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Haar wavelet collocation method for the numerical solution of singular initial value problems

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Abstract In this paper, numerical solutions of singular initial value problems are obtained by the Haar wavelet collocation method (HWCM). The HWCM is a numerical method for solving integral equations, ordinary and partial differential equations. To show the efficiency of the HWCM, some examples are presented. This method provides a fast convergent series of easily computable components. The errors of HWCM are also computed. Through this analysis, the solution is found on the coarse grid points and then converging toward higher accuracy by increasing the level of the Haar wavelet. Comparisons with exact and existing numerical methods (adomian decomposition method (ADM) & variational iteration method (VIM)) solutions show that the HWCM is a powerful numerical method for the solution of the linear and non-linear singular initial value problems. The Haar wavelet adaptive grid method (HWAGM) based solutions show the excellent performance for the proposed problems.

1. Introduction

In recent years, the studies of singular initial value problems in the second order ordinary differential equations (ODEs) have attracted the attention of many mathematicians and physicists. Many methods including numerical and perturbation methods have been used to solve such type of problems. The approximate solutions for these problems were presented by many researchers for example Wazwaz [1–3] using the ADM and Yildirim and Ozis [4] using the VIM.

In numerical analysis, classical discretization methods, such as finite differences, finite elements and spectral elements are powerful tools for solving differential equations. However, singularities and step changes often emerge in many phenomena, such as stress concentration, elastoplasticity, shock wave and crack. Since small-scale features only exist in a small percentage of the solution domain, if one chooses a uniform numerical grid fine enough to resolve the small-scale characteristics, then the solution to the equations will be over-resolved in the majority of the domain. One would like ideally, to have a dense grid where small-scale structure exists and a sparse grid where the solution is only composed of large-scale features

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[5–7]. It demands for the usage of non-uniform grids and adaptive grids or moving elements to dynamically adapt to the changes in the solution [8]. That is where wavelets play a role.

Wavelet is called "numerical microscope" in signal and image processing. It has been 31 years since Morlet proposed the concept of wavelet analysis to automatically reach the best trade-off between time and frequency resolution [9]. Later, this proposition was considered as a generalization of ideas promoted by Haar (1910), Gabor (1946) [10]. Wavelet was in the air in the numerical analysis community in the early 1990s [11]. Generally, wavelet is used to describe a function that features compact support; i.e. it is nonzero only on a finite interval. The representation of a set of time-dependent data on a wavelet basis leads to a unique structure of information that is localized simultaneously in the frequency and time domains. This does not occur in a Fourier representation, where specific frequencies cannot be associated with a particular time interval, since the basis functions have constant resolution on the entire domain. A wavelet basis representation originates a set of wavelet coefficients structured over different levels of resolution. Each coefficient is associated with a resolution level and a point in the time domain. The coefficients involved in the lowest-resolution level describe the low-frequency features of the data spanning over broad time intervals. At the highest level, the coefficients are associated with highly localized high-frequency features [12,13]. These desirable advantages draw sight of researchers to apply wavelets in the resolution of differential equations [14–16].

One of the popular families of wavelets is Haar wavelet. Due to its simplicity, Haar wavelet has become an effective tool for solving differential equations. The previous work in system analysis via Haar wavelet was led by Chen and Hsiao [17], who first derived a Haar operational matrix for the integrals of the Haar function vector and put the applications for the Haar analysis into the dynamic systems. Lepik [18–20] applied the Haar wavelet collocation method for the solution of differential and integral equations. Bujurke et al. [21–23] presented the Haar wavelet method to establish the solution of nonlinear oscillator equations, Stiff systems, regular Sturm–Liouville problems, etc. Chang and Piau [24], designed the numerical solution of ordinary differential equations using Haar wavelet matrices. Islam et al. [25] obtained the numerical solution of second-order boundary-value problems using the Haar wavelet collocation method for the different boundary conditions.

The purpose of this paper is to introduce the HWCM as an alternative to existing methods for solving singular initial value problems. With this method, the given differential equation and its related initial conditions are transformed into a recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of a Haar series solution. This method is useful to obtain the approximate solutions of linear and nonlinear singular initial value problems, no need to linearization or discretization and large computational work. It has been used to solve effectively, easily and accurately a large class of linear and nonlinear problems with approximations.

The present work is organized as follows. In Section 2, Haar wavelet and operational matrix of integration are given. Method of solution of HWCM is presented in Section 3. In Section 4 numerical results and error analysis of the test problems are obtained. Finally conclusion of the proposed work is discussed in Section 5.

2. Haar wavelet and operational matrix of integration

The scaling function $h_1(t)$ for the family of the Haar wavelet is defined as

$$h_1(t) = \begin{cases} 1 & \text{for } t \in [0, 1) \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

The Haar wavelet family for $t \in [0, 1)$ is defined as

$$h_i(t) = \begin{cases} 1 & \text{for } t \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right) \\ -1 & \text{for } t \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right) \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

where $m = 2^l$, $l = 0, 1, \ldots, J$, J is the level of resolution; and $k = 0, 1, \ldots, m-1$ is the translation parameter. Maximum level of resolution is J. The index i in Eq. (2.2) is calculated using $i = m + k + 1$. In case of minimal values $m = 1$, $k = 0$ then $i = 2$. The maximal value of i is $N = 2^{J+1}$.

Let us define the collocation points $t_j = \frac{j-0.5}{N}$, $j = 1, 2, \ldots, N$, Haar coefficient matrix $H(i,j) = h_i(t_j)$ which has the dimension $N \times N$. For instance, $J = 3 \Rightarrow N = 16$, then we have

$$\mathrm{H}(16,16) = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{pmatrix}$$

We establish an operational matrix for integration via Haar wavelet. The operational matrix of integration is obtained by integrating (2.2) is as,

$$Ph_i = \int_0^t h_i(t)dt \tag{2.3}$$

and

$$Qh_i = \int_0^t Ph_i(t)dt \tag{2.4}$$

These integrals can be evaluated by using Eq. (2.2) and they are given by

$$Ph_i(t) = \begin{cases} t - \frac{k}{m} & \text{for } t \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right) \\ \frac{k+1}{m} - t & \text{for } t \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right) \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

$$Qh_i(t) = \begin{cases} \frac{1}{2}\left(t - \frac{k}{m}\right)^2 & \text{for } t \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right) \\ \frac{1}{4m^2} - \frac{1}{2}\left(\frac{k+1}{m} - t\right)^2 & \text{for } t \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right) \\ \frac{1}{4m^2} & \text{for } t \in \left[\frac{k+1}{m}, 1\right) \\ 0 & \text{Otherwise} \end{cases} \tag{2.6}$$

For instance, $J = 3 \Rightarrow N = 16$, from (2.5) then we have

$$Ph(16,16) = \frac{1}{32}\begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 & 21 & 23 & 25 & 27 & 29 & 31 \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 15 & 13 & 11 & 9 & 7 & 5 & 3 & 1 \\ 1 & 3 & 5 & 7 & 7 & 5 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 5 & 7 & 7 & 5 & 3 & 1 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

and from (2.6) we get

$$Qh(16,16) = \frac{1}{2048}\begin{pmatrix} 1 & 9 & 25 & 49 & 81 & 121 & 169 & 225 & 289 & 361 & 441 & 529 & 625 & 729 & 841 & 961 \\ 1 & 9 & 25 & 49 & 81 & 121 & 169 & 225 & 287 & 343 & 391 & 431 & 463 & 487 & 503 & 511 \\ 1 & 9 & 25 & 49 & 79 & 103 & 119 & 127 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 & 25 & 49 & 79 & 103 & 119 & 127 \\ 1 & 9 & 23 & 31 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 \\ 0 & 0 & 0 & 0 & 1 & 9 & 23 & 31 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 & 23 & 31 & 32 & 32 & 32 & 32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 & 23 & 31 \\ 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

3. Method of solution

Consider the singular initial value problem of the form

$$u''(t) + \frac{c}{t}u'(t) + f(u(t)) = 0 \tag{3.1}$$

Subject to the initial condition

$$u(0) = \alpha, \quad u'(0) = \beta \tag{3.2}$$

where c, α & β are real constants and $f(u(t))$ is a real valued function.

Let us assume that

$$u''(t) = \sum_{i=1}^{N} a_i h_i(t) \tag{3.3}$$

where a_i's, $i = 1, 2, \ldots, N$ are Haar coefficients to be determined. Integrating Eq. (3.3) from 0 to t and hence the solution $u(t)$ is expressed in terms of the Haar functions and their integrals.

Integrating Eq. (3.3) and using the given initial condition (3.2) we get,

$$u'(t) = \beta + \sum_{i=1}^{N} a_i Ph_i(t) \tag{3.4}$$

Again integrating (3.4) and substituting the initial condition then we have

$$u(t) = \alpha + \beta t + \sum_{i=1}^{N} a_i Qh_i(t) \tag{3.5}$$

Substituting (3.3)–(3.5) in (3.1), we get

$$\sum_{i=1}^{N} a_i h_i(t) + \frac{a}{t}\left(\beta + \sum_{i=1}^{N} a_i Ph_i(t)\right) + f\left(\alpha + \beta t + \sum_{i=1}^{N} a_i Qh_i(t)\right) = 0 \tag{3.6}$$

Solving (3.6) by using the Inexact Newton's method [26], we get the Haar wavelet coefficients a_i's and then substituting these values in (3.5), we obtain the Haar wavelet collocation method (HWCM) based numerical solution of the given problem (3.1).

4. Numerical examples

In this section, we are implementing the HWCM as discussed in Section 3 to some of the linear and non-linear singular initial value problems.

Test Problem 1. First consider the homogeneous singular value problem [2,4],

$$u'' + \frac{2}{t}u' - (4t^2 + 6)u = 0, \quad 0 < t \leqslant 1 \tag{4.1}$$

Subjected to

$$u(0) = 1, \quad u'(0) = 0 \tag{4.2}$$

Let us assume that

$$u''(t) = \sum_{i=1}^{N} a_i h_i(t) \tag{4.3}$$

Integrating (4.3) twice using (4.2), we have

$$u'(t) = \sum_{i=1}^{N} a_i Ph_i(t) \tag{4.4}$$

and

$$u(t) = 1 + \sum_{i=1}^{N} a_i Qh_i(t) \tag{4.5}$$

Substituting (4.3)–(4.5) in (4.1), we get

$$\sum_{i=1}^{N} a_i h_i(t) + \frac{2}{t}\sum_{i=1}^{N} a_i Ph_i(t) - 2(2t^2 + 3)\left(1 + \sum_{i=1}^{N} a_i Qh_i(t)\right) = 0 \tag{4.6}$$

Solving (4.6) by using Inexact Newton's method [26], we get the Haar coefficients a_i's $= [5.42, -2.86, -0.44, -2.90, -0.10, -0.36, -0.88, -2.16, -0.02, -0.07, -0.14, -0.22, -0.35, -0.54, -0.85, -1.34]$. Substituting these coefficients in (4.5), we get the HWCM solution of (4.1). The obtained numerical solution of (4.1) is presented in comparison with the ADM, VIM and exact solution $y(t) = \exp(-t^2)$ in Table 1 for

Table 1 Comparison of ADM, VIM and HWCM solutions with Exact solution for $N = 16$ of the Test Problem 1.

$t(=1/32)$	ADM	VIM	HWCM	Exact
1	1.000977	1.000977	1.000978	1.000977
3	1.008827	1.008827	1.008826	1.008827
5	1.024714	1.024714	1.024708	1.024714
7	1.049014	1.049014	1.049001	1.049014
9	1.082314	1.082314	1.082290	1.082314
11	1.125428	1.125428	1.125393	1.125428
13	1.179439	1.179439	1.179390	1.179439
15	1.245735	1.245735	1.245672	1.245736
17	1.326078	1.326078	1.326000	1.326079
19	1.422672	1.422672	1.422581	1.422675
21	1.538270	1.538269	1.538171	1.538278
23	1.676296	1.676292	1.676204	1.676321
25	1.841011	1.840999	1.840959	1.841078
27	2.037715	2.037687	2.037779	2.037888
29	2.273012	2.272946	2.273349	2.273428
31	2.555138	2.554991	2.556063	2.556084

$N = 16$ and Fig. 1 for $N = 32$. The error analysis for higher values of N is given in Table 2.

Test Problem 2. Next consider the nonhomogeneous singular value problem [2,4],

$$u'' + \frac{8}{t}u' + tu = t^5 - t^4 + 44t^2 - 30t, \quad 0 < t \leqslant 1 \tag{4.7}$$

with respect to

$$u(0) = 0, \quad u'(0) = 0 \tag{4.8}$$

Applying the method discussed in Section 3 for the problem (4.7), we get the Haar coefficients a_i's = [0.99, −1.50, 0.01, −1.50, 0.19, −0.19, −0.56, −0.94, 0.21, 0.06, −0.05, −0.14, −0.23, −0.33, −0.42, −0.52]. The obtained numerical solution of (4.7) is presented in comparison with the ADM and exact solution $y(t) = t^4 - t^3$ in Table 3 for $N = 16$ and Fig. 2 for $N = 32$.

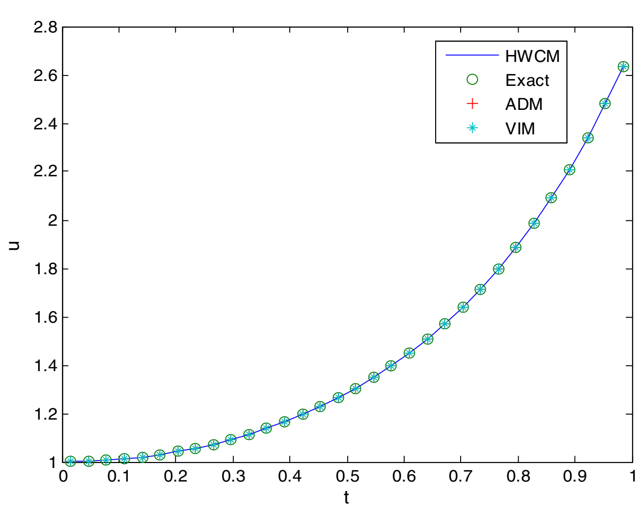

Figure 1 Comparison of ADM, VIM & HWCM solutions with exact solution for $N = 32$ of Test Problem 1.

Table 2 Error analysis of the Test Problem 1.

N	L_∞ (ADM)	L_∞ (VIM)	HWCM L_∞	HWAGM [8] L_∞
8	6.3098e−04	7.3037e−04	3.7226e−04	1.1798e−04
16	9.4542e−04	1.0927e−03	1.1858e−04	3.2938e−05
32	1.1519e−03	1.3304e−03	3.1412e−05	7.9978e−06
64	1.2701e−03	1.4664e−03	7.9072e−06	1.9995e−06
128	1.3334e−03	1.5391e−03	1.9978e−06	5.0090e−07
256	1.3661e−03	1.5767e−03	4.9990e−07	1.0098e−07

Table 3 Comparison of ADM and HWCM solutions with Exact solution for $N = 16$ of the Test Problem 2.

$t(=1/32)$	ADM	HWCM	Exact
1	0	−0.000048	−0.000029
3	0	−0.000640	−0.000746
5	0	−0.003104	−0.003218
7	0	−0.008034	−0.008177
9	0	−0.015832	−0.015990
11	0	−0.026497	−0.026656
13	0	−0.039660	−0.039809
15	0	−0.054590	−0.054717
17	0	−0.070188	−0.070281
19	0	−0.084990	−0.085036
21	0	−0.097163	−0.097151
23	0	−0.104511	−0.104430
25	0	−0.104470	−0.104308
27	0	−0.094110	−0.093855
29	0	−0.070136	−0.069777
31	−0.028411	−0.028885	−0.028410

The error analysis for higher values of N is given in Table 4.

Test Problem 3. Thirdly, consider the singular value problem of the form [3]

$$u'' + \frac{2}{t}u' + u^n = 0, \quad 0 < t \leqslant 1 \tag{4.9}$$

Subjected to

$$u(0) = 1, \quad u'(0) = 0, \tag{4.10}$$

Case (i) when $n = 1$ Eq. (4.9) becomes linear equation,

$$u'' + \frac{2}{t}u' + u = 0, \quad 0 < t \leqslant 1 \tag{4.11}$$

Let us assume that

$$u''(t) = \sum_{i=1}^{N} a_i h_i(t) \tag{4.12}$$

Integrating (4.12) twice with respect to the condition, we have

$$u'(t) = \sum_{i=1}^{N} a_i Ph_i(t) \tag{4.13}$$

and

$$u(t) = 1 + \sum_{i=1}^{N} a_i Qh_i(t) \tag{4.14}$$

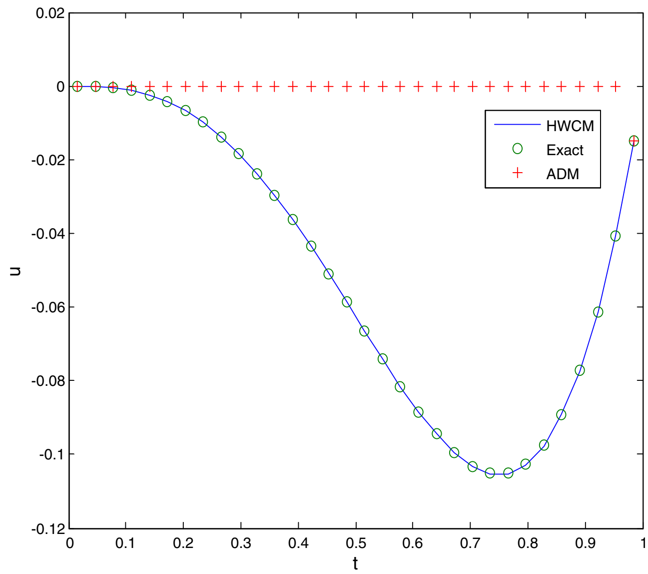

Figure 2 Comparison of ADM & HWCM solutions with exact solution for $N = 32$ of Test Problem 2.

Table 4 Error analysis of the Test Problem 2.

N	ADM L_∞	HWCM L_∞	HWAGM [8] L_∞
8	1.0154e−01	1.6292e−03	4.0839e−04
16	1.0443e−01	4.7476e−04	1.0293e−04
32	1.0520e−01	1.2686e−04	3.0278e−05
64	1.0540e−01	3.2718e−05	8.0095e−06
128	1.0545e−01	8.3037e−06	2.0090e−06
256	1.0546e−01	2.0904e−06	9.0098e−07

Substituting (4.12)–(4.14) in (4.11), we get

$$\sum_{i=1}^{N} a_i h_i(t) + \frac{2}{t}\sum_{i=1}^{N} a_i Ph_i(t) + \left(1 + \sum_{i=1}^{N} a_i Qh_i(t)\right) = 0 \qquad (4.15)$$

Solving (4.15) using Inexact Newton's method, we get the Haar coefficients a_i's [−0.30, −0.02, −0.01, −0.02, −0.00, −0.00, −0.01, −0.01, −0.00, −0.00, −0.00, −0.00, −0.00, −0.00, −0.00, −0.01]. The obtained numerical solution of (4.11) is presented in comparison with the ADM and exact solution $u(t) = \frac{\sin t}{t}$ in Table 5 for $N = 16$ and Fig. 3 for $N = 32$. The error analysis for higher values of N is given in Table 6.

Case (ii) When $n = 5$ Eq. (4.9) takes the nonlinear form,

$$u'' + \frac{2}{t}u' + u^5 = 0, \quad 0 < t \leqslant 1 \qquad (4.16)$$

Substituting (4.12)–(4.14) in (4.16), we get

$$\sum_{i=1}^{N} a_i h_i(t) + \frac{2}{t}\sum_{i=1}^{N} a_i Ph_i(t) + \left(1 + \sum_{i=1}^{N} a_i Qh_i(t)\right)^5 = 0 \qquad (4.17)$$

Solving (4.17) using Inexact Newton's method, we get the Haar coefficients a_i's = [−0.22, −0.08, −0.03, −0.04, −0.01, −0.02, −0.02, −0.02, −0.00, −0.01, −0.01, −0.01, −0.01, −0.01, −0.01, −0.01]. The obtained numerical solution of (4.16) is presented in comparison with the ADM and exact solution $u(t) = (1 + \frac{t^2}{3})^{-1/2}$ in Table 7 for $N = 16$ and Fig. 4 for $N = 32$. The error analysis for higher values of N is given in Table 8.

Table 5 Comparison of ADM and HWCM solutions with Exact solution for $N = 16$ of the Test Problem 3(i).

$t(= 1/32)$	ADM	HWCM	Exact
1	0.999837	0.999837	0.999837
3	0.998536	0.998535	0.998535
5	0.995940	0.995935	0.995935
7	0.992062	0.992043	0.992043
9	0.986920	0.986868	0.986868
11	0.980538	0.980421	0.980422
13	0.972946	0.972718	0.972719
15	0.964181	0.963777	0.963779
17	0.954285	0.953620	0.953621
19	0.943306	0.942268	0.942270
21	0.931298	0.929750	0.929752
23	0.918320	0.916093	0.916096
25	0.904438	0.901331	0.901334
27	0.889723	0.885496	0.885500
29	0.874250	0.868626	0.868630
31	0.858102	0.850759	0.850764

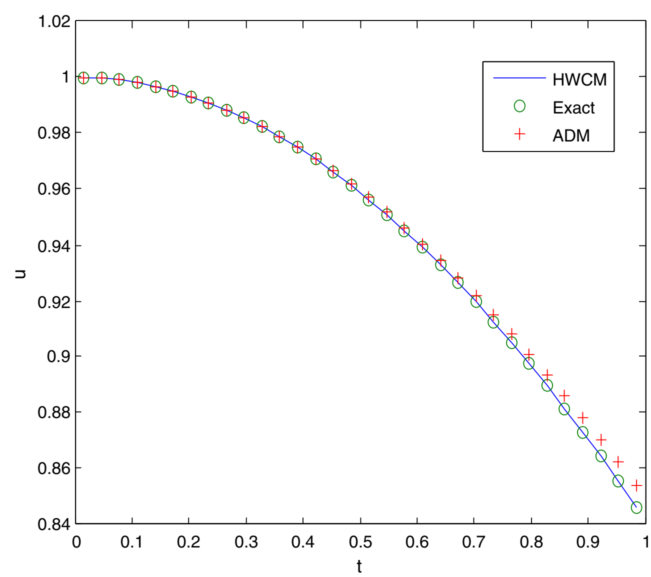

Figure 3 Comparison of ADM & HWCM solutions with exact solution for $N = 32$ of Test Problem 3(i).

Table 6 Error analysis of the Test Problem 3(i).

N	ADM L_∞	HWCM L_∞	HWAGM [8] L_∞
8	6.4356e−03	1.8562e−05	5.0856e−06
16	7.3373e−03	5.0012e−06	1.0012e−06
32	7.8221e−03	1.2932e−06	3.0938e−07
64	8.0733e−03	3.2854e−07	9.1857e−08
128	8.2012e−03	8.2781e−08	3.7787e−08
256	8.2657e−03	2.0776e−08	9.8235e−09

Table 7 Comparison of ADM and HWCM solutions with Exact solution for $N = 16$ of the Test Problem 3(ii).

$t(=1/32)$	ADM	HWCM	Exact
1	0.999837	0.999837	0.999837
3	0.998541	0.998538	0.998538
5	0.995980	0.995955	0.995955
7	0.992214	0.992117	0.992118
9	0.987332	0.987069	0.987071
11	0.981450	0.980866	0.980869
13	0.974711	0.973574	0.973578
15	0.967279	0.965269	0.965275
17	0.959339	0.956032	0.956040
19	0.951093	0.945952	0.945962
21	0.942754	0.935117	0.935129
23	0.934543	0.923618	0.923632
25	0.926687	0.911544	0.911561
27	0.919406	0.898985	0.899004
29	0.912916	0.886023	0.886044
31	0.907415	0.872739	0.872763

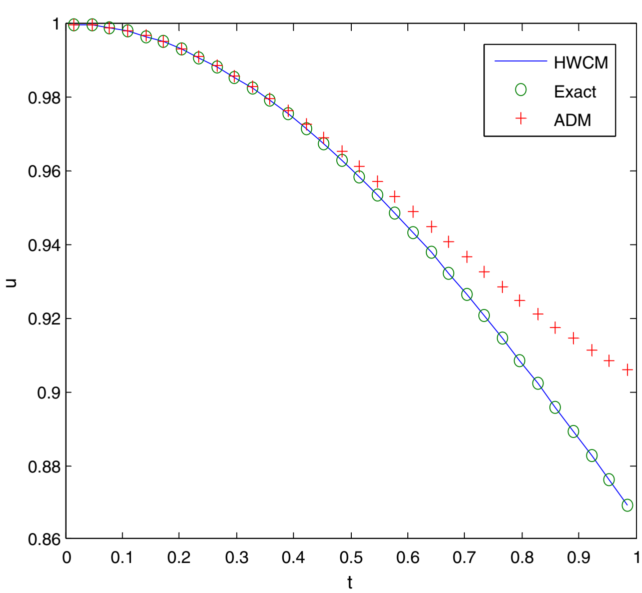

Figure 4 Comparison of ADM & HWCM solutions with exact solution for $N = 32$ of Test Problem 3(ii).

Table 8 Error analysis of the Test Problem 3(ii).

N	ADM L_∞	HWCM L_∞	HWAGM [8] L_∞
8	3.0591e−02	9.2374e−05	2.2745e−05
16	3.4652e−02	2.4231e−05	6.4165e−06
32	3.6814e−02	6.2101e−06	1.0121e−06
64	3.7929e−02	1.5723e−06	3.5372e−07
128	3.8495e−02	3.9558e−07	9.5951e−08
256	3.8780e−02	9.9210e−08	2.2190e−08

Test Problem 4. Fourthly, consider the nonlinear singular initial value problem [2],

$$u'' + \frac{6}{t}u' + 14u + 4u\log u = 0, \quad 0 < t \leqslant 1 \tag{4.18}$$

with respect to

$$u(0) = 1, \quad u'(0) = 0 \tag{4.19}$$

As in the previous examples, applying the method discussed in Section 3 to (4.18), we get the Haar coefficients a_i's = [−0.74, −0.82, −0.32, −0.39, −0.09, −0.22, −0.23, −0.15, −0.02, −0.07, −0.10, −0.12, −0.12, −0.11, −0.09, −0.06]. The obtained numerical solution of (4.18) is presented in comparison with the ADM and exact solution $y(t) = exp(-t^2)$ in Table 9 for $N = 16$ and Fig. 5 for $N = 32$. The error analysis for higher values of N is given in Table 10.

Table 9 Comparison of ADM and HWCM solutions with Exact solution for $N = 16$ of the Test Problem 4.

$t(=1/32)$	ADM	HWCM	Exact
1	0.999023	0.999024	0.999023
3	0.991249	0.991244	0.991249
5	0.975881	0.975866	0.975881
7	0.953275	0.953244	0.953275
9	0.923946	0.923895	0.923946
11	0.888550	0.888476	0.888550
13	0.847860	0.847761	0.847860
15	0.802738	0.802612	0.802738
17	0.754102	0.753949	0.754102
19	0.702898	0.702721	0.702901
21	0.650068	0.649873	0.650077
23	0.596519	0.596318	0.596544
25	0.543093	0.542915	0.543159
27	0.490540	0.490445	0.490704
29	0.439483	0.439595	0.439864
31	0.390388	0.390949	0.391223

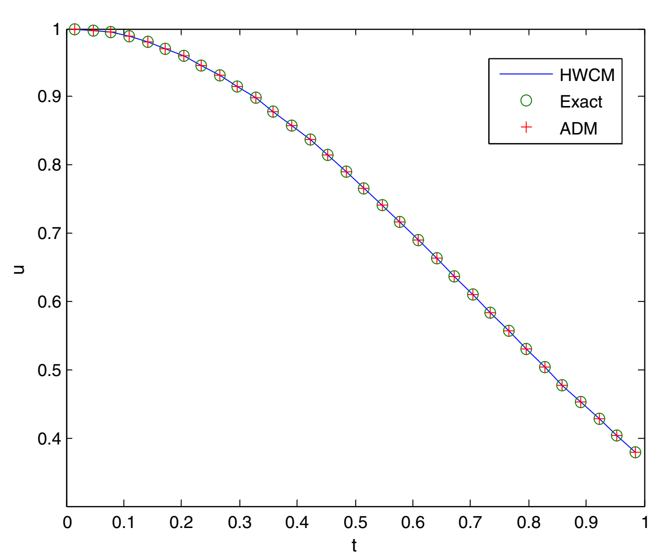

Figure 5 Comparison of ADM & HWCM solutions with exact solution for $N = 32$ of Test Problem 4.

Table 10 Error analysis of the Test Problem 4.

N	ADM L_∞	HWCM L_∞	HWAGM [8] L_∞
8	5.6787e−04	1.1014e−03	2.7158e−04
16	8.3516e−04	2.7389e−04	6.4087e−05
32	1.0079e−03	6.8407e−05	1.0979e−05
64	1.1060e−03	1.7099e−05	4.0426e−06
128	1.1582e−03	4.2747e−06	1.0785e−06
256	1.1852e−03	1.0687e−06	4.0256e−07

Table 11 Comparison of ADM, VIM and HWCM solutions with Exact solution for $N = 16$ of the Test Problem 5.

$t(= 1/32)$	ADM	VIM	HWCM	Exact
1	−0.001952	−0.001952	−0.001949	−0.001952
3	−0.017501	−0.017501	−0.017504	−0.017501
5	−0.048241	−0.048241	−0.048255	−0.048241
7	−0.093483	−0.093483	−0.093513	−0.093483
9	−0.152257	−0.152256	−0.152309	−0.152257
11	−0.223376	−0.223378	−0.223454	−0.223376
13	−0.305508	−0.305466	−0.305619	−0.305509
15	−0.397247	−0.397080	−0.397399	−0.397253
17	−0.497163	−0.496615	−0.497381	−0.497196
19	−0.603819	−0.602281	−0.604194	−0.603967
21	−0.715706	−0.711907	−0.716548	−0.716277
23	−0.831008	−0.822627	−0.833257	−0.832944
25	−0.947015	−0.930367	−0.953261	−0.952905
27	−1.058866	−1.029114	−1.075621	−1.075224
29	−1.157061	−1.109891	−1.199524	−1.199089
31	−1.222860	−1.159399	−1.324275	−1.323804

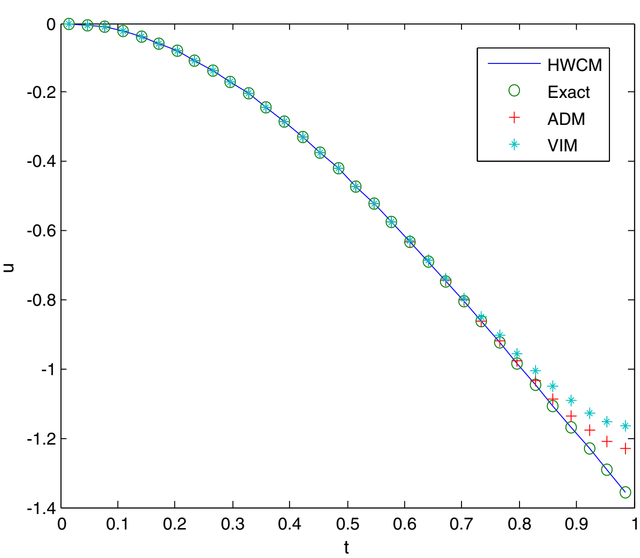

Figure 6 Comparison of ADM, VIM & HWCM solutions with exact solution for $N = 32$ of Test Problem 5.

Test Problem 5. Finally consider another nonlinear singular value problem [2],

$$u'' + \frac{2}{t}u' + 4\left(2e^u + e^{u/2}\right) = 0, \quad 0 < t \leqslant 1 \tag{4.20}$$

Table 12 Error analysis of the Test Problem 5.

N	L_∞ (ADM)	L_∞ (VIM)	HWCM L_∞	HWAGM [8] L_∞
8	6.5646e−02	1.2178e−01	1.8368e−03	4.5110e−04
16	1.0094e−01	1.6440e−01	4.7083e−04	1.0863e−04
32	1.2448e−01	1.9026e−01	1.1945e−04	3.0001e−05
64	1.3804e−01	2.0448e−01	3.0098e−05	7.1538e−06
128	1.4532e−01	2.1193e−01	7.5548e−06	1.2981e−06
256	1.4910e−01	2.1575e−01	1.8926e−06	4.3423e−07

with respect to

$$u(0) = 0, \quad u'(0) = 0 \tag{4.21}$$

As in the previous examples, we get the Haar coefficients a_i's = [−2.00, −1.20, −0.57, −0.48, −0.17, −0.36, −0.30, −0.18, −0.05, −0.13, −0.17, −0.18, −0.17, −0.14, −0.10, −0.07]. The obtained numerical solution of (4.20) is presented in comparison with the ADM, VIM and exact solution $y(t) = -2\log(1 + t^2)$ in Table 11 for $N = 16$ and Fig. 6 for $N = 32$. The error analysis for higher values of N is given in Table 12.

5. Conclusions

In this study, Haar wavelet collocation method is successfully applied to obtain numerical solutions of linear and nonlinear singular initial value problems. A symbolic calculation software package, MATLAB is used for all calculations. The work emphasized our belief that the method is a reliable technique to handle these types of problems. Also, it provides the solutions in terms of convergent Haar series with easily computable components in a direct way without using linearization, discretization or restrictive assumptions. The HWCM offers great advantages of straight forward applicability, computational efficiency and high accuracy. The Haar wavelet adaptive grid method (HWAGM) [8] based solutions show the excellent performance for the proposed problems.

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