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A fourth-order numerical scheme for solving the modified Burgers equation

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ABSTRACT

A finite-difference scheme based on fourth-order rational approximants to the matrix–exponential term in a two-time level recurrence relation is proposed for the numerical solution of the modified Burgers equation. The resulting nonlinear system, which is analyzed for stability, is solved using an already known modified predictor–corrector scheme. The results arising from the experiments are compared with the corresponding ones known from the available literature.

1. Introduction

Burgers [1,2], using earlier studies in Bateman [3], introduced an equation to capture some of the features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion. This equation, which includes nonlinearity and dissipation together in the simplest possible way and may be thought of as a nonlinear version of the heat equation, has in general the form

$$u_t + u\, u_x - \nu\, u_{xx} = 0; \quad L_0 \le x \le L_1,\ t > t_0, \tag{1.1}$$

where $u = u(x, t)$ is a sufficiently often differentiable function of the space and the time variables, respectively, and ν is a constant which can be interpreted as the viscosity, controlling the balance between convection and diffusion. When $\nu = 0$ it becomes the inviscid Burgers equation, which is a prototype for equations that develop shock waves, with important applications in physics [4]. The Burgers equation, which is a fundamental equation in fluid mechanics, is used as a model in fields as wide apart as acoustics, continuous stochastic processes, dispersive water waves, gas dynamics, heat conduction, longitudinal elastic waves in an isotropic solid, number theory, shock waves, turbulence and so forth [5]. Since Hopf's [6] and Cole's [7] independent proof that Eq. (1.1) can be reduced to the linear heat equation by a proper nonlinear transformation, numerous studies have approached its solution [8–18]. Formal generalizations of the Burgers equation are the Burgers–Huxley [19–23], Fisher [24–26], Korteweg–de Vries–Burgers [27–29] and Kuramoto–Sivashinsky [30–32] equations.

The modified Burgers equation (MBE), which is examined here, has the form

$$u_t + u^{\mu}\, u_x - \nu\, u_{xx} = 0; \quad L_0 \le x \le L_1, t > t_0, \tag{1.2}$$

where μ is a positive integer with $\mu \ge 2$. The cases to be examined where $\mu = 2$ and $\mu = 3$ will be denoted from now on as MBE2 and MBE3 respectively. The MBE equation, which has the strong nonlinear aspects of the governing equation (1.1),

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is used in many practical transport problems such as those of nonlinear waves in a medium with low-frequency pumping or absorption, ion reflection at quasi-perpendicular shocks, turbulence transport, wave processes in thermoelastic media, transport and dispersion of pollutants in rivers and sediment transport, etc. Theoretical solutions of the MBE equation are found in, among other references, Harris [33], Sachdev and Rao [34], Sachdev et al. [35], Inan and Ugurlu [36], while numerical solutions are found in Ramadan and El-Danaf [37] using the collocation method with quintic splines, Ramadan et al. [38] with finite elements with a method based on the collocation of septic B-splines, Dağ et al. [39] with finite elements with a method based on the collocation of cubic B-splines, Dağ et al. [40] using both quadratic and cubic B-spline Galerkin finite-element methods, Saka and Dağ [41] with the use of a quartic B-spline collocation method, Saka and Dağ [42] with the use of a quintic B-spline collocation method, Irk [43] with the sextic B-spline collocation method, Temsah [44] using the El-Gendi method, Duan et al. [45] using a lattice Boltzmann model, etc.

The initial condition associated with Eq. (1.2) will be

$$u\left(x, t_0\right) = f(x); \quad L_0 \le x \le L_1, \tag{1.3}$$

while the boundary conditions are

$$u_x|_{x=L_i} = 0; \quad i = 0, 1;\ t > t_0. \tag{1.4}$$

2. The numerical method

2.1. Development of the method

To obtain numerical solutions, the region $R = \{(x, t) \in [L_0 < x < L_1] \times]t_0, T]\}$ with its boundary ∂R consisting of the lines $x = L_0, x = L_1$ and $t = t_0$ is covered with a rectangular mesh of points, G, with coordinates $(x, t) = (x_m, t_n) = (L_0 + mh, t_0 + n\ell)$ with $m = 0, 1, \ldots, N + 1$ and $n = 1, 2, \ldots$. The theoretical solution of Eq. (1.2) at the typical mesh point (x_m, t_n) will be denoted by u_m^n and that relevant to an approximating difference scheme by U_m^n.

Let the solution vector at time $t = t_n$ be

$$\mathbf{U}^n = \mathbf{U}\left(t_n\right) = \left[U_0^n, U_1^n, \ldots, U_{N+1}^n\right]^\top. \tag{2.1}$$

If Eq. (1.4) is used in the second-order approximation, it gives to second order $U_{-1}^n = U_1^n$ and $U_{N+2}^n = U_N^n$. Then we apply Eq. (1.2) at each point of the grid G at time level $t = t_0 + n\ell$ with $n = 1, 2, \ldots$ using for the first-order space derivative the usual central-difference approximant at each point of (L_0, L_1) and the forward one at $x = L_0, L_1$; while for the second-order space derivative, the second-order central-difference approximant leads to a first-order initial-value problem, which is written in a matrix–vector form as

$$D\,\mathbf{U}(t) = -\Delta\, A\,\mathbf{U}(t) + \nu\, B\,\mathbf{U}(t); \quad t > t_0 \tag{2.2}$$

$$\mathbf{U}^0 = \mathbf{U}\left(t_0\right) = \left[f\left(x_0\right), f\left(x_1\right), \ldots, f\left(x_{N+1}\right)\right]^\top = \mathbf{f}$$

in which $D = \{\mathrm{d}/\mathrm{d}t\}$,

$$\Delta = \Delta^n = \Delta(t) = \operatorname{diag}\left\{\left(U_m^n\right)^\mu\right\}; \quad m = 0, 1, \ldots, N + 1 \tag{2.3}$$

are diagonal matrices,

$$A = \frac{1}{2h}\begin{bmatrix} -2 & 2 & & & & \\ -1 & 0 & 1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & & -1 & 0 & 1 \\ & & & & 2 & -2 \end{bmatrix}, \qquad B = \frac{1}{h^2}\begin{bmatrix} -2 & 2 & & & & \\ 1 & -2 & 1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & & 1 & -2 & 1 \\ & & & & 2 & -2 \end{bmatrix} \tag{2.4}$$

and tridiagonal matrices, and $\mathbf{f}$ is the vector of the initial condition, all of order $N + 2$.

Relation (2.2) gives

$$D = -\Delta\, A + \nu\, B, \tag{2.5}$$

so

$$D^2 = (\Delta\, A)^2 - \nu\,(\Delta\, A\, B + B\,\Delta A) + \nu^2\, B^2 \tag{2.6}$$

matrices of order $N + 2$, which are easily obtained from (2.3)–(2.4).

Numerical methods will be developed by replacing the matrix–exponential term in the recurrence relation

$$\mathbf{U}\left(t + \ell\right) = \exp\left(\ell D\right)\mathbf{U}\left(t\right); \quad t = t_0, t_0 + \ell, \ldots \tag{2.7}$$

where $D\,\mathbf{U}(t)$ is given by (2.2), by rational replacements, which are also known as the (p, q) Padé approximants of order $p + q$ to $\exp\left(\ell D\right)$ [46], of the form

$$\exp\left(\ell D\right) \approx \left(I + a_1\ell D + b_1\ell^2 D^2\right)^{-1}\left(I + c_1\ell D + d_1\ell^2 D^2\right) \tag{2.8}$$

Table 1
Parameters of the (p, q) Padé approximants to the exponential function.

Method	(p, q) Padé	a_1	b_1	c_1	d_1	Principal error term
I	(0,2)	0	0	1	1/2	$\ell^3 D^3/6$
II	(2,2)	−1/2	1/12	1/2	1/12	$\ell^5 D^5/720$

Table 2
Expression of Eq. (2.9) arising from the parameters in Table 1.

Method	Expression
I	$\mathbf{U}(t+\ell) = \left(I + \frac{1}{2}\ell^2 D^2\right)\mathbf{U}(t)$
II	$\left(I - \frac{1}{2}\ell D + \frac{1}{12}\ell^2 D^2\right)\mathbf{U}(t+\ell) = \left(I + \frac{1}{2}\ell D + \frac{1}{12}\ell^2 D^2\right)\mathbf{U}(t)$

in which a_1, b_1, c_1, d_1 are real parameters having appropriate values for each type of the approximants given in Table 1. Eq. (2.7), using Eq. (2.8), is written as

$$\left(I + a_1\ell D + b_1\ell^2 D^2\right)\mathbf{U}(t+\ell) = \left(I + c_1\ell D + d_1\ell^2 D^2\right)\mathbf{U}(t); \quad t = t_0, t_0+\ell, \ldots. \tag{2.9}$$

The expression for Eq. (2.9) arising from the use of these parameters for the methods to be examined in this paper is given in Table 2.

2.2. The proposed method

The method arises from Method II in Table 2, which, using the notation of (2.3) and (2.5)–(2.6), leads to the following nonlinear system:

$$\begin{aligned}
&\mathbf{U}(t+\ell) - \frac{1}{2}\ell[-\Delta^{n+1}A + \nu B]\mathbf{U}(t+\ell) + \frac{1}{12}\ell^2[(\Delta^{n+1}A)^2 + \nu^2 B^2 - \nu(\Delta^{n+1}AB + B\Delta^{n+1}A)]\mathbf{U}(t+\ell) \\
&\quad = \mathbf{U}(t) + \frac{1}{2}\ell[-\Delta^n A + \nu B]\mathbf{U}(t) + \frac{1}{12}\ell^2[(\Delta^n A)^2 + \nu^2 B^2 - \nu(\Delta^n AB + B\Delta^n A)]\mathbf{U}(t).
\end{aligned} \tag{2.10}$$

Let $r_1 = \ell\nu/2h^2, r_2 = \ell/4h, r_3 = \ell^2/48h^2, r_4 = \ell^2\nu/24h^3$ and $r_5 = \ell^2\nu^2/12h^4$. Eq. (2.10), when applied to the general mesh point of the grid G, gives

$$\begin{aligned}
&U_m^{n+1} - r_1\left(U_{m-1}^{n+1} - 2U_m^{n+1} + U_{m+1}^{n+1}\right) - r_2\left(U_{m-1}^{n+1} - U_{m+1}^{n+1}\right)\left(U_m^{n+1}\right)^\mu \\
&\quad + r_3\left\{\left(U_{m-1}^{n+1}\right)^\mu\left(U_m^{n+1}\right)^\mu U_{m-2}^{n+1} - \left[\left(U_{m-1}^{n+1}\right)^\mu + \left(U_{m+1}^{n+1}\right)^\mu\right]\left(U_m^{n+1}\right)^\mu U_m^{n+1}\right. \\
&\quad + \left.\left(U_m^{n+1}\right)^\mu\left(U_{m+1}^{n+1}\right)^\mu U_{m+2}^{n+1}\right\} - r_4\left\{4\left(U_m^{n+1}\right)^\mu U_{m-1}^{n+1} - \left[\left(U_{m-1}^{n+1}\right)^\mu + \left(U_m^{n+1}\right)^\mu\right]U_{m-2}^{n+1}\right. \\
&\quad - 4\left(U_m^{n+1}\right)^\mu U_{m+1}^{n+1} + \left[\left(U_{m-1}^{n+1}\right)^\mu - \left(U_{m+1}^{n+1}\right)^\mu\right]U_m^{n+1} \\
&\quad + \left.\left[\left(U_m^{n+1}\right)^\mu + \left(U_{m+1}^{n+1}\right)^\mu\right]U_{m+2}^{n+1}\right\} + r_5\left(U_{m-2}^{n+1} - 4U_{m-1}^{n+1} + 6U_m^{n+1} - 4U_{m+1}^{n+1} + U_{m+2}^{n+1}\right) \\
&= U_m^n + r_1\left(U_{m-1}^n - 2U_m^n + U_{m+1}^n\right) + r_2\left(U_{m-1}^n - U_{m+1}^n\right)\left(U_m^n\right)^\mu \\
&\quad + r_3\left\{\left(U_{m-1}^n\right)^\mu\left(U_m^n\right)^\mu U_{m-2}^n - \left[\left(U_{m-1}^n\right)^\mu + \left(U_{m+1}^n\right)^\mu\right]\left(U_m^n\right)^\mu U_m^n + \left(U_m^n\right)^\mu\left(U_{m+1}^n\right)^\mu U_{m+2}^n\right\} \\
&\quad + r_4\left\{4\left(U_m^n\right)^\mu U_{m-1}^n - \left[\left(U_{m-1}^n\right)^\mu + \left(U_m^n\right)^\mu\right]U_{m-2}^n - 4\left(U_m^n\right)^\mu U_{m+1}^n\right. \\
&\quad + \left.\left[\left(U_{m-1}^n\right)^\mu - \left(U_{m+1}^n\right)^\mu\right]U_m^n + \left[\left(U_m^n\right)^\mu + \left(U_{m+1}^n\right)^\mu\right]U_{m+2}^n\right\} \\
&\quad + r_5\left(U_{m-2}^n - 4U_{m-1}^n + 6U_m^n - 4U_{m+1}^n + U_{m+2}^n\right); \quad m = 0, 1, \ldots, N+1.
\end{aligned} \tag{2.11}$$

2.2.1. Stability analysis

Following the Fourier method of analyzing the stability [46, p. 142], if $\xi = e^{\alpha\ell}$ is the amplification factor and $\tilde{U}_m^n$ the numerical value of U_m^n actually obtained, an error of the form $U_m^n - \tilde{U}_m^n = \xi^n e^{im\beta h}$, $i = \sqrt{-1}$, with α a complex number and β real is considered. Then Eq. (2.11) leads to the following stability equation:

$$\begin{aligned}
&\left\{1 + 4r_1\sin^2\omega - 4r_3\left(U_0\right)^{2\mu}\sin^2 2\omega + 16r_5\sin^4\omega + i\left[2r_2\left(U_0\right)^\mu\sin 2\omega - 4r_4\left(U_0\right)^\mu\left(-2\sin 2\omega + \sin 4\omega\right)\right]\right\}\xi \\
&\quad = 1 - 4r_1\sin^2\omega - 4r_3\left(U_0\right)^{2\mu}\sin^2 2\omega + 16r_5\sin^4\omega \\
&\qquad + i\left[-2r_2\left(U_0\right)^\mu\sin 2\omega + 4r_4\left(U_0\right)^\mu\left(-2\sin 2\omega + \sin 4\omega\right)\right]
\end{aligned} \tag{2.12}$$

in which U_0 is a typical value of U_m^n; U_m^{n+1}, $m = 0, 1, \ldots, N+1$, is used for the linearization of the nonlinear terms and $\omega = \beta\, h/2$. Eq. (2.12) is of the form

$$\breve{A}\,\xi = \breve{B}; \quad \breve{A},\ \breve{B} \in \mathcal{C}, \tag{2.13}$$

where $\breve{A} \neq 0$ and $\mathcal{C}$ is the set of the complex numbers. The von Neumann necessary criterion for stability $|\xi| \leq 1$ will always be satisfied when $\left|\breve{B}\right| \leq \left|\breve{A}\right|$; otherwise, if $4\, r_3\, (U_0)^{2\mu} + 16\, r_5 \leq 1$, this leads to

$$\ell \leq 2\sqrt{3}h^2 \left[16\, \nu^2 + h^2\, (U_0)^{2\mu}\right]^{-1/2}. \tag{2.14}$$

Expanding the right-hand side of inequality (2.14) after using Maclaurin's expansion gives

$$\sqrt{3}\, h^2 \left[16\, \nu^2 + h^2\, (U_0)^{2\mu}\right]^{-1/2} = \frac{\sqrt{3}h^2}{2\nu} + \mathcal{O}\left(h^4\right).$$

Then assuming that $h \ll 1$ inequality (2.14), finally, leads to the following restriction for the time step:

$$\ell \leq \frac{\sqrt{3}h^2}{2\nu}. \tag{2.15}$$

2.3. *The predictor–corrector scheme*

To avoid the necessity of solving the nonlinear system (2.11), the following predictor–corrector scheme is proposed.

2.3.1. *The predictor*

The value $\hat{\mathbf{U}}\,(t+\ell)$ is evaluated from Method I in Table 2, using again the notation (2.3) and subject to (2.4)–(2.5), as follows:

$$\hat{\mathbf{U}}\,(t+\ell) = \mathbf{U}\,(t) + \ell\left(-\Delta^n A + \nu\, B\right)\mathbf{U}\,(t) + \frac{1}{2}\ell^2\Big[\left(\Delta^n A\right)^2 + \nu^2\, B^2 - \nu\left(\Delta^n A\, B + B\, \Delta^n A\right)\Big]\mathbf{U}\,(t)\,. \tag{2.16}$$

Let $p_1 = \ell\, \nu/h^2$, $p_2 = \ell/2\, h$, $p_3 = \ell^2/8\, h^2$, $p_4 = \ell^2\, \nu/4\, h^3$ and $p_5 = \ell^2 \nu^2/2\, h^4$. Eq. (2.16), when applied to the general mesh point of the grid G, gives

$$\begin{aligned}
\hat{U}_m^{n+1} &= U_m^n + p_1\left(U_{m-1}^n - 2U_m^n + U_{m+1}^n\right) + p_2\left(U_{m-1}^n - U_{m+1}^n\right)\left(U_m^n\right)^\mu \\
&\quad + p_3\left\{U_{m-2}^n\left(U_{m-1}^n\right)^\mu - U_m^n\left[\left(U_{m-1}^n\right)^\mu + \left(U_{m+1}^n\right)^\mu\right] + U_{m+2}^n\left(U_{m+1}^n\right)^\mu\right\}\left(U_m^n\right)^\mu \\
&\quad + p_4\left\{-4U_{m-1}^n\left(U_m^n\right)^\mu + U_{m-2}^n\left[\left(U_{m-1}^n\right)^\mu + \left(U_m^n\right)^\mu\right] + 4\left(U_m^n\right)^\mu U_{m+1}^n\right. \\
&\quad + \left. U_m^n\left[-\left(U_{m-1}^n\right)^\mu + \left(U_{m+1}^n\right)^\mu\right] - U_{m+2}^n\left[\left(U_m^n\right)^\mu + \left(U_{m+1}^n\right)^\mu\right]\right\} \\
&\quad + p_5\left(U_{m-2}^n - 4U_{m-1}^n + 6U_m^n - 4U_{m+1}^n + U_{m+2}^n\right); \quad m = 0, 1, \ldots, N+1.
\end{aligned} \tag{2.17}$$

2.3.1.1. *Stability analysis.* Following again the Fourier method of analyzing stability, Eq. (2.17) leads to the following stability equation:

$$\begin{aligned}
\xi &= 1 - 4\, p_1 \sin^2\omega - 4\, p_3 U_0^{2\mu} \sin^2 2\omega + 16 p_5 \sin^4\omega + \mathrm{i}\, U_0^\mu\left[-2\, p_2 \sin 2\omega + p_4\left(8 \sin 2\omega - 4 \sin 4\omega\right)\right] \\
&= K(\omega) + \mathrm{i}\, \Lambda(\omega).
\end{aligned} \tag{2.18}$$

Then the von Neumann necessary criterion for stability $|\xi| \leq 1$ leads to

$$\sqrt{K^2(\omega) + \Lambda^2(\omega)} \leq 1. \tag{2.19}$$

If $\omega = 0$, inequality (2.19) is always satisfied, while when $\omega = \pi/2$, it leads to the following restriction:

$$\left|1 - \frac{4\,\ell\,\nu}{h^2} + \frac{8\,\ell^2\nu^2}{h^4}\right| \leq 1. \tag{2.20}$$

Inequality (2.20) will always be satisfied when $-4\,\ell\,\nu/h^2 + 8\,\ell^2\nu^2/h^4 \leq 0$, which leads to the following restriction for the time step:

$$l \leq \frac{h^2}{2\,\nu}. \tag{2.21}$$

Since inequality (2.21) is more restrictive than inequality (2.15), it will be used for the experiments.

2.3.2. The corrector

The corrector arises from Method II in Table 2 as follows:

$$\mathbf{U}(t+\ell)=\left(\frac{1}{2}\ell D-\frac{1}{12}\ell^2D^2\right)\hat{\mathbf{U}}\,(t+\ell)+\left(I+\frac{1}{2}\ell D+\frac{1}{12}\ell^2D^2\right)\mathbf{U}\,(t)\,. \tag{2.22}$$

Instead of the classical substitution of $\mathbf{U}\,(t+\ell)$ in the right-hand side of (2.22) by $\hat{\mathbf{U}}\,(t+\ell)$, the modified predictor–corrector method (MPC) already known in the literature [47–50] was applied. The MPC method, which is applied *once*, consists of considering (2.22) componentwise and using an updated component in the corrector vector as soon as it becomes available. Hence in computing U_m^{n+1} the corrected values U_{m-1}^{n+1} instead of the predicted value $\hat{U}_{m-1}^{n+1}$ are used. The stability analysis of the corrector is given in Section 2.2.1.

3. Numerical results

For the linearization, $U_0=\max_{m=0,1,\ldots,N+1}\left\{u_m^{t_0}\right\}$ was given. Let $e_m^n=\left|u_m^n-U_m^n\right|$, $m=0,1,\ldots,N+1$. Then the error at time level $t=t_0+n\ell$, $n=1,2,\ldots$, is $e(t)=L_\infty=\max_{m=0,1,\ldots,N+1}\,e_m^n$ and the corresponding error $L_2=\sqrt{h\sum_{m=0}^{N+1}\left(e_m^n\right)^2}$.

It is known from calculus that:

Theorem 3.1. *Consider the series* $\sum_{k=0}^{+\infty}|u_k(x,t)|$ *with* $u_k(x,t)\neq 0$, $k=0,1,\ldots$. *If* $\rho_{k+1}=|u_{k+1}(x,t)/u_k(x,t)|$, $k=0,1,\ldots$, *with* $\lim_{k\to+\infty}\rho_{k+1}=\theta$, *then the series converges only if* $\theta<1$.

On the basis of the conclusion of Theorem 3.1, to examine the *speed* of the convergence of the method from time level $t=t_1$ to $t=t_2$ the ratio

$$\rho=\rho_{t_1,t_2}=\frac{e\,(t_2)}{e\,(t_1)} \tag{3.1}$$

is used. Obviously when $\rho\ll 1$ the convergence is fast.

3.1. MBE2 equation

Following [33] the MBE2 equation has the analytic solution

$$u(x,t)=\frac{x}{t}\left[1+\frac{\sqrt{t}}{t_0}\exp\left(\frac{x^2}{4\,\nu\,t}\right)\right]^{-1},\quad 0\le x\le 1;\ t\ge 1 \tag{3.2}$$

with $t_0\in(0,1)$. In the numerical solution the case with $t_0=0.5$ was considered. The initial condition $u(x,1)$ was given from Eq. (3.2). For reasons of comparison with the corresponding work in [37,38,42–44] the values

- $U_0^n=0$ and $U_{N+1}^N=u(1,t)$ with $u(1,t)$ given by Eq. (3.2), and
- $\nu=0.01, 0.001$ and 0.005

were used.

Experiments proved that the most accurate results are obtained for $h\ge 0.001$ and $\ell=10^{-5}$. From the results given in Table 3 it is deduced that the method introduced gives more accurate results for all time levels used than the corresponding results in [37,38]—finally, better than [42] and of equivalent accuracy to those in [43,44]. In Fig. 1 the curves of U and u for $\nu=0.005, 0.001$ at various time levels are shown. It is seen that as the time increases there is no visible difference between the curves.

Results using other values of the viscosity parameter ν at analogous time levels and the corresponding values of the ratio $\rho=\rho_{2,10}$ are given in Table 4. The following are deduced:

- the accuracy depends on the viscosity parameter ν increasing when ν is smaller,
- $\rho\ll 1$ for all ν examined,
- for $\nu>0.001$ the convergence becomes faster.

3.2. MBE3 equation

The MBE3 equation [35] using as initial condition

$$u(x,0)=A\,\sin\left(\frac{\pi x}{d}\right) \tag{3.3}$$

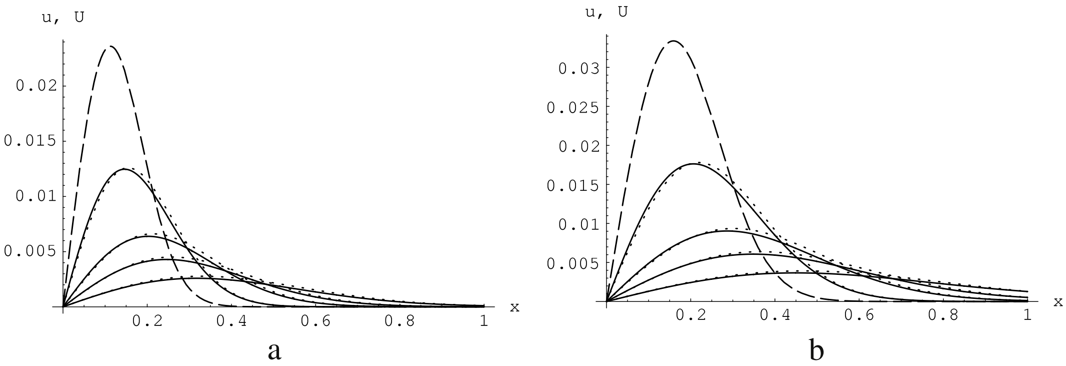

Fig. 1. MBE2 equation: in (a) the dashed curves show the theoretical solution u at $t = 1$ (initial condition—higher curve), the full curves the numerical solution U at $t = 2, 4, 6$ and 10 (lowest curve) and the dotted curves the corresponding theoretical solution u when $\nu = 0.005$, while (b) shows the corresponding curves when $\nu = 0.01$.

Table 3
MBE2 equation. Comparisons for the proposed method ($h = 0.001$ and $\ell = 10^{-5}$).

	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$
$\nu = 0.01$	$t = 2$	$t = 2$	$t = 4$	$t = 4$	$t = 6$	$t = 6$	$t = 10$	$t = 10$
MPC	0.81669	0.37920	0.60556	0.31548	0.46499	0.27314	0.30183	0.19337
[37]	1.21698	0.52308	0.93136	0.51625	0.72249	0.49023	1.28124	0.64007
[38]	1.70309	0.79043	0.99645	0.55767	0.76105	0.51672	1.80329	0.80026
[42]	0.81680	0.37932	0.60537	0.31724	0.52579	0.32602	1.28125	0.54701
[43]	0.81502	0.41321					1.28127	0.55095
[44]	0.758		0.564		0.459		0.300	
$\nu = 0.005$								
MPC	0.58027	0.22653	0.42949	0.18819	0.32993	0.16461	0.22874	0.13524
[37]	0.72264	0.25786	0.55445	0.25277	0.43082	0.22569	0.30006	0.18735
[42]	0.57998	0.22651	0.42940	0.18816	0.32897	0.16460	0.22885	0.13959
[43]	0.58424	0.23397					0.22626	0.13871
$\nu = 0.001$								
MPC	0.26109	0.06817	0.19289	0.05652	0.14809	0.04942	0.10263	0.04067
[37]	0.27967	0.06703	0.21856	0.06670	0.17176	0.06046	0.12129	0.05010
[38]	0.81852	0.18355	0.35635	0.11441	0.21348	0.08142	0.13943	0.05512
[42]	0.26094	0.06811	0.19288	0.05652	0.14810	0.04942	0.10264	0.04067
[43]	0.25975	0.07220					0.09872	0.03871
[44]	0.273		0.157		0.139		0.0936	

Table 4
MBE2 equation. Results with $h = 0.001$, $\ell = 10^{-5}$ and $\rho = \rho_{2,10}$.

	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	ρ
ν	$t = 2$	$t = 2$	$t = 4$	$t = 4$	$t = 6$	$t = 6$	$t = 10$	$t = 10$	
0.0004	0.16550	0.03437	0.12217	0.02847	0.09377	0.02488	0.06496	0.02048	0.39
0.007	0.68523	0.29096	0.50749	0.24188	0.38993	0.21154	0.26934	0.16960	0.39
0.001	0.26109	0.06817	0.19289	0.05652	0.14809	0.04942	0.10263	0.04067	0.39
0.02	1.14826	0.63313	0.83628	0.49074	0.52278	0.31136	0.14668	0.08999	0.13
0.05	1.62900	1.94553	0.43890	0.25466	0.35248	0.23769	0.18987	0.13375	0.12

with $d = \pi, A = 1$ subject to

$$U_0^n = u(x_0, t) = u(0, t) = 0$$
$$U_{N+1}^n = u(x_{N+1}, t) = u(d, t) = 0 \tag{3.4}$$

has an asymptotic solution of the form

$$u(x, t) = e^{-kt} f_0(x, t) + e^{-4kt} f_1(x, t) + e^{-7kt} f_2(x, t) + \cdots, \tag{3.5}$$

where

$$f_0(x, t) = A_1 \sin\left(\frac{\pi x}{d}\right),$$

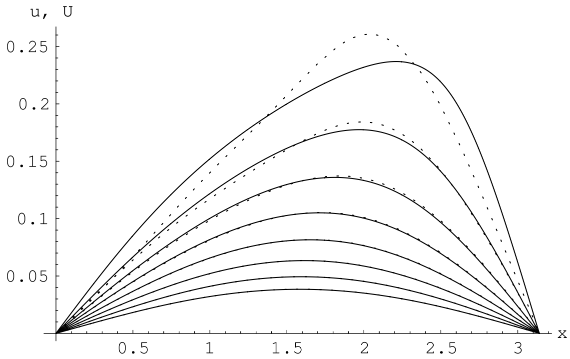

Fig. 2. MBE3 equation: the dotted curves show the theoretical solution u and the full curves the numerical solution U at $t = 100$ (higher curve), 150, 200, 250, 300, 350, 400 and 450 (lowest curve) when $\nu = 0.005$.

Table 5
MBE3 equation. Comparisons for the proposed method ($h = 0.005$, $\ell = 10^{-5}$ and $\nu = 0.005$).

	t	100	150	200	250
MPC	L_∞	0.33976E−01	0.68400E−02	0.20416E−02	0.83351E−03
	L_2	0.32761E−01	0.61258E−02	0.22273E−02	0.91238E−03
[45]	L_∞		5.172E−03	1.671E−03	1.400E−03
	L_2		3.227E−03	9.912E−04	5.031E−04
	t	300	350	400	450
MPC	L_∞	0.39559E−03	0.21860E−03	0.14160E−03	0.10361E−03
	L_2	0.41341E−03	0.23070E−03	0.16168E−03	0.12836E−03
[45]	L_∞	1.452E−03	1.488E−03	1.513E−03	1.531E−03
	L_2	5.939E−04	6.940E−03	7.567E−04	7.990E−04

$$
\begin{aligned}
f_1(x,t) &= -\frac{A_1^4\pi}{4d} t \sin\left(\frac{2\pi x}{d}\right) + \frac{A_1^4 d}{96\nu\pi}\sin\left(\frac{4\pi x}{d}\right) \\
&= B_1 t \sin\left(\frac{2\pi x}{d}\right) + B_2 \sin\left(\frac{4\pi x}{d}\right),
\end{aligned}
$$

$$
f_2(x,t) = g_3(t)\sin\left(\frac{\pi x}{d}\right) + g_4(t)\sin\left(\frac{3\pi x}{d}\right) + g_5(t)\sin\left(\frac{5\pi x}{d}\right) + g_6(t)\sin\left(\frac{7\pi x}{d}\right),
$$

$$
g_3(t) = -\frac{d^2}{6\nu\pi^2}\left[D_1 t + E_1 + \frac{d^2 D_1}{6\nu\pi^2}\right], \qquad g_4(t) = \frac{d^2}{2\nu\pi^2}\left[D_2 t + E_2 - \frac{d^2 D_2}{2\nu\pi^2}\right],
$$

$$
g_5(t) = \frac{d^2}{18\nu\pi^2}\left[D_3 t + E_3 - \frac{d^2 D_3}{18\nu\pi^2}\right], \qquad g_6(t) = \frac{d^2 E_4}{42\nu\pi^2}, \qquad k = \frac{\nu\pi^2}{d^2}
$$

$$
D_1 = \frac{A_1^3 B_1\pi}{4d}, \qquad D_2 = -\frac{9A_1^3 B_1\pi}{8d}, \qquad D_3 = \frac{5A_1^3 B_1\pi}{8d},
$$

$$
E_1 = -\frac{A_1^3 B_2\pi}{8d}, \qquad E_2 = \frac{9A_1^3 B_2\pi}{8d}, \qquad E_3 = -\frac{15A_1^3 B_2\pi}{8d}, \qquad E_4 = \frac{7A_1^3 B_2\pi}{8d}
$$

and old age constant $A_1 = 0.365366$.

Since the ratio $\rho = \rho_{100,150} \approx 0.20$ for $h = 0.001$ and $h = 0.005$, to avoid large vectors, the value $h = 0.005$ was preferred for the experiments. From the comparison of the MPC method with the corresponding method in [45] given in Table 5 the following are derived:

(i) for $t \geq 100$ a convergence of the numerical solution to the analytical one appears,
(ii) for $t \geq 250$ the MPC method has given results more accurate than the corresponding ones in [45],
(iii) from $t = 300$ instead of $t = 450$ in [35] no visible differences in the curves of u and U (see Fig. 2) appear.

4. Conclusions

An implicit finite-difference scheme based on fourth-order rational approximants to the matrix–exponential term was proposed for the numerical solution of the modified Burgers equation. The resulting nonlinear scheme was solved using an already known [47–50] MPC method in which the corrector, which is an explicit scheme, is applied once. The computational efficiency of the proposed method given in detail in Section 3 was tested by comparing the numerical results with the corresponding ones in [37,38,42–45]. It was found that the method for the MBE2 equation gives more accurate results than [37,38] and ones of equivalent accuracy to those of [42–44], while for the MBE3 one [45] more accurate results are given at long time intervals.

Since the varying time step integration methods [51,52] could improve the accuracy of the numerical schemes, the natural next step of future work would be to investigate these methods with reference to the MPC scheme introduced.

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