bar{U}_n$ and Γ by X^*.) If $(y_n)_{n=1}^{\infty} \subset B_Y$ satisfies $(j^{-1}S_n j)(y_n) = (j^{-1}S_n j)\, y_{n+1} = y_n$, for $n = 1, 2,...,$ and (S_n) is boundedly complete, then $(jy_n)_{n=1}^{\infty}$ is convergent in X, hence its limit x belongs to C. It follows from (vii) that (y_n) is $\sigma(Y, Y^*)$ convergent to $j^{-1}(x)$. As above, this is sufficient to establish that $(j^{-1}S_n j)$ be boundedly complete.

(x) The argument of (ix) gives that any countable part of $j^{-1}(\{x_i: i \in I\})$ is a basic sequence in Y in any ordering, as well as that $j^{-1}(\{x_i: i \in I\})$ is fundamental in Y. This proves (x) and completes the proof of the lemma.

In the rest of this section X will be a Banach space with a fixed basis $(x_n)_{n=1}^{\infty}$ or unconditional basis $(x_i)_{i\in I}$, and $\Gamma \subset X^*$ will be the closure of the span of the biorthogonal functionals.

LEMMA 2. *If $V \subset X$ is $\sigma(X, \Gamma)$ compact, then so is*

$$V_s = V \cup \bigcup_n S_n(V).$$

If the basis is unconditional, then also $V_u = \bigcup_{\alpha\subseteq I} P_\alpha(V)$ is $\sigma(X, \Gamma)$ compact.

Proof. Given a net $(P_{\alpha_d} v_d)_{d\in D}$ of elements of V_u, there is a subnet $(d_e)_{e\in E}$ so that $\alpha_{d_e} \to \alpha_0$ in 2^I and $v_{d_e} \to v_0 \in V$ in the $\sigma(X, \Gamma)$ topology of V. Since, for bounded nets in X, the $\sigma(X, \Gamma)$ convergence is equivalent to the coordinatewise convergence, it follows that $(P_{\alpha_{d_e}} v_{d_e})_{e\in E}$ is $\sigma(X, \Gamma)$ convergent to $P_{\alpha_0} v_0 \in V_u$. Therefore V_u is $\sigma(X, \Gamma)$ compact.

The proof for V_s is even simpler since the $\sigma(X, \Gamma)$ topology of V_s is metrizable.

If $\Gamma = X^*$, i.e., the basis is shrinking, we get the following corollary.

COROLLARY 6. *Every weakly compact operator $T: Z \to X$ into a space with a shrinking (resp. shrinking unconditional) basis factors through a reflexive space with a basis (resp. unconditional basis).*

Proof. Use Lemma 2 with $V = \overline{T(B_Z)}$, note that $W = \overline{\text{conv}}\, V_s$ (resp. $\overline{\text{conv}}\, V_u$) is weakly compact, since V_s (resp. V_u) was, and satisfies $S_n(W) \subseteq W$ (resp. $P_\alpha(W) \subseteq W$). Then use (iv) and (ix) (resp. (iv) and (x)) of Lemma 1.

COROLLARY 7. *A weakly compact subset K of a Banach space Z is the image of a bounded unconditional basis in a reflexive space under a linear operator iff 0 is the only weak limit point of K.*

Proof. The necessity is obvious. To prove the sufficiency, consider the (weakly compact) operator $T: Z^* \to c_0(K')$, where $K' = K\backslash\{0\}$, given by $Tz^* = (z^*(k))_{k\in K'}$. Let $Z^* \xrightarrow[S]{} Y \xrightarrow[j]{} c_0(K')$ be the factorization yielded by Corollary 6. It is easy to check that j^* maps the natural basis for $c_0(K')^*$ onto a (bounded) unconditional basis for Y^* and S^* maps the latter onto K', as required.

Remark 5. In every reflexive space Z there is a biorthogonal system $(z_i, z_i^*)_{i\in I}$ so that $\overline{\text{span}}\{z_i\} = Z$, $\overline{\text{span}}\{z_i^*\} = \bar{Z}$ and $\| z_i \| \leqslant 1$, for $i \in I$. For Z separable, this was proved by Markouchevitch [13]; the general case follows from this by transfinite induction if one uses Lindenstrauss' decomposition of reflexive spaces by "long sequences" of projections (cf. [10]). The set $K = \{z_i\}_{i \in I} \cup \{0\}$ has the property described in Corollary 7. Applying the construction, we see that $S(\{z_i^*\}_{i \in I})$ is fundamental in Y (being an unconditional basis), hence S^* is one to one. This, combined with Corollary 3, yields Remark 3.

If $\Gamma \neq X^*$, and $A \subset X$ is $\sigma(X, \Gamma)$ compact, then the $\sigma(X, \Gamma)$ closure of conv A need not be so. One has, however,

LEMMA 3. *If $V \subset X$ is convex, symmetric and $\sigma(X, \Gamma)$ compact, then so is the $\sigma(X, \Gamma)$ closure W of* conv V_s.

Proof. Since conv V_s is $\sigma(\Gamma^*, \Gamma)$ relatively compact and metrizable, it is enough to prove that if a sequence $(w_k)_{k=1}^\infty \subset$ conv V_s is $\sigma(\Gamma^*, \Gamma)$ convergent to a $w \in \Gamma^*$, then $w \in X$. For this write

$$w_k = t_k{}^0 v_k{}^0 + \sum_{n=1}^{\infty} t_k{}^n S_n v_k{}^n,$$

where $v_k{}^n \in V$, $0 \leqslant t_k{}^n \leqslant 1$, $\sum_{n=0}^\infty t_k{}^n = 1$, for $k = 1, 2, \ldots, n = 0, 1, 2, \ldots$, and suppose, passing to a suitable subsequence, that $t_k{}^n \to t_n$, $v_k{}^n \to v_n \in V$ (in the $\sigma(X, \Gamma)$ topology), for $n = 0, 1, 2, \ldots$. Obviously, $\sum_{n=0}^\infty t_n \leqslant 1$.

Let $\epsilon > 0$ be arbitrary, and fix N so that $\sum_{n>N} t_n < \epsilon$. Assume next that $\sum_{n>N} t_n S_n v_k{}^n \to z_\epsilon \in \Gamma^*$ (in the $\sigma(\Gamma^*, \Gamma)$ topology) as $k \to \infty$. Obviously $\| z_\epsilon \| \leqslant \liminf \| \sum_{n>N} t_n S_n v_k{}^n \| \leqslant K\epsilon$, where

$$K = \sup_n \| S_n \| \cdot \sup_{v \in V} \| v \|_{\Gamma^*}.$$

Now put $\tau_k{}^n = t_k{}^n - t_n$, and consider $\omega_k = \sum_{n>N} \tau_k{}^n S_n v_k{}^n$. Since $\tau_k{}^n \to 0$ for each n, we can choose a subsequence $(k_j)_{j=1}^\infty$ so that $\sum_{n=N+1}^{N+j} | \tau_{k_j}^n | < j^{-1}$ for each j. Write

$$\omega_{k_j} = \sum_{n=N+1}^{N+j} \tau_{k_j}^n S_n v_{k_j}^n + \sum_{n>N+j} \tau_{k_j}^n S_{N+j} v_{k_j}^n + \sum_{n>N+j} \tau_{k_j}^n (S_n - S_{N+j}) v_{k_j}^n .$$

The first terms tend strongly to 0, the third ones form a bounded sequence which tends to 0 on each x_n^*, $n = 1, 2, \ldots$, hence also in the $\sigma(X, \Gamma)$ topology. Since, by Lemma 2, V_s is $\sigma(X, \Gamma)$ compact, and, for each j,

$$\sum_{n>N+j} \tau_{k_j}^n S_{N+j} v_{k_j}^n = S_{N+j} \Big(\sum_{n>N+j} \tau_{k_j}^n v_{k_j}^n \Big) \in S_{N+j}((1 + \epsilon) V)) \subseteq (1 + \epsilon) V_s ,$$

the $\sigma(\Gamma^*, \Gamma)$ limit x of this sequence belongs to X. Hence

$$w = \lim_j w_{k_j} = \lim_j \left(t_{k_j}^0 v_{k_j}^0 + \sum_{n=1}^{N} t_{k_j}^n S_n v_{k_j}^n + \sum_{n>N} t_n S_n v_{k_j}^n + \omega_{k_j}\right)$$

$$= t_0 v_0 + \sum_{n=1}^{N} t_n S_n v_n + z_\epsilon + x = x_\epsilon + z_\epsilon .$$

where $x_\epsilon \in X$ and $\| z_\epsilon \| \leqslant K\epsilon$. Therefore, since X is norm closed in Γ^*, we have $w = \lim_{\epsilon \to 0} x_\epsilon \in X$. This completes the proof of the lemma.

It follows from Corollary 6 that any reflexive subspace of a space with shrinking (resp. shrinking unconditional) basis embeds into a reflexive space with basis (resp. unconditional basis). We do not know whether this restriction is essential, or, more generally, whether every Banach space with separable dual embeds into a space with shrinking basis. It is true, however, that separable conjugate Banach spaces embed into spaces with boundedly complete bases. More precisely, one has the following.

COROLLARY 8. *If E is a Banach space with separable dual, then E is a quotient of a space with shrinking basis.*

Proof. Let $Q: l_1 \to E$ be a quotient mapping. By Remark 4.10 of [9] there is a basis $(x_n^*)_{n=1}^{\infty}$ for l_1 so that the space X spanned in l_1^* by the biorthogonal functionals $(x_n)_{n=1}^{\infty}$ contains $Q^*(E^*)$. Obviously, $\Gamma = l_1$, Γ norms X, and $V = Q^*(B_{E^*})$ is $\sigma(X, \Gamma)$ compact. Construct V_s, W and C as in Lemmas 2, 3, 1, respectively. Then C is $\sigma(X, \Gamma)$ compact, hence Y is isometric to the dual of a (unique) Banach space Z so that the $\sigma(Y, Z)$ topology is finer than that induced by $\sigma(X, \Gamma)$. This implies that Q^* regarded as a mapping of E^* into Z^* is w^* continuous, so that E is a quotient of Z. Also the biorthogonal functionals of the basis for Y constructed in Lemma 1(ix), being $\sigma(Y, Z)$ continuous, define a shrinking basis for Z.

COROLLARY 9. *If E^* is separable and $(e_n^*)_{n=1}^{\infty} \subset E^*$ tends to 0 in the $\sigma(E^*, E)$ but not in the norm topology, then (e_n^*) contains a boundedly complete basic subsequence.*

Proof. The previous corollary yields a w^* continuous isomorphic embedding of E^* into a space with boundedly complete basis. Since every block basic sequence of a boundedly complete basis is boundedly complete, the conclusion follows from Theorem 3 of [3].

Remark 6. Corollary 9 is a weaker version of Theorem 3.2 of [8]. To get the full result it is enough to observe that the perturbation used in [3] is, in the present situation, a w^* isomorphism.

4. On Spaces Z^{**}/Z

As our next application of Lemma 1 the following extension of the James–Lindenstrauss theorem will be proved.

Proposition 1. *If F is WCG, then there is a Banach space Z such that F is isomorphic to Z^{**}/Z. The space Z can be chosen so that Z^* has a shrinking Schauder decomposition with all the summands isomorphic to a fixed reflexive space with unconditional basis.*

The way we shall construct Z is reflected in the following lemma.

Lemma 4. *If X, Y, j satisfy* (ii), (iii) *of Lemma* 1 *and E is a closed linear subspace of X^{**} such that*

$$E \cap X = \{0\}, \qquad E \subseteq j^{**}(Y^{**}) \subseteq X + E,$$

then

(a) *the operator $U\colon E \to Y^{**}$ given by $U(e) = (j^{**})^{-1}(e)$ is an isomorphic embedding and $Y^{**} = Y \oplus U(E)$,*

(b) *if, in addition, $E = F^*$ and the inclusion map $i\colon E \to X^{**}$ is w^* continuous, then so is U. Therefore there is a Z so that Z^* is isomorphic to Y and Z^{**}/Z to F.*

Proof. (a) Since Y and $V = (j^{**})^{-1}(E)$ are closed in Y^{**}, and

$$Y^{**} = (j^{**})^{-1}(X + E) = (j^{**})^{-1}(X) + (j^{**})^{-1}(E) = Y + V,$$

$$Y \cap V = (j^{**})^{-1}(X \cap E) = \{0\},$$

both conclusions follow from the inverse mapping theorem. (b) We know that $i^*(X^*) \subseteq F$, and are to prove that $U^*(Y^*) \subseteq F$. This inclusion follows from

$$U^*(j^*(X^*)) = U^*(j^{***}(X^*)) = (j^{**}U)^*(X^*) = i^*(X^*) \subseteq F,$$

since, by (iii), $j^*(X^*)$ is norm dense in Y^*.

It is well known (see [9], the proof of Cor. 1) that the second statement follows from (a) and the first one.

Proof of Proposition 1. Let $T\colon R \to F$ be an operator of Remark 3. For each $n = 1, 2, \ldots$, the formula

$$| r |_n = \max \left(\| Tr \|, \frac{1}{n} | r | \right)$$

defines an equivalent norm on R. Let $R_n = (R, | \cdot |_n)$,

$$X = \left(\sum_{n=1}^{\infty} R_n{}^* \right)_{c_0}, \qquad S\colon X^* \to F, \qquad S((x_n)) = \sum_{n=1}^{\infty} Tx_n .$$

Obviously, $\| S \| \leqslant 1$ and $S(B_{X^*})$ is dense in B_F. Therefore $S^*\colon F^* \to X^{**}$ is an isometric embedding, so that $E = S^*(F^*)$ is closed.

Let $x = (x_n{}^*) \in X$ and $e \in E$, say $e = S^*f^* = (T^*f^*)_{n=1}^{\infty}$. The inequality

$$\| e \| = \lim_{n\to\infty} | T^*f^* |_n = \lim_{n\to\infty} | T^*f^* + x_n{}^* |_n \leqslant \| e + x \|,$$

implies easily that $E \cap X = \{0\}$ and $E + X$ is closed in X^{**}.

Let, in Lemma 1, $W = \{x = (x_n{}^*) \in X\colon \sum_{n=1}^{\infty} | x_{n+1}^* - x_n{}^* |_n \leqslant 1\}$ and let $\Gamma = X^*$. Recall that $j^{**}(B_{Y^{**}}) = \tilde{C}$, and, by (i) and (v), $\tilde{W} \subseteq \tilde{C} \subseteq \overline{\operatorname{span}}\, \tilde{W}$. Hence, $X + E$ being closed, for Y to satisfy the assumptions of Lemma 4 it is enough that $B_E \subseteq \tilde{W} \subseteq X + E$.

For the first inclusion, let $e \in B_E$, say $e = (T^*f^*)_{n=1}^{\infty}$, and let, for $k = 1, 2, \ldots$, w_k have the first k coordinates equal T^*f^* and the others 0. Then the w_k's belong to W and tend $\sigma(X^{**}, X^*)$ to e, so that $e \in \tilde{W}$. Hence $B_E \subseteq \tilde{W}$.

On the other hand, since the functions $p_k(x) = \sum_{n=1}^{k} | x_{n+1}^* - x_n{}^* |_n$ are $\sigma(X^{**}, X^*)$ lower semicontinuous, one has

$$\tilde{W} \subseteq \left\{ x = (x_n{}^*) \in X^{**} : \sum_{n=1}^{\infty} | x_{n+1}^* - x_n{}^* |_n \leqslant 1 \right\}.$$

Let $x = (x_n{}^*) \in \tilde{W}$, $x_0{}^* = \lim_n x_n{}^*$, $x_0 = (x_0{}^*)_{n=1}^{\infty}$. Since

$$| x_0{}^* - x_k{}^* |_k \leqslant \sum_{n=k}^{\infty} | x_{n+1}^* - x_n{}^* |_k \leqslant \sum_{n=k}^{\infty} | x_{n+1}^* - x_n{}^* |_n ,$$

one has $x - x_0 \in X$, so that $x_0 = x - (x - x_0) \in X^{**}$, hence $\sup_n | x_0{}^* |_n < \infty$. Obviously $x_0{}^* = T^*f^*$, where $f^* \in F^*$ is defined on $T(R)$ by $f^*(Tx) = x_0{}^*(x)$, hence $x_0 = S^*(f^*) \in E$. Thus $x = (x - x_0) + x_0 \in X + E$, so that the second inclusion has also been established.

Finally, the decomposition for Y is constructed by applying Lemma 1(ix) to the natural decomposition for X given by

$$S_n((x_k^*)_{k=1}^{\infty}) = (x_1^*,\ldots, x_n^*, 0, 0,\ldots,).$$

Since $(S_n - S_{n-1})(X) \subseteq \operatorname{span} W \subseteq j(Y)$, for $n = 1, 2,\ldots,$ then the summands of $(j^{-1}S_n j)_{n=1}^{\infty}$ are isomorphic to those of (S_n), which were isomorphic to R. This completes the proof.

Remark 7. Clearly, the shrinking Schauder decomposition $(j^{-1}S_n j)_{n=1}^{\infty}$ we have constructed for Y is monotone; i.e., $|||\, j^{-1}S_n j\, ||| = 1$, for $n = 1, 2,\ldots,$. Since the spaces $(j^{-1}S_n j)^*(Y^*)$ have the metric approximation property (this follows from Corollaire 2 to Proposition 40 of [4]), we get easily that so does Y^*.

Remark 8. In the case of a separable F Proposition 1 was proved in [11] in a stronger form (with Z^* having a shrinking basis). We do not know if this is so for the space we have constructed (it would be very interesting if it should not necessarily be the case). It is obvious, however, that Z^{**} is separable, and, by Remark 7, has the metric approximation property. This implies (cf. [7], p. 343, added in proof) that there is a reflexive space C_2 so that, if $Z_1 = C_2 \oplus Z$, then $Z_1{}^*$ has a shrinking basis. Clearly, Z_1^{**}/Z_1 is isomorphic to F.

One can also change the construction in the proof of Proposition 1 to yield readily a Y with shrinking finite dimensional decomposition. Namely, let $(R_n)_{n=1}^{\infty}$ be an increasing sequence of finite dimensional subspaces of Y, whose union is dense in Y, and let $T_n\colon R_n \to Y$ be the identity embedding. We define everything as before, only W is now to be

$$\left\{(x_n^*)_{n=1}^{\infty} \in X : \sum_{n=1}^{\infty} \| x_{n+1}^* |_{R_n} - x_n^* \|_{R_n^*} \leqslant 1\right\}.$$

This approach avoids an appeal to Grothendieck's results.

Our final application needs another portion of Lemma 1.

LEMMA 1. (xi) *If W is (norm) separable, then Y is separable.*

(xii) *If W is $\sigma(X^{**}, X^*)$ sequentially compact in X^{**}, then so is C, hence B_Y is $\sigma(Y^{**}, Y^*)$ sequentially compact in Y^{**}.*

(xiii) *The space Y^{**}/Y is naturally isometric to a subspace of the space Y_0 obtained by applying the construction to $X_0 = X^{**}/X$ and $W_0 = Q(W)$, Q being the quotient map $X^{**} \to X^{**}/X$.*

Proof. (xi) It is an instance of a general principle saying that Y is completely determined by W and $\overline{\text{span}}\ W$ (it follows from the formula $C \subseteq \bigcap_{n=1}^{\infty} U_n \subseteq \overline{\text{span}}\ W$). Indeed, if W is norm separable, then so is C, hence, by (vii), B_Y is $\sigma(Y, Y^*)$ separable, which implies the separability of Y.

(xii) Let $(c_n)_{n=1}^{\infty}$ be a sequence of elements of C. Write $c_n = 2^k w_n{}^k + 2^{-k} b_n{}^k$, with $w_n{}^k \in W$, $b_n{}^k \in B_X$. Using the diagonal procedure we can choose a subsequence $(n_i)_{i=1}^{\infty}$ so that, for $k = 1, 2, \ldots$, $(w_{n_i}^k)_{i=1}^{\infty}$ is $\sigma(X^{**}, X^*)$ convergent to an $x_k^{**} \in X^{**}$. Obviously, $(2^k x_k^{**})$ is a Cauchy sequence in X^{**} and $(c_{n_i})_{i=1}^{\infty}$ is $\sigma(X^{**}, X^*)$ convergent to $\lim_k 2^k x_k^{**}$. This proves the first statement, the second one follows now from (vii).

(xiii) Observe first that one may identify the spaces X_n^{**}/X_n and $(X_0)_n$. Indeed, for the corresponding open unit balls (denoted here b_E) one has

$$b_{X_n^{**}/X_n} = Q(b_{X_n^{**}}) = Q(2^n \tilde{W} + 2^{-n} b_{X^{**}}),$$

$$b_{(X_0)_n} = 2^n Q(\tilde{W}) + 2^{-n} b_{X_0} = 2^n Q(\tilde{W}) + 2^{-n} Q(b_{X^{**}}).$$

The isometry φ: $Y \to Z$ defined in the proof of (iii) induces the natural isometry of Y^{**}/Y into Z^{**}/Z. The natural quotient map of Z^{**} onto $Z_0 = (\sum (X_n^{**}/X_n))_{l_2}$ annihilates the elements of Z and only them, so that it induces the natural isometry of Z^{**}/Z onto Z_0. The composition of these isometries maps Y^{**}/Y into the space of "diagonal elements" of Z_0 which, as we know, is naturally isometric to Y_0.

Let us remark that (xiii) and the principle mentioned in the proof of (xi) imply that if E is a closed subspace of X^{**} containing $X \cup \tilde{W}$, then Y_0 is naturally isometric to the space Y_1 obtained using $Q(\tilde{W})$ and E/X in the construction.

In the following example any bounded Borel function f on a compact space K is identified with a functional on $C(K)^*$ via the formula $f(\mu) = \int_K f\, d\mu$, for $\mu \in C(K)^*$. Recall that a uniformly bounded sequence of Borel functions is $\sigma(C(K)^{**}, C(K)^*)$ convergent if it converges pointwise. (This follows from the Riesz representation theorem and the dominated convergence theorem.)

EXAMPLE. Let $X = C(\Delta)$, where Δ is the Cantor set and let $(h_n)_{n=1}^{\infty}$ be the normalized "Haar system" for Δ. Let W be the closure of the convex symmetric hull of the h_n's.

The total variation for functions on Δ is defined as for those on [0, 1] and has analogous properties. In particular, since the variation of any f in W is $\leqslant 4$ (it was so for the h_n's), each sequence in W has a subsequence convergent everywhere (cf. [5]). Therefore, by (xii) and our previous remark B_Y is $\sigma(Y^{**}, Y^*)$ sequentially compact in Y^{**}, so that no subspace of Y can be isomorphic to l_1. Needless to say, Y is separable (in fact $(j^{-1}h_n)_{n=1}^{\infty}$ is a Schauder basis), whereas Y^* is not since the evaluation functionals $j^*\delta_s$, $s \in \Delta$, form an uncountable discrete subset of Y^*.

Proofs of other properties of Y depend on the following description of $\tilde{W}$ (to be proved below),

$$(*) \qquad \tilde{W} = \operatorname{conv}(W \cup B_{l_1(\Delta)}).$$

Denote by E the subspace $C(\Delta) + c_0(\Delta)$ of $C(\Delta)^{**}$. E is closed since the obvious projection P of E onto $C(\Delta)$ is contractive. By (*) $\overline{\operatorname{span}}\, \tilde{W} = E$, so that, by (v), $j^{**}(Y^{**}) \subseteq E$. Also by (*) $P(\tilde{W}) \subseteq W$, hence using (viii) we conclude that Y is norm 1 complemented in Y^{**}.

Now go back to the remark following the proof of (xiii). Since the present E/X is naturally isomorphic to $c_0(\Delta)$ with $Q(\tilde{W})$ corresponding, by (*), to $B_{l_1(\Delta)}$, we get using (iv) and (x) that Y_1 is reflexive and has a symmetric unconditional basis of the cardinality of the continuum. Obviously, the natural isometry $Y^{**}/Y \to Y_1$ is onto, since, by (*), the basis vectors in Y_1 belong to its range.

The kernel Y_2 of the contractive projection from Y^{**} onto Y is reflexive (being isomorphic to Y^{**}/Y), hence it is $\sigma(Y^{**}, Y^*)$ closed. Let F be the annihilator of Y_2 in Y^*. A standard duality argument gives that F^* is isometric to Y. Furthermore, the biorthogonal functionals of the basis $(j^{-1}h_n)_{n=1}^{\infty}$ annihilate Y_2, hence they are $\sigma(Y, F)$ continuous, so that the basis is boundedly complete.

We are still to prove (*). The inclusion "$\supset$" is obvious. For the other let $\tilde{w}$ be an element of $\tilde{W}$. There is a net $(c_d)_{d\in D}$ in B_{l_1}, $c_d = (c_n{}^d)_{n=1}^{\infty}$ for $d \in D$, so that $[w^* - \lim_d] \sum c_n{}^d h_n = \tilde{w}$. We may suppose, passing to a subnet, that (c_d) be $\sigma(l_1, c_0)$ convergent to a $c = (c_n) \in B_{l_1}$. Clearly, $\overline{\lim} \| c_d - c \| \leqslant 1 - \| c \|$. Therefore it would be enough for us to know that if (γ_d) is a net in B_{l_1}, $\sigma(l_1, c_0)$ convergent to 0 so that $v = [w^* - \lim_d] \sum_n \gamma_n{}^d h_n$ exists, then $v \in B_{l_1(\Delta)}$.

Let $s_1, \ldots, s_k$ be a finite sequence of pairwise different elements of Δ and pick an N so that if $n > N$, then $h_n(s_i) \neq 0$ for at most one i, $1 \leqslant i \leqslant k$. Then

$$\sum_{i=1}^{k} |v(\delta_{s_i})| = \sum_{i=1}^{k} \lim_d \left| \sum_{n=1}^{\infty} \gamma_n{}^d h_n(s_i) \right|$$

$$\leqslant \varlimsup_d \sum_{i=1}^{k} \sum_{n=1}^{\infty} |\gamma_n{}^d| \, |h_n(s_i)|$$

$$\leqslant \varlimsup_d k \sum_{n=1}^{N} |\gamma_n{}^d| + \varlimsup_d \sum_{n>N} |\gamma_n{}^d| \leqslant 0 + 1 = 1.$$

Since the s_i's could be chosen arbitrarily, it follows that the formula $v_1(s) = v(\delta_s)$ defines an element v_1 of $B_{l_1(\Delta)}$. Clearly, $\int v_1 \, d\mu = \lim_d \int (\sum_{n=1}^{\infty} \gamma_n{}^d h_n) \, d\mu = v(\mu)$, if $\mu \in C(\Delta)^*$ is purely atomic. On the other hand, if μ is atomless, then $\int v_1 \, d\mu = 0$, but also $(\int h_n \, d\mu)_{n=1}^{\infty} \in c_0$ (the latter fact depends only on the diameters of the supports of the h_n's tending to zero), so that

$$v(\mu) = \lim_d \int \left(\sum_{n=1}^{\infty} \gamma_n{}^d h_n \right) d\mu = \lim_d \sum_{n=1}^{\infty} \gamma_n{}^d \int h_n \, d\mu = 0 = \int v_1 \, d\mu.$$

Since any $\mu \in C(\Delta)^*$ is the sum of an atomless and a purely atomic measures, we obtain that $v = v_1 \in B_{l_1(\Delta)}$. This completes the proof.

References

1. D. Amir and J. Lindenstrauss, The structure of weakly compact sets in Banach spaces, *Ann. Math.* **88** (1968), 35–46.
2. A. Baernstein II, On reflexivity and summability, *Studia Math.* **42** (1972), 91–94.
3. C. Bessaga and A. Pelczynski, On bases and unconditional convergence of series in Banach spaces, *Studia Math.* **17** (1958), 151–164.
4. A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, *Mem. Amer. Math. Soc.* **16** (1955).
5. E. Helly, "Über lineare Funktionaloperationen, *S.-B. K.Akad. Wiss. Wien Math.-Naturwiss. Kl.* 121, IIa, 1912), 265–297.
6. R. C. James, "A nonreflexive Banach space that is uniformly non-octahedral" (unpublished).
7. W. B. Johnson, Factoring compact operators, *Israel J. Math.* **9** (1971), 337–345.
8. W. B. Johnson and H. P. Rosenthal, On w^*-basic sequences and their applications to the study of Banach spaces, *Studia Math.* **43** (1972), 77–92.
9. W. B. Johnson, H. P. Rosenthal, and M. Zippin, On bases, finite dimensional decompositions and weaker structures in Banach spaces, *Israel J. Math.* **9** (1971), 488–506.
10. J. Lindenstrauss, On nonseparable reflexive Banach spaces, *Bull. Amer. Math. Soc.* **72** (1966), 967–970.
11. J. Lindenstrauss, On James' paper "Separable conjugate spaces," *Israel J. Math.* **9** (1971), 279–284.

12. J. LINDENSTRAUSS AND C. STEGALL, "Remarks on a space of R. C. James" (unpublished).
13. A. MARKOUCHEVITCH, Sur les bases (au sens large) dans es espaces linéaires, *Doklady Akad. Nauk SSSR* (N.S.) **41** (1943), 227–229.
14. A. PELCZYNSKI, Some problems on bases in Banach and Fréchet spaces, *Israel J. Math.* 2 (1964), 132–138.
15. J. SCHREIER, Ein Gegenbeispiel zur Theorie der schwachen Konvergenz, *Studia Math.* 2 (1930), 58–62.