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ScienceDirect
Journal of Functional Analysis 259 (2010) 2814–2855
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

On differentiable vectors for representations of infinite dimensional Lie groups

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Received 9 February 2010; accepted 30 July 2010

Available online 17 August 2010

Communicated by P. Delorme

Abstract

In this paper we develop two types of tools to deal with differentiability properties of vectors in continuous representations $\pi : G \to \mathrm{GL}(V)$ of an infinite dimensional Lie group G on a locally convex space V. The first class of results concerns the space V^∞ of smooth vectors. If G is a Banach–Lie group, we define a topology on the space V^∞ of smooth vectors for which the action of G on this space is smooth. If V is a Banach space, then V^∞ is a Fréchet space. This applies in particular to C^*-dynamical systems $(\mathcal{A}, G, \alpha)$, where G is a Banach–Lie group. For unitary representations we show that a vector v is smooth if the corresponding positive definite function $\langle \pi(g)v, v\rangle$ is smooth. The second class of results concerns criteria for C^k-vectors in terms of operators of the derived representation for a Banach–Lie group G acting on a Banach space V. In particular, we provide for each $k \in \mathbb{N}$ examples of continuous unitary representations for which the space of C^{k+1}-vectors is trivial and the space of C^k-vectors is dense.

Keywords: Infinite dimensional Lie group; Representation; Differentiable vector; Smooth vector; Derived representation

Contents

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[1] Supported by DFG-grant NE 413/7-1, Schwerpunktprogramm “Darstellungstheorie”.

doi:10.1016/j.jfa.2010.07.020

1. Introduction

Let G be a Lie group modeled on a locally convex space (cf. [43] for a survey on locally convex Lie theory). A representation $\pi : G \to \mathrm{GL}(V)$ of G on the locally convex space V is called *continuous* if it defines a continuous action of G on V. We call an element $v \in V$ a *C^k-vector* ($k \in \mathbb{N}_0 \cup \{\infty\}$) if the orbit map $\pi^v : G \to V$, $g \mapsto \pi(g)v$, is a C^k-map and write $V^k = V^k(\pi)$ for the space of C^k-vectors (cf. Section 2 for precise definitions).

It is a fundamental problem in the representation theory of Lie groups to understand the interplay between differentiability properties of group representations and representations of the Lie algebra $\mathfrak{g} = \mathbf{L}(G)$ of G. In particular, the following questions are of interest:

(1) How to find C^k-vectors? Can they be characterized in term of the Lie algebra?
(2) Does the space V^∞ of smooth vectors carry a natural topology for which the action of G on this space is smooth?
(3) Under which circumstances do differentiable, resp., smooth vectors exist, resp., form a dense subspace?

In this note we provide answers to some of these questions for infinite dimensional Lie groups G. For finite dimensional groups most of these results are either trivial or well known (see [14] for (1), [32] for (2) and [10] for (3)), so that the main issue is to identify and deal with the subtleties of infinite dimensional Lie groups.

After discussing some preliminaries on Lie groups and their representations in Sections 2 and 3, we turn in Section 4 to the space V^∞ of smooth vectors of the representation of a Banach–Lie group G on a locally convex space V. Here our main result is that, by embedding V^∞ into a product of the Banach spaces $\mathrm{Mult}^p(\mathfrak{g}, V)$ of continuous p-linear maps $\mathfrak{g}^p \to V$, we obtain a locally convex topology on V^∞ for which the G-action on V^∞ is smooth (Theorem 4.4). For finite dimensional groups, this follows easily from the smoothness of the translation action of G on the space $C^\infty(G, V)$ (cf. Proposition 4.6), but for infinite dimensional groups there seems to be no "good" topology on the space $C^\infty(G, V)$. The smooth compact open topology is too coarse to ensure continuity of the action. For the Fréchet–Lie group $G = \mathbb{R}^{\mathbb{N}}$ and the unitary representation on $\mathcal{H} = \ell^2(\mathbb{N}, \mathbb{C})$ defined by $(\pi(x)y)_n = e^{ix_n} y_n$, there exists no locally convex topology on $\mathcal{H}^\infty = \mathbb{C}^{(\mathbb{N})}$ for which the G-action is smooth. In view of this example, Theorem 4.4 on the smoothness of the action on V^∞ does not extend to unitary representations of

Fréchet–Lie groups. The locally convex space V^∞ has a topological dual space $V^{-\infty} := (V^\infty)'$, whose elements are called *distribution vectors*. For finite dimensional groups they correspond to equivariant embeddings of V into the space of distributions on G, and in our context they still parametrize equivariant embeddings $V^\infty \to C^\infty(G)$, where $C^\infty(G)$ carries the left regular representation. In this sense the spaces $V^{-\infty}$ provide natural candidates for spaces of distributions on Banach–Lie groups. For discussions of distributions on infinite dimensional vector spaces, we refer to [27,26,25].

We continue the discussion of V^∞ in Section 5, where we show that if V is a Banach space, then V^∞ is complete, hence a Fréchet space. In Section 6 we briefly discuss an application of these results to C^*-dynamical systems $(\mathcal{A}, G, \alpha)$, where G is a Banach–Lie group and $\mathcal{A}$ is a unital Banach algebra. Then the Fréchet space $\mathcal{A}^\infty$ is shown to be a continuous inverse algebra, i.e., its unit group is open and the inversion is a continuous map. The main result of Section 7 is that, for a unitary representation $(\pi, \mathcal{H})$, a vector $v \in \mathcal{H}$ is smooth if and only if the corresponding matrix coefficient $\pi^{v,v}(g) = \langle \pi(g)v, v\rangle$ is smooth.

Although smooth vectors form the natural domain for the derived representation of the Lie algebra $\mathfrak{g}$ of G, in many situations it is desirable to have some information on C^k-vectors for $k < \infty$. This motivates the discussion of C^1-vectors for continuous representations of Banach–Lie groups on Banach spaces in Section 8. The main result is that the intersection $\mathcal{D}_\mathfrak{g} \subseteq V$ of the domains of the generators $\overline{\mathrm{d}\pi}(x)$ of the one-parameter groups $t \mapsto \pi(\exp_G(tx))$ on V coincides with the space of C^1-vectors (Theorem 8.5). Here the main difficulty lies in the verification of the additivity of the map $\omega_v : \mathfrak{g} \to V$, $x \mapsto \overline{\mathrm{d}\pi}(x)v$, for which we use quite recent refinements of Chernoff's Theorem by Neklyudov [52]. If ω_v is assumed to be continuous, its linearity is easily verified (Lemma 3.3). It is also easily verified for finite dimensional groups [15, p. 221]. All this is applied in Section 9 to C^k-vectors, which are shown to coincide (for a representation of a Banach–Lie group on a Banach space) with the space $\mathcal{D}^k_\mathfrak{g}$, the common domain of k-fold products of the operators $\overline{\mathrm{d}\pi}(x)$, $x \in \mathfrak{g}$ (Theorem 9.4). Here an interesting point is that the C^k-concept we use is weaker than the C^k-concept used in the context of Fréchet differentiable maps between Banach spaces. The discussion of the examples in Section 10 shows that, in general, the space of Fréchet-C^k-vectors is a proper subspace of $\mathcal{D}^k_\mathfrak{g}$ which can be trivial.

A crucial problem one has to face in the representation theory of infinite dimensional Lie groups is that a continuous representation need not have any differentiable vector, as is the case for finite dimensional Lie groups [10]. Refining a construction from [3], we discuss in Section 10 the continuous unitary representation of the Banach–Lie group $G = (L^p([0,1], \mathbb{R}), +)$ on the Hilbert space $L^2([0,1], \mathbb{C})$ by $\pi(g)f = e^{ig}f$ and determine for every $k \in \mathbb{N}$ the space of C^k-vectors. For $p = 1$ we thus obtain a continuous representation with no non-zero C^1-vector, and for $p = 2k$, the space of C^k-vectors is non-zero, but there is no non-zero C^{k+1}-vector.

The situation is much better for Lie groups which are direct limits of finite dimensional ones. For these groups strong existence results on smooth vectors are available ([62,7] for unitary representations and [65] for Banach representations), and we explain in Section 11 how they fit into our general framework. The density of $\mathcal{H}^\infty$ for particular unitary representations of diffeomorphism groups is shown in [64]. In Section 12 we show that, for rather trivial reasons, the space of smooth vectors of a continuous unitary representation is also dense for projective limits of finite dimensional Lie groups. We conclude this paper in Section 13 with a discussion of some open problems.

The class of groups for which the theory developed in this article has the strongest impact is the class of Banach–Lie groups and we hope in particular that our results on smooth vectors will lead to a better understanding of their unitary representations. To make this more concrete, we recall some of the major classes of Banach–Lie groups and what is known on their unitary representation theory.

For a Banach–Lie group G, the most regular class of unitary representations are the *bounded* ones, i.e., those for which $\pi : G \to \mathrm{U}(\mathcal{H})$ is a morphism of Banach–Lie groups with respect to the norm topology on $\mathrm{U}(\mathcal{H})$. It is easy to see that faithful representations of this kind exist only when $\mathbf{L}(G)$ carries an $\mathrm{Ad}(G)$-invariant norm, a property which for finite dimensional Lie algebras is equivalent to compactness. If G has a universal complexification $\eta_G : G \to G_{\mathbb{C}}$ (see [16] for criteria on the existence), then every bounded unitary representation extends to a holomorphic representation $\widehat{\pi} : G_{\mathbb{C}} \to \mathrm{GL}(\mathcal{H})$. Holomorphic representations of Banach–Lie groups have been studied systematically in the context of the large class of root graded Lie groups in [35] and [49]. The class of root graded Lie groups contains in particular analogs of classical groups over unital Banach algebras and the most typical example is $\mathrm{GL}_n(\mathcal{A})$ for a unital Banach algebra $\mathcal{A}$. In [35] we developed a Borel–Weil theory for such groups, showing in particular that representations in spaces of holomorphic sections of bundles over the natural analogs G/P of flag manifolds are always holomorphic representation by bounded operators and that all holomorphic irreducible representations are of this form. In this context the most intriguing problem is to decide for which holomorphic representations $\rho : P \to \mathrm{GL}(V)$, the associated holomorphic vector bundle has nonzero holomorphic sections. For root graded groups whose Lie algebra has the form $\mathfrak{g} \otimes \mathcal{A}$, $\mathfrak{g}$ complex semisimple and $\mathcal{A}$ a unital commutative Banach algebra, this question has been answered completely in [49] for the special case of line bundles, i.e., $\rho : P \to \mathbb{C}^\times$ is a holomorphic character. From this classification result it follows in particular that the spaces of holomorphic sections of these line bundles are always finite dimensional. As a special case, this theory leads to a complete classification of the bounded unitary representations of the unitary groups $\mathrm{U}_n(\mathcal{A})$, where $\mathcal{A} \cong C(X)$ is a commutative unital C^*-algebra and all these representation factor through some quotient group $\mathrm{U}_n(\mathbb{C})^d$.

In [4] the Schur–Weyl theory of representations of $\mathrm{U}_n(\mathbb{C})$ is extended to unitary groups $\mathrm{U}(\mathcal{A})$ of a C^*-algebra. It is shown that tensor products of irreducible representations of $\mathcal{A}$ and their duals decompose into finitely many bounded unitary representations of $\mathrm{U}(\mathcal{A})$ according to the classical Schur–Weyl pattern. If $\mathcal{A}$ is commutative, then the aforementioned results from [49] imply that we thus obtain all irreducible unitary representations of $\mathrm{SU}_n(\mathcal{A}) \subseteq \mathrm{U}(M_n(\mathcal{A}))$, but if $\mathcal{A}$ is non-commutative the classification problem for irreducible unitary representations of $\mathrm{U}_n(\mathcal{A})$ is open. Thanks to the existence of group complexifications, the main tool in the analysis of bounded unitary representations of Banach–Lie groups is complex analysis, resp., holomorphic extension. This method is also exploited extensively in the classification of bounded unitary representations of the operator groups

$$\mathrm{U}_p(\mathcal{H}) := \mathrm{U}(\mathcal{H}) \cap \big(\mathbf{1} + \mathcal{L}_p(\mathcal{H})\big), \quad 1 \leqslant p \leqslant \infty,$$

where $\mathcal{L}_p(\mathcal{H})$ is the *p-th Schatten ideal* ([36], see also [42]). Here a remarkable result is that the classification does not depend on p in the range $1 < p \leqslant \infty$, so that the picture is the same as for $p = \infty$, where $\mathcal{L}_\infty(\mathcal{H}) = K(\mathcal{H})$ is the C^*-algebra of compact operators, and the irreducible representations are obtained from Schur–Weyl theory by decomposing tensor products of the form $\mathcal{H}^{\otimes n} \otimes \overline{\mathcal{H}}^{\otimes m}$. For $p = 1$, the bounded representation theory of $\mathrm{U}_1(\mathcal{H})$ is much richer but not of type I. A classification of the bounded irreducible, resp., factor representations of this group is still an interesting open problem.

In this context a remarkable result of D. Pickrell asserts that for the full unitary group $\mathrm{U}(\mathcal{H})$ of an infinite dimensional separable Hilbert space over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, every separable continuous unitary representation is automatically bounded, a direct sum of irreducible representations, and these are obtained by Schur–Weyl theory from the decomposition of the tensor products $\mathcal{H}_{\mathbb{C}}^{\otimes n}$ [57]. Basically this means that, as far as separable representations are concerned, the full unitary groups and finite products thereof behave exactly like compact groups.

It is a rule of thumb that complex analytic methods apply well to bounded and semibounded unitary representations, where the latter class is defined by the condition that the operators $i\mathrm{d}\pi(x)$ are uniformly bounded above on some open subset of the Lie algebra (see [45] for a survey on semibounded representations). Beyond semibounded representations one has to rely on real analytic techniques.

Typical analogs of finite dimensional non-compact semisimple groups are the automorphism groups of Hilbert–Riemannian symmetric spaces such as restricted Graßmannians and symmetric Hilbert domains [48,40,42]. For the same reason that finite dimensional groups with non-compact Lie algebra have no faithful finite dimensional unitary representation, these groups have no faithful bounded unitary representation. An important example is the *symplectic group* $\mathrm{Sp}(\mathcal{H})$ of all real linear symplectic automorphisms of $(\mathcal{H}, \omega)$, where $\mathcal{H}$ is a complex Hilbert space and $\omega(v, w) = \mathrm{Im}\langle v, w\rangle$. Even more important is the *restricted symplectic group* $\mathrm{Sp}_{\mathrm{res}}(\mathcal{H})$ consisting of all elements for which $g^\top g - \mathbf{1}$ is Hilbert–Schmidt (cf. [45]). The latter group has a by far richer (projective) representation theory than the former. It is the prototype of a hermitian Lie groups, i.e., an automorphism group of an infinite dimensional hermitian symmetric space. For hermitian Lie groups and their central extensions, the irreducible semibounded unitary representations have recently been classified in [46] by a combination of complex analytic methods with Pickrell's Theorem [57] and some results on unitarity of highest weight modules [48]. For $\mathrm{Sp}_{\mathrm{res}}(\mathcal{H})$ it turns out that all separable semibounded representations are trivial, but it has a central extension with a rich semibounded representation theory. For the full symplectic group $\mathrm{Sp}(\mathcal{H})$ it seems quite likely that all its unitary representation are trivial. So it becomes an interesting issue which structural features of a Banach–Lie group lead to obstructions for the existence of non-trivial unitary representations.

An area which is still largely unexplored is the theory of $(\mathfrak{g}, K)$-modules for pairs such as $(\mathfrak{sp}_{\mathrm{res}}(\mathcal{H}), \mathrm{U}(\mathcal{H}))$. Related problems have been addressed by G. Olshanski from the point of view of topological groups in terms of (G, K)-pairs, where K is a subgroup of G, such as $(\mathrm{Sp}_{\mathrm{res}}(\mathcal{H}), \mathrm{U}(\mathcal{H}))$ (cf. [54,58]), and we hope that, eventually, a better understanding of differentiability properties as discussed in the present paper and integration techniques as in [44] and [31] will lead to a more transparent Lie theoretic understanding of the representations of such groups.

The groups $G = C^k(M, K)$ of C^k-maps, $k \geqslant 1$, on a compact manifold M with values in a finite dimensional Lie group K form a another class of interesting Banach–Lie groups; more generally one considers C^k-gauge transformations of principal bundles. These groups have interesting central extensions $\widehat{G}$ [47], and for the case $M = \mathbb{S}^1$, where G is a loop group, the central extension $\widehat{G}$ has an interesting family of unitary representations extending to a semidirect product $\widehat{G} \rtimes \mathbb{T}$, where $\mathbb{T}$ acts on $\mathbb{S}^1$ by rigid rotations (cf. [60,39,1]). These groups are Banach manifolds and topological groups, but their multiplication is not smooth. We expect that a thorough analysis of such semidirect products $N \rtimes H$, where H is a Banach–Lie group acting *continuously* on the Banach–Lie group N, can be based on the results of the present paper. For the case where $(\mathcal{A}, H, \alpha)$ is a C^*-dynamical system, the corresponding covariant representations lead in particular to unitary representations of the groups $\mathrm{U}(\mathcal{A}) \rtimes_\alpha H$ which are of this type (cf. [56]). Another interesting class of such groups providing testing cases for a general theory are the semidirect products $\mathrm{Heis}(\mathcal{H}) \rtimes_\alpha \mathbb{R}$, where $\mathcal{H}$ is a complex Hilbert space and $\alpha(t) = e^{itH}$ a strongly continuous unitary one-parameter group (cf. [51]).

2. Locally convex Lie groups

In this section we briefly recall the basic concepts related to infinite dimensional Lie groups.

Definition 2.1. Let E and F be locally convex spaces, $U \subseteq E$ open and $f : U \to F$ a map. Then the *derivative of f at x in the direction h* is defined as

$$\mathtt{d}f(x)(h) := (\partial_h f)(x) := \frac{d}{dt}\bigg|_{t=0} f(x+th) = \lim_{t\to 0} \frac{1}{t}\big(f(x+th) - f(x)\big)$$

whenever it exists. The function f is called *differentiable at x* if $\mathtt{d}f(x)(h)$ exists for all $h \in E$. It is called *continuously differentiable*, if it is differentiable at all points of U and

$$\mathtt{d}f : U \times E \to F, \quad (x,h) \mapsto \mathtt{d}f(x)(h)$$

is a continuous map. Note that this implies that the maps $\mathtt{d}f(x)$ are linear (cf. [13, Lemma 2.2.14]). The map f is called a *C^k-map*, $k \in \mathbb{N} \cup \{\infty\}$, if it is continuous, the iterated directional derivatives

$$\mathtt{d}^j f(x)(h_1, \ldots, h_j) := (\partial_{h_j} \cdots \partial_{h_1} f)(x)$$

exist for all integers $j \leqslant k$, $x \in U$ and $h_1, \ldots, h_j \in E$, and all maps $\mathtt{d}^j f : U \times E^j \to F$ are continuous. As usual, C^∞-maps are called *smooth*.

Once the concept of a smooth function between open subsets of locally convex spaces is established (cf. [43,33,13]), it is clear how to define a locally convex smooth manifold. A (*locally convex*) *Lie group* G is a group equipped with a smooth manifold structure modeled on a locally convex space for which the group multiplication and the inversion are smooth maps. We write $\mathbf{1} \in G$ for the identity element and $\lambda_g(x) = gx$, resp., $\rho_g(x) = xg$ for the left, resp., right multiplication on G. Then each $x \in T_\mathbf{1}(G)$ corresponds to a unique left invariant vector field x_l with $x_l(g) := T_\mathbf{1}(\lambda_g)x$, $g \in G$. The space of left invariant vector fields is closed under the Lie bracket of vector fields, hence inherits a Lie algebra structure. In this sense we obtain on $\mathfrak{g} := T_\mathbf{1}(G)$ a continuous Lie bracket which is uniquely determined by $[x,y]_l = [x_l, y_l]$ for $x, y \in \mathfrak{g}$. We shall also use the functorial notation $\mathbf{L}(G) := (\mathfrak{g}, [\cdot,\cdot])$ for the Lie algebra of G and, accordingly, $\mathbf{L}(\varphi) = T_\mathbf{1}(\varphi) : \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ for the Lie algebra morphism associated to a morphism $\varphi : G_1 \to G_2$ of Lie groups. Then $\mathbf{L}$ defines a functor from the category of locally convex Lie groups to the category of locally convex topological Lie algebras. The adjoint action of G on $\mathfrak{g}$ is defined by $\mathrm{Ad}(g) := \mathbf{L}(c_g)$, where $c_g(x) = gxg^{-1}$ is the conjugation map. The adjoint action is smooth and each $\mathrm{Ad}(g)$ is a topological isomorphism of $\mathfrak{g}$. If $\mathfrak{g}$ is a Fréchet, resp., a Banach space, then G is called a *Fréchet-*, resp., a *Banach–Lie group*.

For every Lie group G, the tangent bundle TG is a Lie group with respect to the tangent map $T(m_G)$ of the multiplication $m_G : G \times G \to G$ on G. It contains G as the zero section, which is a Lie subgroup, and the projection $TG \to G$ is a morphism of Lie groups whose kernel is the additive group of $\mathfrak{g} \cong T_{\mathbf{1}}(G)$. In this sense we write $g.x = T_{(g,\mathbf{1})}(m_G)(0,x) = T_{\mathbf{1}}(\lambda_g)x$ and $x.g = T_{(\mathbf{1},g)}(m_G)(x,0) = T_{\mathbf{1}}(\rho_g)x$ for $g \in G$ and $x \in \mathfrak{g}$. The two maps

$$G \times \mathfrak{g} \to TG, \quad (g,x) \mapsto g.x \quad \text{and} \quad G \times \mathfrak{g} \to TG, \quad (g,x) \mapsto x.g \tag{1}$$

trivialize the tangent bundle.

A smooth map $\exp_G : \mathfrak{g} \to G$ is called an *exponential function* if each curve $\gamma_x(t) := \exp_G(tx)$ is a one-parameter group with $\gamma_x'(0) = x$. The Lie group G is said to be *locally exponential* if it has an exponential function for which there is an open 0-neighborhood U in $\mathfrak{g}$ mapped diffeomorphically by $\exp_G$ onto an open subset of G. All Banach–Lie groups are locally exponential [43, Prop. IV.1.2]. Not every infinite dimensional Lie group has an exponential function [43, Ex. II.5.5], but exponential functions are unique whenever they exist.

If $\pi : G \to \mathrm{GL}(V)$ is a representation of G on a locally convex space V, the exponential function permits us to associate to each element x of the Lie algebra a one-parameter group $\pi_x(t) := \pi(\exp_G tx)$. We therefore **assume** in the following that G has an exponential function.

3. Basic facts and definitions

In this section we introduce some basic notation and derive some general results for representations of Lie groups on locally convex spaces.

Definition 3.1. Let (π, V) be a representation of the Lie group G (with a smooth exponential function) on the locally convex space V.

(a) We say that π is *continuous* if the action of G on V defined by $(g,v) \mapsto \pi(g)v$ is continuous.

(b) An element $v \in V$ is a C^k-vector, $k \in \mathbb{N}_0 \cup \{\infty\}$, if the orbit map $\pi^v : G \to V, g \mapsto \pi(g)v$ is a C^k-map. We write $V^k := V^k(\pi)$ for the linear subspace of C^k-vectors and we say that the representation π is *smooth* if the space V^∞ of smooth vectors is dense.

If G is a Banach–Lie group and V a Banach space, then we write $FV^k \subseteq V^k$ for the subspace of those C^k-vectors whose orbit map is also C^k in the Fréchet sense.

(c) For each $x \in \mathfrak{g}$, we write

$$\mathcal{D}_x := \left\{ v \in V\colon \left.\frac{d}{dt}\right|_{t=0} \pi(\exp_G tx)v \text{ exists} \right\}$$

for the domain of the infinitesimal generator

$$\overline{\mathrm{d}\pi}(x)v := \left.\frac{d}{dt}\right|_{t=0} \pi\big(\exp_G(tx)\big)v$$

of the one-parameter group $\pi(\exp_G(tx))$. We write $\mathcal{D}_\mathfrak{g} := \bigcap_{x\in\mathfrak{g}} \mathcal{D}_x$ and $\omega_v(x) := \overline{\mathrm{d}\pi}(x)v$ for $v \in \mathcal{D}_\mathfrak{g}$. Each $\mathcal{D}_x$ and therefore also $\mathcal{D}_\mathfrak{g}$ are linear subspaces of V, but at this point we do not know whether ω_v is linear (cf. Theorem 8.2 for a positive answer for Banach–Lie groups).

(d) We define inductively $\mathcal{D}^1_\mathfrak{g} := \mathcal{D}_\mathfrak{g}$ and

$$\mathcal{D}^n_\mathfrak{g} := \big\{ v \in \mathcal{D}_\mathfrak{g}\colon (\forall x \in \mathfrak{g})\overline{\mathrm{d}\pi}(x)v \in \mathcal{D}^{n-1}_\mathfrak{g} \big\} \quad \text{for } n > 1,$$

so that

$$\omega_v^n(x_1,\dots,x_n) := \overline{\mathrm{d}\pi}(x_1)\cdots\overline{\mathrm{d}\pi}(x_n)v$$

is defined for $v \in \mathcal{D}_{\mathfrak{g}}^n$ and $x_1,\dots,x_n \in \mathfrak{g}$. We further put $\mathcal{D}_{\mathfrak{g}}^\infty := \bigcap_{n\in\mathbb{N}} \mathcal{D}_{\mathfrak{g}}^n$.

(e) Let A be an operator with domain $\mathcal{D}(A)$ on the locally convex space V. An element in the space $\mathcal{D}^\infty(A) := \bigcap_{n=1}^\infty \mathcal{D}(A^n)$ is called a *smooth vector for* A.

(f) If V is a Banach space, then we say that (π, V) is *locally bounded* if there exists a $\mathbf{1}$-neighborhood $U \subseteq G$ for which $\pi(U)$ is a bounded set of linear operators on V. It is easy to see that this requirement implies boundedness of π on a neighborhood of any compact subset of G.

Remark 3.2. (a) For every representation (π, V) we have

$$V^1(\pi) \subseteq \mathcal{D}_{\mathfrak{g}} \quad \text{and} \quad V^k(\pi) \subseteq \mathcal{D}_{\mathfrak{g}}^k \quad \text{for } k \in \mathbb{N}.$$

Note that ω_v^k is continuous and k-linear for every $v \in V^k(\pi)$.

(b) By definition, we have $\mathcal{D}_{\mathfrak{g}}^2 \subseteq \mathcal{D}_{\mathfrak{g}}^1$, so that we obtain by induction that $\mathcal{D}_{\mathfrak{g}}^{n+1} \subseteq \mathcal{D}_{\mathfrak{g}}^n$ for every $n \in \mathbb{N}$.

(c) If $v \in \mathcal{D}_x$, then $\gamma(t) := \pi(\exp_G(tx))v$ is a C^1-curve in V with

$$\gamma'(t) = \pi\big(\exp_G(tx)\big)\overline{\mathrm{d}\pi}(x)v,$$

so that

$$\pi(\exp_G x)v - v = \int_0^1 \pi(\exp_G tx)\overline{\mathrm{d}\pi}(x)v\,dt. \tag{2}$$

Lemma 3.3. *Suppose that (π, V) is a continuous representation of the Lie group G on V. Then a vector $v \in \mathcal{D}_{\mathfrak{g}}$ is a C^1-vector if and only if the following two conditions are satisfied*:

(i) *For every smooth curve $\gamma : [-\varepsilon, \varepsilon] \to G$ with $\gamma(0) = \mathbf{1}$ and $\gamma'(0) = x$, the derivative $\frac{d}{dt}|_{t=0}\pi(\gamma(t))v$ exists and equals $\overline{\mathrm{d}\pi}(x)v$.*

(ii) *$\omega_v : \mathfrak{g} \to V, x \mapsto \overline{\mathrm{d}\pi}(x)v$ is continuous.*

If G is locally exponential, then (i) *follows from* (ii)*, and if G is finite dimensional, then* (i) *and* (ii) *hold for every $v \in \mathcal{D}_{\mathfrak{g}}$.*

Proof. If π^v is a C^1-map, then $\omega_v = T_{\mathbf{1}}(\pi^v)$ is a continuous linear map satisfying $\frac{d}{dt}|_{t=0}\pi(\gamma(t))v = \omega_v(x)$ for each smooth curve in G with $\gamma(0) = \mathbf{1}$ and $\gamma'(0) = x$.

Suppose, conversely, that (i) and (ii) are satisfied. Then the relation

$$\pi\big(\gamma(t+h)\big) = \pi\big(\gamma(t)\big)\pi\big(\gamma(t)^{-1}\gamma(t+h)\big)$$

implies that for each smooth curve γ in G, we have

$$\frac{d}{dt}\pi\big(\gamma(t)\big)v = \pi\big(\gamma(t)\big)\omega_v\big(\delta(\gamma)_t\big),$$

where $\delta(\gamma)_t = \gamma(t)^{-1}.\gamma'(t)$ is the left logarithmic derivative of γ (cf. (1) in Section 2).

We conclude that the orbit map π^v has directional derivatives in each point $g \in G$, and that its tangent map is given by

$$T_g(\pi^v)(g.x) = \pi(g)\omega_v(x)$$

(cf. (1)). Since ω_v and the action of G on V are continuous, the map $T(\pi^v) : TG \cong G \times \mathfrak{g} \to V$ is continuous, i.e., $\pi^v \in C^1(G, V)$. From [13, Lemma 2.2.14] it now follows that ω_v is linear.

Now we assume, in addition, that G is locally exponential and claim that (ii) implies (i). Any smooth curve γ with $\gamma(0) = \mathbf{1}$ and $\gamma'(0) = x$ can be written for sufficiently small values of $t \in \mathbb{R}$ as $\exp_G(\eta(t))$ with a smooth curve $\eta : [-\varepsilon, \varepsilon] \to \mathfrak{g}$ satisfying $\eta(0) = 0$ and $\eta'(0) = x$. Therefore it suffices to show that for any such curve η, we have

$$\frac{d}{ds}\bigg|_{s=0} \pi\big(\exp_G \eta(s)\big)v = \omega_v(x).$$

First we note that $\eta(0) = 0$ implies that

$$\frac{\eta(s)}{s} = \frac{1}{s}\int_0^1 \eta'(ts)s\,dt = \int_0^1 \eta'(st)\,dt$$

extends by the value $x = \eta'(0)$ to a smooth curve on $[-\varepsilon, \varepsilon]$.

Next, we derive for $v \in \mathcal{D}_{\mathfrak{g}}$ from (2) the relation

$$\pi\big(\exp_G \eta(s)\big)v - v = \int_0^1 \pi\big(\exp_G t\eta(s)\big)\omega_v\big(\eta(s)\big)\,dt, \tag{3}$$

which leads to

$$\begin{aligned}\frac{d}{ds}\bigg|_{s=0} \pi\big(\exp_G \eta(s)\big)v &= \lim_{s\to 0}\int_0^1 \pi\big(\exp_G t\eta(s)\big)\omega_v\big(\eta(s)/s\big)\,dt \\ &= \int_0^1 \omega_v(x)\,dt = \omega_v(x)\end{aligned}$$

because, in view of (ii), the integrand in (3) is a continuous function of (s, t), so that we may exchange integration and the limit. This means that (i) is satisfied.

If G is finite dimensional, then Goodman's argument in [15, p. 221] implies that every $v \in \mathcal{D}_{\mathfrak{g}}$ is a C^1-vector, so that (i) and (ii) are satisfied. □

Lemma 3.4. *If G is locally exponential, then a vector $v \in V$ is a C^k-vector if and only if $v \in \mathcal{D}_{\mathfrak{g}}^k$ and the maps ω_v^n, $n \leqslant k$, are continuous and n-linear. In particular, v is a smooth vector if and only if $v \in \mathcal{D}_{\mathfrak{g}}^\infty$ and all the maps ω_v^n are continuous and n-linear.*

Proof. If v is a C^k-vector, then

$$\omega_v^n(x_1, \ldots, x_k) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n}\bigg|_{t_1=\cdots=t_n=0} \pi\big(\exp_G(t_1 x_1) \cdots \exp_G(t_n x_n)\big)v$$

is a continuous n-linear map for any $n \leqslant k$.

Suppose, conversely, that this is the case for some $k \geqslant 1$. Lemma 3.3 takes care of the case $k = 1$. We may therefore assume that $k > 1$ and that the assertion holds for $k - 1$. From Lemma 3.3 we derive that v is a C^1-vector with $T(\pi^v)(g.x) = \pi(g)\omega_v^1(x)$. We have to show that $T(\pi^v) : TG \cong G \times \mathfrak{g} \to V$ is a C^{k-1}-map. For each fixed $x \in \mathfrak{g}$, the element $\omega_v^1(x) = \overline{\mathtt{d}\pi}(x)v$ is contained in $\mathcal{D}_{\mathfrak{g}}^{k-1}$, so that, in view of

$$\omega^n_{\omega_v^1(x)}(x_1, \ldots, x_n) = \omega_v^{n+1}(x_1, \ldots, x_n, x) \quad \text{for } n \leqslant k-1,$$

our induction hypothesis implies that $\omega_v^1(x)$ is a C^{k-1}-vector, hence that $T(\pi^v)$ has directional derivatives of all orders $\leqslant k - 1$, and that they are sums of terms of the form

$$\pi(g)\overline{\mathtt{d}\pi}(x_1) \cdots \overline{\mathtt{d}\pi}(x_j)v = \pi(g)\omega_v^j(x_1, \ldots, x_j),$$

which are continuous functions on $G \times \mathfrak{g}^j$, $j \leqslant k - 1$. This proves that $T(\pi^v)$ is a C^{k-1}-map, and hence that v is a C^k-vector. □

4. A topology on the space of smooth vectors

Throughout this section, we assume that G is a Banach–Lie group and that $\|\cdot\|$ is a compatible norm on $\mathfrak{g} = \mathbf{L}(G)$ satisfying

$$\|[x, y]\| \leqslant \|x\| \cdot \|y\|, \quad x, y \in \mathfrak{g}.$$

We shall define a topology on V^∞ for which the action

$$\sigma : G \times V^\infty \to V^\infty, \quad (g, v) \mapsto \pi(g)v$$

is smooth.

Let

$$\mathtt{d}\pi : \mathfrak{g} \to \operatorname{End}\big(V^\infty\big), \qquad \mathtt{d}\pi(x)v := \frac{d}{dt}\bigg|_{t=0} \pi(\exp tx)v$$

denote the derived action of $\mathfrak{g}$ on V^∞. That this is indeed a representation of $\mathfrak{g}$ follows by observing that the map $V^\infty \to C^\infty(G, V)$, $v \mapsto \pi^v$ intertwines the action of G with the right translation action on $C^\infty(G, V)$, and this implies that the derived action corresponds to the action of $\mathfrak{g}$ on $C^\infty(G, V)$ by left invariant vector fields (cf. [38, Rem. IV.2] for details).

Definition 4.1. (a) For $n \in \mathbb{N}_0$, let $\operatorname{Mult}^n(\mathfrak{g}, V)$ be the space of continuous n-linear maps $\mathfrak{g}^n \to V$ (for $n = 0$ we interpret this as the constant maps) and write $\mathcal{P}(V)$ for the set of continuous seminorms on V. The space $\mathrm{Mult}^n(\mathfrak{g}, V)$ carries a natural locally convex topology defined by the seminorms

$$p(\omega) := \sup\{p(\omega(x_1, \ldots, x_n)) \colon \|x_1\|, \ldots, \|x_n\| \leqslant 1\}, \quad p \in \mathcal{P}(V).$$

Note that $p(\omega)$ is the smallest constant $c \geqslant 0$ for which we have the estimate

$$p(\omega(x_1, \ldots, x_n)) \leqslant c\|x_1\| \cdots \|x_n\| \quad \text{for } x_1, \ldots, x_n \in \mathfrak{g}. \tag{4}$$

(b) To topologize V^∞, we define for each $n \in \mathbb{N}_0$ a map

$$\Psi_n : V^\infty \to \mathrm{Mult}^n(\mathfrak{g}, V), \qquad \Psi_n(v)(x_1, \ldots, x_n) := \mathtt{d}\pi(x_1) \cdots \mathtt{d}\pi(x_n)v.$$

That $\Psi_n(v)$ defines a continuous n-linear map follows from the smoothness of the orbit map $\pi^v : G \to V$, $g \mapsto \pi(g)v$ and the fact that $\Psi_n(v)$ is obtained by n-fold partial derivatives in $\mathbf{1}$. We thus obtain an injective linear map

$$\Psi : V^\infty \to \prod_{n \in \mathbb{N}_0} \mathrm{Mult}^n(\mathfrak{g}, V), \quad v \mapsto (\Psi_n(v))_{n \in \mathbb{N}_0},$$

and define the topology on V^∞ such that Ψ is a topological embedding. This means that the topology on V^∞ is defined by the seminorms

$$p_n(v) := \sup\{p(\mathtt{d}\pi(x_1) \cdots \mathtt{d}\pi(x_n)v) \colon x_i \in \mathfrak{g},\ \|x_i\| \leqslant 1\}, \quad p \in \mathcal{P}(V),\ n \in \mathbb{N}_0.$$

Note that $p_n(v) = p(\Psi_n(v))$ is the smallest constant $c \geqslant 0$ for which we have the estimate

$$p(\mathtt{d}\pi(x_1) \cdots \mathtt{d}\pi(x_n)v) \leqslant c\|x_1\| \cdots \|x_n\|, \quad x_1, \ldots, x_n \in \mathfrak{g}. \tag{5}$$

We endow V^∞ with the locally convex topology defined by the seminorms p_n, $n \in \mathbb{N}_0$, $p \in \mathcal{P}(V)$.

Lemma 4.2. *The linear map* $\mathtt{d}\pi : \mathfrak{g} \times V^\infty \to V^\infty$, $(x, v) \mapsto \mathtt{d}\pi(x)v$ *is continuous.*

Proof. This follows from $p_n(\mathtt{d}\pi(x)v) \leqslant p_{n+1}(v)\|x\|$, which is a consequence of (5). □

Lemma 4.3. *The group G acts by continuous linear operators on V^∞. More precisely, we have*

$$p_n(\pi(g)v) \leqslant (p \circ \pi(g))_n(v) \|\mathrm{Ad}(g)^{-1}\|^n. \tag{6}$$

Proof. For $x_1, \ldots, x_n \in \mathfrak{g}$, we have

$$\mathtt{d}\pi(x_1) \cdots \mathtt{d}\pi(x_n)\pi(g)v = \pi(g)\mathtt{d}\pi(\mathrm{Ad}(g^{-1})x_1) \cdots \mathtt{d}\pi(\mathrm{Ad}(g^{-1})x_n)v,$$

and, in view of (5), this implies (6). □

Theorem 4.4. *If (π, V) is a representation of the Banach–Lie group G on the locally convex space V defining a continuous action of G on V, then the action $\sigma(g, v) := \pi(g)v$ of G on V^∞ is smooth.*

Proof. First we show that σ is continuous. In view of Lemma 4.3, it suffices to show that σ is continuous in $(\mathbf{1}, v)$ for each $v \in V^\infty$. For $w \in V^\infty$, the relation

$$\pi(g)w - v = \pi(g)(w - v) + \pi(g)v - v$$

permits us to break the argument into a proof for the continuity of σ in $(\mathbf{1}, 0)$ and the continuity of the orbit map $\pi^v : G \to V^\infty$.

First we use (6) to obtain

$$p_n\big(\pi(g)w\big) \leqslant \big(p \circ \pi(g)\big)_n(w)\big\|\mathrm{Ad}(g)^{-1}\big\|^n.$$

Since $U^p := \{w \in V \colon\ p(w) \leqslant 1\}$ is a 0-neighborhood in V, the continuity of the action of G on V implies the existence of a $\mathbf{1}$-neighborhood $U_G \subseteq G$ and a continuous seminorm q on V with $\pi(U_G)U^q \subseteq U^p$. This means that $p(\pi(g)v) \leqslant q(v)$ for $v \in V$ and $g \in U_G$, i.e., $p \circ \pi(g) \leqslant q$. We thus obtain for $g \in U_G$ the estimate

$$p_n\big(\pi(g)(w)\big) \leqslant q_n(w)\big\|\mathrm{Ad}(g)^{-1}\big\|^n,$$

and since $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is locally bounded, σ is continuous in $(\mathbf{1}, 0)$.

To verify the continuity of the orbit maps π^v, $v \in V^\infty$, we consider the smooth function $f : G \times G \to V$, $(g, h) \mapsto \pi(g)\pi(h)v$ and observe that $f^h(g) := f(g, h) = \pi^{\pi(h)v}(g)$. Using the family $\mathcal{P}(V)$ of all continuous seminorms on V, we embed V into the topological product $\prod_{p\in\mathcal{P}(V)} V_p$, where V_p is the Banach space obtained by completing $V/p^{-1}(0)$ with respect to the norm on this space induced by p. We write $v \mapsto [v]_p$ for the corresponding quotient map. Then the smoothness of the map f is equivalent to the smoothness of the component mappings $f_p : G \times G \to V_p$, $(g, h) \mapsto [f(g, h)]_p$, and since smoothness for Banach space-valued maps implies smooth dependence of higher derivatives (cf. [38, Thm. I.7], [29,13]), it follows that the Banach space-valued maps

$$f_p^n : G \to \mathrm{Mult}^n(\mathfrak{g}, V_p), \quad h \mapsto \Psi_n\big(\pi(h)v\big)_p$$

are smooth. Here we use that for each open subset U in a Banach space E and a smooth map $F : U \to V_p$, the map $F^n : U \to \mathrm{Mult}^n(E, V_p)$, defined by

$$F^n(x)(v_1, \ldots, v_n) := (\partial_{v_1} \cdots \partial_{v_n} F)(x),$$

is smooth. This implies that the corresponding map

$$f_p : G \to \prod_{n\in\mathbb{N}_0} \mathrm{Mult}^n(\mathfrak{g}, V_p), \quad h \mapsto \Psi\big(\pi(h)v\big)_p = \Psi\big(\pi^v(h)\big)_p$$

is smooth, and this in turn means that the orbit map $\pi^v : G \to V^\infty$ is smooth, hence in particular continuous. This completes the proof of the continuity of σ.

The preceding argument already implies that the partial derivatives of the action map σ exist and that they are given by

$$T_{(g,v)}(\sigma)(g.x, w) = \pi(g)\mathrm{d}\pi(x)v + \pi(g)w.$$

From the continuity of σ and the action of $\mathfrak{g}$ on V^∞ (Lemma 4.2), it now follows that $T\sigma$ is continuous, so that σ is actually C^1. Iterating this argument, we see that whenever σ is C^n, then $T\sigma$ also is C^n, and this shows that σ is smooth. $\square$

Corollary 4.5. *If G is a Banach–Lie group and $(\pi, \mathcal{H})$ a continuous unitary representation of G on $\mathcal{H}$, then the induced action of G on $\mathcal{H}^\infty$ is smooth.*

It is instructive to compare our topology on V^∞ with the standard construction for finite dimensional Lie groups:

Proposition 4.6. *Suppose that G is finite dimensional, V is a locally convex space, and $C^\infty(G, V)$ is endowed with the smooth compact open topology, i.e., the topology of uniform convergence of all derivatives on compact subsets. Then for any continuous representation (π, V) of G, the injection*

$$\eta : V^\infty \to C^\infty(G, V), \quad v \mapsto \pi^v$$

is a topological embedding whose range is the subspace of smooth maps $f : G \to V$ which are equivariant in the sense that

$$f(gh) = \pi(g) f(h) \quad \text{for } g, h \in G.$$

Proof. We recall that the smooth compact open topology is defined by the property that the embedding $C^\infty(G, V) \hookrightarrow \prod_{n\in\mathbb{N}_0} C(T^n G, T^n V)$, $f \mapsto (T^n(f))_{n\in\mathbb{N}_0}$, defined by the tangent maps, is a topological embedding, where the factors on the right are endowed with the compact open topology. It is easy to see that this is a locally convex topology for which the action of G on $C^\infty(G, V)$ by right translations is smooth (cf. [38, Sect. III]). Here the key observation is that the smoothness of the map

$$G \times C^\infty(G, V) \to C^\infty(G, V), \quad (g, f) \mapsto f \circ \rho_g$$

follows from the smoothness of the corresponding map

$$G \times C^\infty(G, V) \times G \to V, \quad (g, f, x) \mapsto f(xg)$$

[50, Lemma A.2]. This in turn follows from the smoothness of the evaluation map ev: $C^\infty(G, V) \times G \to V$, whose smoothness follows from the smoothness of $\mathrm{id}_{C^\infty(G,V)}$ (cf. [50, Lemma A.3(1)]; resp., [12, Prop. 12.2]).[2]

If (π, V) is a continuous representation of G on V, then $\eta(v) := \pi^v$ injects V^∞ into $C^\infty(G, V)$ and the image of η clearly is the subspace of equivariant map because any such map f satisfies $f = \pi^{f(\mathbf{1})}$. Since $\mathrm{im}(\eta)$ is a closed subspace invariant under right translation, we obtain by restriction a smooth action on this space.

[2] In the setting of convenient calculus, one finds similar arguments for the smoothness of the G-action on V^∞ in [32, Thm. 5.2].

For $d = \dim \mathfrak{g}$, we also have a topological isomorphism $\mathrm{Mult}^p(\mathfrak{g}, V) \cong V^{pd}$ by evaluating p-linear maps on p-tuples of elements of a fixed basis of $\mathfrak{g}$. Therefore the topology on V^∞ is defined by the seminorms

$$v \mapsto p\big(\mathtt{d}\pi(x_1)\cdots\mathtt{d}\pi(x_n)v\big), \quad p \in \mathcal{P}(V),$$

where $x_1, \dots, x_n$ are n elements of a fixed basis of $\mathfrak{g}$ (Definition 4.1).

Since $\mathfrak{g}$ acts on $C^\infty(G, V)$ by continuous linear maps, all the operators $\mathtt{d}\pi(x)$ are continuous with respect to the smooth compact open topology, and therefore the evaluation map

$$\mathrm{ev}_{\mathbf{1}} : \mathrm{im}(\eta) \to V^\infty$$

is a continuous bijection. Conversely, the smoothness of the action of G on V^∞ (Theorem 4.4) implies that the linear map η is continuous [50, Lemma A.2], hence a topological embedding. □

Remark 4.7. (a) If G is finite dimensional with countably many connected components, which is equivalent to being a countable union of compact subsets, and V is a Fréchet space, then $C^\infty(G, V)$ is also Fréchet, and therefore V^∞ is a Fréchet space.

(b) For a unitary representation $(\pi, \mathcal{H})$ of a finite dimensional Lie group G, Goodman uses in [14] Sobolev space techniques to show that the space $\mathcal{H}^\infty$ of smooth vectors is the intersection of the spaces $\mathcal{D}^\infty(\overline{\mathtt{d}\pi}(x_j))$, $j = 1, \dots, n$, where $x_1, \dots, x_n$ is a basis for $\mathfrak{g}$ ([14, Thm. 1.1]; see also [63, Thm. 10.1.9]). This implies in particular that $\mathcal{H}^\infty$ is complete and that the topology on this space is compatible with our construction (cf. Proposition 4.6).

It seems that the chances that the results in this section extend to some classes of Fréchet–Lie groups are not very high, as the following two examples show.

Example 4.8. For the Fréchet–Lie group $G = (\mathbb{R}^{\mathbb{N}}, +)$ (endowed with the product topology), we consider the continuous unitary representation on $\mathcal{H} = \ell^2(\mathbb{N}, \mathbb{C})$ given by $(\pi(g)x)_n = e^{ig_n}x_n$. Then

$$\mathcal{D}_{\mathfrak{g}} = \big\{x \in \mathcal{H}\colon \big(\forall y \in \mathbb{R}^{\mathbb{N}}\big)(x_n y_n) \in \mathcal{H}\big\} = \mathrm{span}\{e_n\colon n \in \mathbb{N}\} = \mathbb{C}^{(\mathbb{N})}$$

is a countably dimensional vector space. Since $\mathcal{D}_{\mathfrak{g}}$ is spanned by eigenvectors, we see that $\mathcal{D}_{\mathfrak{g}} = \mathcal{H}^\infty$.

We claim that for no locally convex topology on $\mathcal{H}^\infty$ the bilinear map

$$\beta : \mathfrak{g} \times \mathcal{H}^\infty \to \mathcal{H}, \quad (x, v) \mapsto \mathtt{d}\pi(x)v$$

is continuous, which implies in particular that the action of G on $\mathcal{H}^\infty$ is not C^1, hence in particular not smooth. To verify our claim, let $B \subseteq \mathcal{H}$ denote the closed unit ball. If β is continuous, then there exist 0-neighborhoods $U_{\mathfrak{g}} \subseteq \mathfrak{g}$ and $U \subseteq \mathcal{H}^\infty$ with $\beta(U_{\mathfrak{g}} \times U) \subseteq B$. Next we observe that $U_{\mathfrak{g}}$ contains a subspace of the form $\mathbb{R}^M$, $M := \mathbb{N} \setminus \{1, \dots, n\}$. In particular $\mathbb{R}e_{n+1} \subseteq U_{\mathfrak{g}}$. Since we also have $\varepsilon e_{n+1} \in U$ for some $\varepsilon > 0$, we arrive at the contradiction $\mathbb{R} \cdot \varepsilon e_{n+1} = \mathbb{R}e_{n+1} \subseteq B$.

This proves that Theorem 4.4 does not generalize to unitary representations of general Fréchet–Lie groups.

Example 4.9. We have seen in Proposition 4.6 that, for finite dimensional Lie groups, we obtain the natural topology on V^∞ by embedding it into the space $C^\infty(G, V)$, endowed with the smooth compact open topology. In this case the smoothness of the G-action on $C^\infty(G, V)$ yields the smoothness on the invariant subspace V^∞. For any infinite dimensional Lie group G, the smooth compact open topology still makes sense, but, as the following example shows, this topology is too weak to guarantee the continuity of the G-action.

To substantiate this claim, we consider a locally convex space G, considered as a Lie group, and the space $E := \mathrm{Aff}(G, \mathbb{R})$ of affine real-valued functions on G. Then G acts on E by $(\pi(g)f)(x) := f(x+g)$. Identifying E with $\mathbb{R} \times G'$, we see that for $f = c + \alpha$ ($c \in \mathbb{R}, \alpha \in G'$) we have

$$\pi(g)(c+\alpha) = \alpha(g) + c + \alpha.$$

Therefore π is continuous with respect to a locally convex topology on E, resp., G', if and only if the evaluation map

$$G' \times G \to \mathbb{R}, \quad (\alpha, x) \mapsto \alpha(x)$$

is continuous.

The smooth compact open topology on $C^\infty(G, \mathbb{R})$ corresponds on G' to the topology of uniform convergence on compact subsets of G. The continuity of the evaluation map with respect to this topology is equivalent to the existence of a compact subset $C \subseteq G$ for which the closed convex hull $\overline{\mathrm{conv}(C)}$ is a 0-neighborhood. If V is complete, then the precompactness of $\overline{\mathrm{conv}(C)}$ [5, Ch. II, §4, no. 2, Prop. 3] implies that G has a compact 0-neighborhood, hence that it is finite dimensional.

A similar argument shows that, for the finer topology of uniform convergence on bounded subsets of G', the evaluation map is continuous if and only if G has a bounded 0-neighborhood which means that it is a normed space.

4.1. Distribution vectors

An element of the topological dual space $V^{-\infty} := (V^\infty)'$ is called a *distribution vector*. The main property of these functionals is that they correspond to G-morphisms $V^\infty \to C^\infty(G)$. Note that $(g.\alpha) := \alpha \circ \pi(g)^{-1}$ defines a natural action of G on $V^{-\infty}$.

Lemma 4.10. *We have an injective map*

$$\Phi : V^{-\infty} \to \mathrm{Hom}_G\big(V^\infty, C^\infty(G)\big), \qquad \Phi(\alpha)(v)(g) := \alpha\big(\pi(g)^{-1}v\big),$$

where G acts on $C^\infty(G)$ by $(g.f)(h) := f(g^{-1}h)$. The range of Φ consists of all those G-morphisms $\varphi : V^\infty \to C^\infty(G)$ for which the composition $\mathrm{ev}_{\mathbf{1}} \circ \varphi$ is continuous, i.e., an element of $V^{-\infty}$.

Proof. The smoothness of the functions $\Phi(\alpha)(v)$ follows from the smoothness of the G-action on V^∞, and the equivariance of $\Phi(\alpha)$ is immediate from the definition.

For $\varphi := \Phi(\alpha)$ we have $\alpha = \mathrm{ev}_{\mathbf{1}} \circ \varphi \in V^{-\infty}$. If, conversely, $\varphi \in \mathrm{Hom}_G(V^\infty, C^\infty(G))$ is such that $\alpha := \mathrm{ev}_{\mathbf{1}} \circ \varphi$ is continuous, we have

$$\varphi(v)(g) = \big(g^{-1}.\varphi(v)\big)(\mathbf{1}) = \varphi\big(\pi(g)^{-1}v\big)(\mathbf{1}) = (\mathrm{ev}_{\mathbf{1}} \circ \varphi)\big(\pi(g)^{-1}v\big) = \Phi(\alpha)(v)(g),$$

i.e., $\varphi = \Phi(\alpha)$. □

Remark 4.11. (a) For finite dimensional Lie groups, compactly supported distributions are defined as the elements of the topological dual $C^\infty(G)'$ of the space of smooth functions. Since the G-action on $C^\infty(G)$ is smooth (cf. the proof of Proposition 4.6), compactly supported distributions are distribution vectors in the sense defined above.

(b) We do not know if there exists a locally convex topology on $C^\infty(G)$ for which the left or right translation action is smooth and therefore it is not clear what a good concept of a distribution on a Banach–Lie group is. However, the topology on V^∞ for a continuous representation (π, V) provides a natural concept of a distribution vector.

In harmonic analysis on homogeneous spaces G/H, distribution vectors of unitary representations play an important role. If $H \subseteq G$ is a Lie subgroup, so that the quotient G/H carries a manifold structure for which the quotient map $q : G \to G/H$, $g \mapsto gH$, is a submersion, then we may identify $C^\infty(G/H)$ with the subspace of $C^\infty(G)$ consisting of all functions constant on the cosets gH. For the map Φ in Lemma 4.10 we immediately see that

$$\Phi(\alpha)\big(V^\infty\big) \subseteq C^\infty(G/H)$$

is equivalent to the invariance of α under H. Therefore the space $(V^\infty)^H$ of H-invariant distribution vectors parametrizes the G-morphisms $\varphi : V^\infty \to C^\infty(G/H)$ which are continuous in the very weak sense that their composition with point evaluations is continuous (Lemma 4.10, see also [17, p. 137]). This situation is of particular interest if these embeddings are essentially unique: A pair (G, H) of a Banach–Lie group G and a subgroup H is called a *generalized Gelfand pair* if

$$\dim\big(\mathcal{H}^{-\infty}\big)^H \leqslant 1$$

holds for every irreducible continuous unitary representation $(\pi, \mathcal{H})$ of G. For finite dimensional Lie groups there exists a rich theory of generalized Gelfand pairs with many applications in harmonic analysis (cf. [8]). It is a very worthwhile project to explore this concept also for Banach–Lie groups, e.g., for automorphism groups of infinite dimensional semi-Riemannian symmetric spaces such as hyperboloids

$$X = \big\{(t, v) \in \mathbb{R} \times \mathcal{H}\colon\ \|v\|^2 - t^2 = 1\big\},$$

where $\mathcal{H}$ is a real Hilbert space (cf. [9] for the case $\dim \mathcal{H} < \infty$).

5. Smooth vectors in Banach spaces

In this section we assume that G is a Banach–Lie group and that V is a Banach space. Our goal is to show that the space V^∞ is complete, hence a Fréchet space.

The following proposition generalizes the well-known fact that separately continuous bilinear maps on Fréchet spaces are continuous.

Proposition 5.1 *(Continuity criterion). Let X be a first countable topological space, F a Fréchet space and V a topological vector space. Let $\alpha : X \to \mathcal{L}(F, V)$ (the space of continuous linear operators $F \to V$) be a map such that, for each $f \in F$, the map*

$$\alpha_f : X \to V, \quad x \mapsto \alpha(x) f$$

is continuous. Then the map $\widehat{\alpha} : X \times F \to V$, $(x, f) \mapsto \alpha(x) f$ is continuous.

Proof. Since X is first countable, the same holds for the product space $X \times F$, so that we only have to show that, for any sequence $(x_n, f_n) \to (x_0, f_0)$ in $X \times F$, we have $\alpha(x_n) f_n \to \alpha(x_0) f_0$. Our assumption implies that the sequence $\alpha(x_n)$ of linear maps converges pointwise to $\alpha(x_0)$. In particular, for each $f \in F$, the sequence $\alpha(x_n) f$ in V is bounded. Now the Banach–Steinhaus Theorem [61, Thm. 2.6] implies that the sequence $\alpha(x_n)$ is equicontinuous, which leads to $\alpha(x_n)(f_n - f_0) \to 0$. We thus obtain

$$\alpha(x_n) f_n - \alpha(x_0) f_0 = \alpha(x_n) f_n - \alpha(x_n) f_0 + \alpha(x_n) f_0 - \alpha(x_0) f_0 \to 0.$$

This proves the continuity of $\widehat{\alpha}$. □

Lemma 5.2. *Let V be a Banach space, G be a topological group and $\pi : G \to \mathrm{GL}(V)$ be a homomorphism. Then the following are equivalent*:

(i) *The linear action $\sigma : G \times V \to V$, $(g, v) \mapsto \pi(g)v$ is continuous.*
(ii) *σ is continuous in* $(\mathbf{1}, 0)$.
(iii) *π is locally bounded and π is strongly continuous, i.e., all orbit maps $\pi^v(g) = \sigma(g, v)$ are continuous.*

If, in addition, G is metrizable, then (i)–(iii) *are equivalent to*

(iv) *π is strongly continuous.*

Proof. (i) ⇒ (ii) is trivial.

(ii) ⇒ (iii): If σ is continuous in $(\mathbf{1}, 0)$, then there exists a $\mathbf{1}$-neighborhood $U \subseteq G$ and a ball $B_\varepsilon(0)$ in V with $\pi(U) B_\varepsilon(0) \subseteq B_1(0)$. This implies that $\|\pi(g)\| \leqslant \varepsilon^{-1}$ for every $g \in U$.

(iii) ⇒ (i) follows from $\|\pi(g)v - \pi(h)w\| \leqslant \|\pi(g)\| \|v - w\| + \|\pi(g)w - \pi(h)w\|$.

(iv) ⇒ (i): If, in addition, G is metrizable, then it is first countable, so that Proposition 5.1 shows that (iv) implies (i). □

Remark 5.3. If $\pi : G \to \mathrm{GL}(V)$ is a representation of the topological group G on the Banach space V which is locally bounded, then the subspace V^0 of *continuous vectors*, i.e., of those $v \in V$ for which the orbit map $\pi^v : G \to V$, $g \mapsto \pi(g)v$, is continuous is closed. In fact, suppose that $v_n \to v$ holds for some sequence $v_n \in V^0$ and that U is a neighborhood of $g_0 \in G$ on which π is bounded. Then $\pi^{v_n} \to \pi^v$ holds uniformly on U. Therefore the continuity of the maps π^{v_n} implies the continuity of π^v on U. Since g_0 was arbitrary, it follows that $v \in V^0$, and hence that V^0 is closed. Since V^0 is G-invariant, we obtain a locally bounded representation $\pi^0 : G \to \mathrm{GL}(V^0)$, and its continuity follows from Lemma 5.2.

In view of the preceding lemma, the continuity of the action of G on V implies that π is locally bounded. The following proposition can also be derived from the results in Section 9 (cf. Remark 9.6).

Proposition 5.4. *If (π, V) is a continuous representation of the Banach–Lie group G on the Banach space V, then V^∞ is complete, i.e., a Fréchet space.*

Proof. First we note that $\prod_{n\in\mathbb{N}_0} \mathrm{Mult}^n(\mathfrak{g}, V)$ is a countable product of Banach spaces, hence a Fréchet space. Therefore the completion $\widehat{V}^\infty$ of the topological vector space V^∞ can be identified with the closure of the subspace $\Psi(V^\infty)$. Since V is complete, we have a continuous linear map $\iota : \widehat{V}^\infty \to V$, extending the inclusion $V^\infty \hookrightarrow V$, and, with respect to the realization of $\widehat{V}^\infty$ as the closure of the image of Ψ, this map is given by $\iota((\alpha_n)_{n\in\mathbb{N}_0}) = \alpha_0$. In the course of the proof we shall see that ι is injective and that its range coincides with V^∞. This implies that $\widehat{V}^\infty$ is not larger than V^∞, so that V^∞ is complete.

Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in V^∞ for which $\Psi_k(v_n)$ converges to some $\omega^k \in \mathrm{Mult}^k(\mathfrak{g}, V)$ for each $k \in \mathbb{N}_0$. We have to show that $v := \omega^0$ is a smooth vector and that $\omega^k = \Psi_k(v)$ for each k. Since the continuous linear maps $\mathtt{d}\pi(x) : V^\infty \to V^\infty$ extend to continuous linear maps

$$\widehat{\mathtt{d}}\pi(x) : \widehat{V}^\infty \to \widehat{V}^\infty$$

on the completion, the convergence $v_n \to v$ in $\widehat{V}^\infty$ implies that

$$\mathtt{d}\pi(x_1)\cdots\mathtt{d}\pi(x_k)v_n \to \widehat{\mathtt{d}}\pi(x_1)\cdots\widehat{\mathtt{d}}\pi(x_k)v,$$

and hence that

$$\omega^k(x_1, \ldots, x_k) = \widehat{\mathtt{d}}\pi(x_1)\cdots\widehat{\mathtt{d}}\pi(x_k)v \in \widehat{V}^\infty. \tag{7}$$

This proves in particular that $v = 0$ implies $\omega^k = 0$ for each $k > 0$, and hence that the map $\iota : \widehat{V}^\infty \to V$ is injective.

For $x \in \mathfrak{g}$, we obtain with (2) that

$$\pi(\exp_G x)v_n - v_n = \int_0^1 \pi(\exp_G tx)\mathtt{d}\pi(x)v_n\, dt$$

for each n. Since $\mathtt{d}\pi(x)v_n \to \omega^1(x)$ and the linear map $w \mapsto \int_0^1 \pi(\exp_G(tx))w\, dt$ is continuous (π is locally bounded), we obtain

$$\pi(\exp_G x)v - v = \int_0^1 \pi(\exp_G tx)\omega^1(x)\, dt, \tag{8}$$

which leads to

$$\frac{d}{ds}\bigg|_{s=0} \pi(\exp_G sx)v = \lim_{s\to 0} \int_0^1 \pi(\exp_G stx)\omega^1(x)\,dt = \int_0^1 \omega^1(x)\,dt = \omega^1(x).$$

This implies that $v \in \mathcal{D}_{\mathfrak{g}}$ with $\omega_v = \omega^1$. In particular, ω_v is continuous and linear. Since Banach–Lie groups are locally exponential, Lemma 3.3 now implies that v is a C^1-vector.

To see that π^v is C^2, we have to show that $T(\pi^v)$ is C^1. From (7) we recall that, for each $x \in \mathfrak{g}$, $\omega^1(x) = \widehat{\mathtt{d}}\pi(x)v \in \widehat{V}^\infty$ is a C^1-vector by the preceding argument. Therefore

$$T\big(\pi^v\big) : TG \cong G \times \mathfrak{g} \to V, \quad (g, x) \mapsto \pi(g)\widehat{\mathtt{d}}\pi(x)v$$

has directional derivatives given by

$$T_{g,x}T\big(\pi^v\big)(g.y, w) = \pi(g)\widehat{\mathtt{d}}\pi(y)\widehat{\mathtt{d}}\pi(x)v + \pi(g)\widehat{\mathtt{d}}\pi(y)v.$$

We have already seen above that the second term is continuous, and the continuity of the first term follows from the continuity of the action of G on V and the continuity of the bilinear map

$$\omega^2(y, x) = \widehat{\mathtt{d}}\pi(y)\widehat{\mathtt{d}}\pi(x)v.$$

This proves that each π^v is C^2. Iterating this argument implies that π^v has directional derivatives of any order k, and that they are sums of terms of the form

$$\pi(g)\widehat{\mathtt{d}}\pi(x_1)\cdots\widehat{\mathtt{d}}\pi(x_j) = \pi(g)\omega^j(x_1, \ldots, x_j),$$

which are continuous on $G \times \mathfrak{g}^j$. Therefore v is a C^k-vector for any k, hence smooth, and thus $\widehat{V}^\infty \subseteq V^\infty$ implies that V^∞ is complete. □

6. An applications to C^*-dynamical systems

In this section we show that in the special situation where a Banach–Lie group G acts by automorphisms on a unital C^*-algebra $\mathcal{A}$, i.e., for C^*-dynamical systems, the Fréchet space $\mathcal{A}^\infty$ is a continuous inverse algebra, i.e., its unit group is open and the inversion is a continuous map.

Definition 6.1. (a) A locally convex unital algebra $\mathcal{A}$ is called a *continuous inverse algebra* if its group of units $\mathcal{A}^\times$ is open and the inversion map $\eta : \mathcal{A}^\times \to \mathcal{A}, a \mapsto a^{-1}$ is continuous.

(b) Let G be a topological group and $\mathcal{A}$ be a C^*-algebra. A *C^*-dynamical system* is a triple $(\mathcal{A}, G, \alpha)$, where $\alpha : G \to \mathrm{Aut}(\mathcal{A}), g \mapsto \alpha_g$, is a homomorphism defining a continuous action of G on $\mathcal{A}$.

Theorem 6.2. *If G is a Banach–Lie group and $(\mathcal{A}, G, \alpha)$ a C^*-dynamical system with a unital C^*-algebra $\mathcal{A}$, then the space $\mathcal{A}^\infty$ of smooth vectors is a subalgebra, which is a topological algebra with respect to its natural topology, and the action of G on the Fréchet space $\mathcal{A}^\infty$ is smooth.*

If, in addition, $\mathcal{A}$ is unital, then $\mathcal{A}^\infty$ is a continuous inverse algebra.

Proof. In view of Theorem 4.4, the action of G on $\mathcal{A}^\infty$ is smooth, and it is a Fréchet space by Proposition 5.4. It therefore remains to verify that $\mathcal{A}^\infty$ is a topological algebra, i.e., a subalgebra on which the product and the involution are continuous.

If $a, b \in \mathcal{A}^\infty$, then the orbit map of ab is given by $\alpha_g(ab) = \alpha_g(a)\alpha_g(b)$, and since the multiplication in $\mathcal{A}$ is continuous bilinear, hence smooth, $ab \in \mathcal{A}^\infty$. We also derive from the smoothness of the inversion in $\mathcal{A}^\times$ that if $a \in \mathcal{A}^\times \cap \mathcal{A}^\infty$, then also $a^{-1} \in \mathcal{A}^\infty$.

The continuity of the multiplication in $\mathcal{A}^\infty$ follows from the fact that the operators $\mathtt{d}\pi(x) : \mathcal{A}^\infty \to \mathcal{A}^\infty$ are derivations. If we write for a subset

$$S = \{i_1 < i_2 < \cdots < i_k\} \subseteq \{1, \ldots, n\},$$
$$\mathtt{d}\pi(x_S) := \mathtt{d}\pi(x_{i_1}) \cdots \mathtt{d}\pi(x_{i_k}),$$

then

$$\mathtt{d}\pi(x_1) \cdots \mathtt{d}\pi(x_n)(ab) = \sum_S \mathtt{d}\pi(x_S)(a)\mathtt{d}\pi(x_{S^c})(b). \tag{9}$$

In view of the submultiplicativity of the norm, this leads to

$$\left\|\mathtt{d}\pi(x_1) \cdots \mathtt{d}\pi(x_n)(ab)\right\| \leqslant \sum_S \left\|\mathtt{d}\pi(x_S)(a)\right\| \cdot \left\|\mathtt{d}\pi(x_{S^c})(b)\right\|,$$

so that the seminorm $p(a) := \|a\|$ satisfies $p_n(ab) \leqslant \sum_S p_{|S|}(a)p_{|S^c|}(b)$. Since the sequence $(p_n)_{n\in\mathbb{N}}$ of seminorms defines the topology on $\mathcal{A}^\infty$, it follows that the multiplication in $\mathcal{A}$ is continuous.

The continuity of the involution on $\mathcal{A}^\infty$ follows from the fact that the operators $\mathtt{d}\pi(x)$ commute with $*$:

$$\mathtt{d}\pi(x_1) \cdots \mathtt{d}\pi(x_n)\big(a^*\big) = \mathtt{d}\pi(x_1) \cdots \mathtt{d}\pi(x_n)(a)^*,$$

which leads to $p_n(a^*) = p_n(a)$ for $a \in \mathcal{A}^\infty$.

Since the inclusion $\mathcal{A}^\infty \hookrightarrow \mathcal{A}$ is continuous, the unit group

$$\big(\mathcal{A}^\infty\big)^\times = \mathcal{A}^\infty \cap \mathcal{A}^\times$$

is open. To prove the continuity of the inversion, we show by induction that it is continuous with respect to the seminorms p_n, $n \leqslant N$. For $N = 0$, the assertion follows from the continuity of the inversion on $\mathcal{A}^\times$. For $N > 0$, we apply (9) to $b = a^{-1}$ to obtain

$$0 = \sum_S \mathtt{d}\pi(x_S)(a)\mathtt{d}\pi(x_{S^c})\big(a^{-1}\big),$$

where the sum is extended over all subsets of $\{1, \ldots, N\}$. This leads to

$$\mathtt{d}\pi(x_1) \cdots \mathtt{d}\pi(x_n)\big(a^{-1}\big) = -a^{-1} \sum_{S \neq \emptyset} \mathtt{d}\pi(x_S)(a)\mathtt{d}\pi(x_{S^c})\big(a^{-1}\big).$$

That this sum depends, as an element of $\mathrm{Mult}^N(\mathfrak{g}, \mathcal{A})$, continuously on a follows by induction because $|S^c| < N$ whenever $S \neq \emptyset$. □

7. Smooth vectors for unitary representations

In this section we provide a remarkably effective criterion for the smoothness of vectors in unitary representations of infinite dimensional Lie groups, namely that $v \in \mathcal{H}^\infty$ if and only if the matrix coefficient $\pi^{v,v}(g) = \langle \pi(g)v, v\rangle$ is smooth in an identity neighborhood. This will be a simple consequence of the following observation.[3]

Theorem 7.1. *Let M be a smooth manifold, $\mathcal{H}$ a Hilbert space and $\gamma : M \to \mathcal{H}$ be a map. Then γ is a smooth map if and only if the kernel $K(x, y) := \langle \gamma(x), \gamma(y)\rangle$ is a smooth function on $M \times M$.*

Proof. Since the smoothness only refers to the real structure on $\mathcal{H}$, we may assume that $\mathcal{H}$ is a real Hilbert space. If γ is smooth, then K is also smooth because the scalar product on $\mathcal{H}$ is real bilinear and continuous, hence smooth.

Step 1: γ is continuous: This follows from

$$\|\gamma(x) - \gamma(y)\|^2 = K(x, x) + K(y, y) - 2K(x, y).$$

Step 2: Now we consider the case where $M \subseteq \mathbb{R}$ is an open interval, so that $\gamma : M \to \mathcal{H}$ is a curve in $\mathcal{H}$. From Step 1 we know that γ is continuous. For a fixed $t \in M$ we now consider on $M - t$ the function

$$f(h) := \|\gamma(t + h) - \gamma(t)\|^2 = K(t + h, t + h) + K(t, t) - 2K(t + h, t).$$

It is smooth with $f(0) = 0$ and

$$f'(0) = (\partial_1 K)(t, t) + (\partial_2 K)(t, t) - 2(\partial_1 K)(t, t) = 0$$

because K is symmetric. This implies that

$$\lim_{h\to 0} \frac{2}{h^2} f(h) = f''(0)$$

exists, and from the Chain Rule and the symmetry of K we obtain

$$\begin{aligned} f''(0) &= \left(\partial_1^2 K\right)(t, t) + \left(\partial_2^2 K\right)(t, t) + 2(\partial_1\partial_2 K)(t, t) - 2\left(\partial_1^2 K\right)(t, t) \\ &= 2(\partial_1\partial_2 K)(t, t). \end{aligned}$$

We conclude that

$$\lim_{h\to 0} \left\| \frac{1}{h}\left(\gamma(t + h) - \gamma(t)\right)\right\|^2 = \lim_{h\to 0} \frac{1}{h^2} f(h) = \frac{1}{2} f''(0) = (\partial_1\partial_2 K)(t, t) \tag{10}$$

[3] A finer analysis of the situation shows that if K has continuous k-fold derivatives in both variables separately in some neighborhood of the diagonal, then γ is a C^k-map, $k \in \mathbb{N} \cup \{\infty\}$ (cf. [28, p. 78] for the case where M is a real interval).

exists. We also note that, for each $s \in M$,

$$\frac{d}{dt}\langle \gamma(t), \gamma(s)\rangle = (\partial_1 K)(t, s)$$

exists. Since $\frac{1}{h}(\gamma(t+h) - \gamma(t))$ is bounded by (10), it follows that $\frac{d}{dt}\langle \gamma(t), v\rangle$ exists for each v in the closed subspace generated by $\gamma(M)$. For $v \in \gamma(M)^\perp$, the expression vanishes anyway, so that $\gamma'(t)$ exists weakly and satisfies

$$\langle \gamma'(t), \gamma(s)\rangle = (\partial_1 K)(t, s).$$

In particular, we have

$$\begin{aligned} \|\gamma'(t)\|^2 &= \lim_{h\to 0}\left\langle \gamma'(t), \frac{\gamma(t+h) - \gamma(t)}{h}\right\rangle = \lim_{h\to 0}\frac{1}{h}\big((\partial_1 K)(t, t+h) - (\partial_1 K)(t, t)\big) \\ &= (\partial_1\partial_2 K)(t, t) = \lim_{h\to 0}\left\|\frac{1}{h}\big(\gamma(t+h) - \gamma(t)\big)\right\|^2, \end{aligned}$$

and this implies that

$$\lim_{h\to 0}\frac{1}{h}\big(\gamma(t+h) - \gamma(t)\big) = \gamma'(t)$$

holds in the norm topology of $\mathcal{H}$. We conclude that γ is a C^1-curve.

Next we observe that

$$\langle \gamma'(t), \gamma'(s)\rangle = \partial_1\partial_2\langle \gamma(t), \gamma(s)\rangle = (\partial_1\partial_2 K)(t, s)$$

is also a smooth kernel. Therefore the argument above implies that γ' is C^1, so that γ is C^2. Repeating this argument shows that γ is C^k for every $k \in \mathbb{N}$, hence smooth.

Step 3: Now we consider a general locally convex manifold M. As the assertion of the proposition is local, we may assume that M is an open subset of a locally convex space. From Step 2 we derive that the map γ has directional derivatives in all directions, which leads to a "tangent map"

$$\mathtt{d}\gamma : TM \to \mathcal{H}, \quad (\mathtt{d}\gamma)(x, v) := \frac{d}{dt}\Big|_{t=0}\gamma(x + tv).$$

Since the kernel

$$\begin{aligned} \langle (\mathtt{d}\gamma)(x, v), (\mathtt{d}\gamma)(y, w)\rangle &= \left\langle \frac{d}{dt}\Big|_{t=0}\gamma(x + tv), \frac{d}{ds}\Big|_{s=0}\gamma(y + sw)\right\rangle \\ &= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\langle \gamma(x + tv), \gamma(y + sw)\rangle \\ &= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}K(x + tv, y + sw) = (\partial_{(v,0)}\partial_{(0,w)}K)(x, y), \end{aligned}$$

is smooth, Step 1 shows that $\mathrm{d}\gamma$ is continuous, so that γ is a C^1-map. Applying the same argument to $\mathrm{d}\gamma$ instead of γ, it follows that $\mathrm{d}\gamma$ is C^1 and hence that γ is C^2. Iterating this argument implies that γ is smooth. □

The following theorem substantially sharpens the well-known criterion (for finite dimensional groups) that $v \in \mathcal{H}^\infty$ if its orbit map π^v is weakly smooth, i.e., all matrix coefficients $\pi^{v,w}(g) = \langle \pi(g)v, w\rangle$ are smooth (cf. [63, Cor. 10.1.3]). In [59, p. 278] one finds a remark suggesting its validity for finite dimensional groups, which is proved in [37, Prop. X.6.4] by using Goodman's characterization of smooth vectors [14].

Theorem 7.2. *If $(\pi, \mathcal{H})$ is a unitary representation of a Lie group G, then $v \in \mathcal{H}$ is a smooth vector if and only if the corresponding matrix coefficient $\pi^{v,v}(g) := \langle \pi(g)v, v\rangle$ is smooth on a $\mathbf{1}$-neighborhood in G.*

Proof. Clearly, $\pi^{v,v}$ is smooth if v is a smooth vector. Suppose, conversely, that $\pi^{v,v}$ is smooth in a $\mathbf{1}$-neighborhood $U \subseteq G$ and let U' be a $\mathbf{1}$-neighborhood with $h^{-1}g \in U$ for $g, h \in U'$. In view of Theorem 7.1, the smoothness of π^v on xU', $x \in G$, is equivalent to the smoothness of the function

$$(g, h) \mapsto \langle \pi(xg)v, \pi(xh)v\rangle = \pi^{v,v}\left(h^{-1}g\right)$$

on $U' \times U'$, which follows from the smoothness of $\pi^{v,v}$ on U. □

Corollary 7.3. *If the continuous unitary representation $(\pi, \mathcal{H})$ of the Lie group G has a cyclic vector v for which the function $\pi^{v,v}(g) := \langle \pi(g)v, v\rangle$ is smooth on some identity neighborhood, then the representation $(\pi, \mathcal{H})$ is smooth, i.e., $\mathcal{H}^\infty$ is dense.*

Proof. The preceding theorem implies that v is a smooth vector, and therefore $\operatorname{span} \pi(G)v$ consists of smooth vectors. Hence $\mathcal{H}^\infty$ is dense. □

Corollary 7.4. *If φ is a positive definite function on a Lie group G which is smooth in a $\mathbf{1}$-neighborhood, then φ is smooth.*

Proof. Since φ is positive definite, the GNS construction provides a unitary representation $(\pi, \mathcal{H})$ of G and a vector $v \in \mathcal{H}$ with $\varphi = \pi^{v,v}$. Now Theorem 7.2 implies that $v \in \mathcal{H}^\infty$, but this implies that φ is smooth on all of G. □

8. C^1-vectors for Banach representations

In this section we consider a continuous representation (π, V) of the Banach–Lie group G on the Banach space V, which implies in particular that π is locally bounded (Lemma 5.2). Since Banach–Lie groups are locally exponential, Lemma 3.3 implies $v \in V$ is a C^1-vector if and only if $v \in \mathcal{D}_\mathfrak{g}$ and the map $\omega_v : \mathfrak{g} \to V$, $x \mapsto \overline{\mathrm{d}\pi}(x)$ is continuous, which implies in particular that it is linear. The goal of this section is to see that the space $\mathcal{D}_\mathfrak{g}$ coincides with the space of C^1-vectors for the action of G on V (Theorem 8.5). In view of what we know already, the main point is that, for every $v \in \mathcal{D}_\mathfrak{g}$, the map ω_v is continuous. Surprisingly (for us), the most difficult part in our argument is to see that the maps $\omega_v : \mathfrak{g} \to V$, $x \mapsto \overline{\mathrm{d}\pi}(x)v$, are additive for each $v \in \mathcal{D}_\mathfrak{g}$.

Lemma 8.1. *Let $F : [0, \infty[\to \mathcal{L}(V)$ be a curve with $F(0) = \mathbf{1}$ and $M > 0$ with*

$$\|F(t)\| \leqslant Me^{at} \quad \text{for } t \geqslant 0 \quad \text{and} \quad \|F(t)^n\| \leqslant M \quad \text{for } tn \leqslant 1.$$

Then there exists an $\omega \in \mathbb{R}$ with

$$\|F(t)^n\| \leqslant Me^{\omega nt} \quad \text{for } t \geqslant 0,\ n \in \mathbb{N}.$$

Proof. We put $\omega := 2\max(a, \log M)$ and observe that $M \geqslant \|F(0)\| = 1$ implies $\omega \geqslant 0$. The assertion clearly holds for $t = 0$, so that we may w.l.o.g. assume that $t > 0$.

First we discuss the case where $t \leqslant 1$. Then we define $k \in \mathbb{N}$ by $\frac{1}{2^k} < t \leqslant \frac{1}{2^{k-1}}$ and write n as $n = q2^{k-1} + r$ with $q, r \in \mathbb{N}_0$ and $r < 2^{k-1}$. Now $rt < 2^{k-1}t \leqslant 1$ leads to

$$\|F(t)^n\| \leqslant \|F(t)^{2^{k-1}}\|^q \|F(t)^r\| \leqslant M^q \cdot M.$$

On the other hand, $q \leqslant \frac{n}{2^{k-1}} < 2nt$, so that

$$\|F(t)^n\| \leqslant Me^{q\log M} \leqslant Me^{nt2\log M} \leqslant Me^{nt\omega}.$$

Now we consider the case where $t \geqslant 1$ and observe that

$$\|F(t)^n\| \leqslant M^n e^{nat} = e^{n\log M + nat} \leqslant e^{nt\log M + nta} = e^{nt(\log M + a)} \leqslant e^{nt\omega} \leqslant Me^{nt\omega}. \qquad \square$$

Theorem 8.2. *Let (π, V) be a representation of the Banach–Lie group G on the Banach space V for which the corresponding action is continuous and $v \in \mathcal{D}_{\mathfrak{g}}$. Then the map*

$$\omega_v : \mathfrak{g} \to V, \quad v \mapsto \overline{\mathrm{d}\pi}(x)v$$

is linear.

Proof. Step 1: Since G is a Banach–Lie group, the Trotter Product Formula

$$\lim_{n\to\infty}\big(\exp_G(tx/n)\exp_G(ty/n)\big)^n = \exp_G\big(t(x+y)\big)$$

holds uniformly for $|t| \leqslant N$ and any $N \in \mathbb{N}$.

Below we need a slight refinement which can be obtained as follows. First we observe that in a Banach–Lie algebra, we have for each real sequence $\alpha_n \to s > 0$ and the Baker–Campbell–Hausdorff multiplication $*$ the relation

$$n\left(\frac{\alpha_n}{n}x * \frac{\alpha_n}{n}y\right) = \alpha_n \frac{n}{\alpha_n}\left(\frac{\alpha_n}{n}x * \frac{\alpha_n}{n}y\right) \to s(x+y),$$

so that applying the exponential function on both sides leads to

$$\big(\exp_G(\alpha_n x/n)\exp_G(\alpha_n y/n)\big)^n \to \exp_G\big(s(x+y)\big). \tag{11}$$

Step 2: For $x, y \in \mathfrak{g}$, we consider the strongly continuous curve

$$F : \mathbb{R} \to \mathrm{GL}(V), \quad F(t) := \pi(\exp_G tx)\pi(\exp_G ty).$$

Since π is locally bounded, it is bounded in a neighborhood of $\exp_G([0,1](x+y))$, so that we find with Step 1 a $M_0 > 0$ with

$$\left\| F(t/n)^n \right\| \leqslant M_0 \quad \text{for } 0 \leqslant t \leqslant 1.$$

The strongly continuous one-parameter semigroups $\pi(\exp_G(tx))$ and $\pi(\exp_G(ty))$ satisfy

$$\left\|\pi\left(\exp_G(tx)\right)\right\| \leqslant M_1 e^{t\omega_1}, \qquad \left\|\pi\left(\exp_G(ty)\right)\right\| \leqslant M_2 e^{t\omega_2}$$

for suitable $M_1, M_2 > 0$, $\omega_1, \omega_2 \in \mathbb{R}$ and all $t \geqslant 0$ [55, Thm. 2.2]. This leads to the estimate

$$\left\|F(t)\right\| \leqslant M_1 M_2 e^{t(\omega_1+\omega_2)}.$$

Applying Lemma 8.1, we now find a constant $M > 0$ and $\omega \in \mathbb{R}$ with

$$\left\|F(t)^n\right\| \leqslant M e^{\omega n t} \quad \text{for } t \geqslant 0,\ n \in \mathbb{N}. \tag{12}$$

Step 3: From $\pi(g)\mathcal{D}_x = \mathcal{D}_{\mathrm{Ad}(g)x}$ for $g \in G$ and $x \in \mathfrak{g}$ it follows that $\mathcal{D}_{\mathfrak{g}}$ is a G-invariant subspace of V. Since we are only interested in elements of $\mathcal{D}_{\mathfrak{g}}$, we may w.l.o.g. assume that $\mathcal{D}_{\mathfrak{g}}$ is dense in V. From the definition it immediately follows that $\overline{\mathrm{d}\pi}(\lambda x)v = \lambda\overline{\mathrm{d}\pi}(x)v$ for $\lambda \in \mathbb{R}$.

For the operators $C := \overline{\mathrm{d}\pi}(x)$ and $D := \overline{\mathrm{d}\pi}(y)$ we then find that $\mathcal{D}_{\mathfrak{g}} \subseteq \mathcal{D}(C) \cap \mathcal{D}(D)$ is dense in V. Further, (12) is the estimate required in [52, Cor. 5.2(ii)], which asserts that the unbounded operator $C + D$, defined on $\mathcal{D}(C) \cap \mathcal{D}(D)$, is closable and its closure generates a C_0-semigroup if and only if there exists a dense linear subspace $A \subseteq \mathcal{D}(C) \cap \mathcal{D}(D)$ such that for all $f \in A$ and $s \geqslant 0$ there exist relatively compact sequences $(f_n^s)_{n\in\mathbb{N}}$ in $\mathcal{D}(C) \cap \mathcal{D}(D)$ such that

(a) The set $\{(C+D)f_n^s\colon n \in \mathbb{N}\}$ is precompact for every $s \geqslant 0$.
(b) $\lim_{n\to\infty} \|F(t/n)^{[ns]} f - f_n^s\| = 0$.

We put $A := \mathcal{D}_{\mathfrak{g}}$, and for $f \in \mathcal{D}_{\mathfrak{g}}$, we put $f_n^s := \pi(\exp_G(ts(x+y)))f$. Then (a) is trivially satisfied. To verify (b), we note that for $\alpha_n := \frac{[ns]}{n}$ the relation $ns - 1 \leqslant [ns] \leqslant ns$ implies $\alpha_n = [ns]/n \to s$. Hence (11) leads to

$$F\left(\frac{\alpha_n t}{n}\right)^n f \to \pi\left(\exp_G\left(st(x+y)\right)\right)f \quad \text{for } t \in \mathbb{R},\ f \in \mathcal{D}_{\mathfrak{g}}.$$

We thus obtain

$$\lim_{n\to\infty} F(t/n)^{[ns]} f = \lim_{n\to\infty} F\left(\alpha_n t/[ns]\right)^{[ns]} f = \pi\left(\exp_G\left(st(x+y)\right)\right)f = f_n^s.$$

Now [52, Cor. 5.2] (see also the concluding Remark in [53]) applies and yields

$$\overline{\overline{\mathrm{d}\pi}(x) + \overline{\mathrm{d}\pi}(y)} = \overline{C+D} = \overline{\mathrm{d}\pi}(x+y).$$

In particular, we obtain $\omega_v(x+y) = \omega_v(x) + \omega_v(y)$ for $v \in \mathcal{D}_{\mathfrak{g}}$. $\square$

Remark 8.3. If $\pi(G)$ consists of isometries, there are more direct arguments for the additivity of ω_v. For the strongly continuous curve

$$F : \mathbb{R} \to \mathrm{GL}(V), \quad F(t) := \pi(\exp_G tx)\pi(\exp_G ty),$$

we obtain for each $v \in \mathcal{D}_{\mathfrak{g}}$ the relation

$$\frac{1}{t}\big(F(t)v - v\big) = \pi(\exp_G tx)\frac{1}{t}\big(\pi\big(\exp_G(ty)\big)v - v\big) + \frac{1}{t}\big(\pi\big(\exp_G(tx)\big)v - v\big)$$
$$\to \overline{\mathrm{d}\pi}(y)v + \overline{\mathrm{d}\pi}(x)v,$$

so that the unbounded operator $F'(0)$ is defined on $\mathcal{D}_{\mathfrak{g}}$, where it coincides with $\overline{\mathrm{d}\pi}(x) + \overline{\mathrm{d}\pi}(y)$.

On the other hand, the Trotter–Product Formula in G yields

$$\lim_{n\to\infty} F(t/n)^n = \lim_{n\to\infty} \pi\big(\big(\exp_G(x/t)\exp_G(y/t)\big)^n\big) = \pi\big(\exp_G\big(t(x+y)\big)\big)$$

in the strong operator topology. Now [6, Theorem 3.1] implies that $\overline{\mathrm{d}\pi}(x+y)$ extends the operator $F'(0)$ on $\mathcal{D}_{\mathfrak{g}}$. In particular, we obtain for $v \in \mathcal{D}_{\mathfrak{g}}$ that $\overline{\mathrm{d}\pi}(x+y)v = \overline{\mathrm{d}\pi}(x)v + \overline{\mathrm{d}\pi}(y)v$.

We now take a closer look at the continuity of the linear maps ω_v from Theorem 8.2.

Lemma 8.4. *Let (π, V) be a continuous representation of the Fréchet–Lie group G on the Fréchet space V. Then, for each $v \in \mathcal{D}_{\mathfrak{g}}$ for which ω_v is linear, it is continuous.*

Proof. (S. Merigon) Assume that ω_v is a linear map. In view of the Closed Graph Theorem [61, Thm. 2.15], it suffices to show that the graph of ω_v is closed. Suppose that $x_n \to x$ in $\mathfrak{g}$ such that $\omega_v(x_n) \to w$. Then we obtain from (2) the relation

$$\pi\big(\exp_G(tx_n)\big)v - v = t\int_0^1 \pi\big(\exp_G(stx_n)\big)\omega_v(x_n)\,ds.$$

Since the function

$$[0,1]^2 \times \mathfrak{g} \times V \to V, \quad (s,t,x,v) \mapsto \pi\big(\exp_G(stx)\big)v$$

is continuous, integration over $s \in [0,1]$ leads to a continuous function

$$[0,1] \times \mathfrak{g} \times V \to V, \quad (t,x,v) \mapsto \int_0^1 \pi\big(\exp_G(stx)\big)v\,ds.$$

We thus obtain in the limit $n \to \infty$:

$$\pi\big(\exp_G(tx)\big)v - v = t\int_0^1 \pi\big(\exp_G(stx)\big)w\,ds.$$

For the derivative in $t = 0$ this leads to

$$\omega_v(x) = \overline{\mathrm{d}\pi}(x)v = \int_0^1 \pi\big(\exp_G(0x)\big)w\,ds = \int_0^1 w\,ds = w. \qquad \square$$

For continuous actions of Banach–Lie groups on Banach spaces we thus obtain with Theorem 8.2 and Lemmas 8.4 and 3.3.

Theorem 8.5. *Let (π, V) be a representation of the Banach–Lie group G on the Banach space V for which the corresponding action is continuous. Then $\mathcal{D}_{\mathfrak{g}}$ coincides with the space of C^1-vectors.*

9. C^k-vectors for Banach representations

We continue our discussion of differentiable vectors for a continuous representation (π, V) of a Banach–Lie group G on a Banach space V. We know already that for each $v \in \mathcal{D}_{\mathfrak{g}}$ the map $\omega_v : \mathfrak{g} \to V$ is continuous and linear (Theorems 8.2, Lemma 8.4). We thus obtain a norm on $\mathcal{D}_{\mathfrak{g}}$ by

$$\|v\|_1 := \|v\| + \|\omega_v\| = \|v\| + \sup_{\|x\|\leqslant 1} \big\|\omega_v(x)\big\|$$

(cf. [15] and [23] for similar constructions for finite dimensional Lie algebras). With respect to this norm on $\mathcal{D}_{\mathfrak{g}}$, the bilinear map

$$\mathfrak{g} \times \mathcal{D}_{\mathfrak{g}} \to V, \quad (x, v) \mapsto \omega_v(x) = \overline{\mathrm{d}\pi}(x)v$$

satisfies $\|\omega_v(x)\| \leqslant \|\omega_v\|\|x\| \leqslant \|v\|_1\|x\|$, so that it is continuous.

For the sake of completeness, we recall the following variant of [41, Lemmas A.1/2]:

Lemma 9.1. *Let $X_1, \ldots, X_n$, Y and Z be Banach spaces and $\eta : Y \to Z$ a continuous injection. Suppose that $A : X_1 \times \cdots \times X_n \to Z$ is a continuous n-linear map with* $\operatorname{im}(A) \subseteq \operatorname{im}(\eta)$. *Then the induced n-linear map*

$$\widetilde{A} : X_1 \times \cdots \times X_n \to Y \quad \textit{with } \eta \circ \widetilde{A} = A$$

is continuous.

Proof. We argue by induction on n. First we consider the case $n = 1$. By the Closed Graph Theorem, it suffices to show that the graph of $\widetilde{A} : X_1 \to Y$ is closed. Assume that

$$(x_n, \widetilde{A}x_n) \to (x, y) \in X \times Y.$$

Then $\eta(\widetilde{A}x_n) = Ax_n \to Ax$ implies that $\eta(y) = Ax = \eta(\widetilde{A}x)$, and therefore $\widetilde{A}x = y$. Thus $\widetilde{A}$ is continuous.

Now we consider the general case $n > 1$. From the case $n = 1$, we know that for fixed elements $x_j \in X_j$, $j < n$, the map

$$X_n \to Y, \quad x \mapsto \widetilde{A}(x_1, \cdots, x_{n-1}, x)$$

is continuous. By the induction hypothesis, applied to the continuous inclusion

$$\mathcal{L}(X_n, Y) \to \mathcal{L}(X_n, Z), \quad B \mapsto \eta \circ B,$$

the $(n-1)$-linear map

$$\widetilde{C} : X_1 \times \cdots \times X_{n-1} \to \mathcal{L}(X_n, Y), \quad (x_1, \ldots, x_{n-1}) \mapsto \widetilde{A}(x_1, \ldots, x_{n-1}, \cdot)$$

is continuous because the corresponding map

$$C : X_1 \times \cdots \times X_{n-1} \to \mathcal{L}(X_n, Z), \quad (x_1, \ldots, x_{n-1}) \mapsto A(x_1, \ldots, x_{n-1}, \cdot)$$

is continuous. Since the bilinear evaluation map

$$\mathcal{L}(X_n, Z) \times X_n \to Z, \quad (\varphi, x) \mapsto \varphi(x)$$

is continuous, it follows that $\widetilde{A}$ is a continuous n-linear map. □

Lemma 9.2. *$\mathcal{D}_{\mathfrak{g}}$ is complete with respect to $\|\cdot\|_1$.*

Proof. The map $\iota : \mathcal{D}_{\mathfrak{g}} \to V \times \mathcal{L}(\mathfrak{g}, V)$, $v \mapsto (v, \omega(v))$ is isometric with respect to $\|\cdot\|_1$ on $\mathcal{D}_{\mathfrak{g}}$ and the norm $\|(v, \alpha)\| := \|v\| + \|\alpha\|$ on the product Banach space $V \times \mathcal{L}(\mathfrak{g}, V)$. Therefore the completeness of $\mathcal{D}_{\mathfrak{g}}$ is equivalent to the closedness of the graph $\Gamma(\omega) = \mathrm{im}(\iota)$ of the linear map $\omega : \mathcal{D}_{\mathfrak{g}} \to \mathcal{L}(\mathfrak{g}, V)$, $v \mapsto \omega_v$. We now show that $\Gamma(\omega)$ is closed.

Assume that $(v_n, \omega_{v_n}) \to (v, \alpha)$. Then (2) yields for each $x \in \mathfrak{g}$ the relation

$$\pi\big(\exp_G(tx)\big)v_n - v_n = t \int_0^1 \pi\big(\exp_G(stx)\big)\omega_{v_n}(x)\, ds.$$

Passing to the limit on both sides yields

$$\pi(\exp_G(tx)v - v = t \int_0^1 \pi\big(\exp_G(stx)\big)\alpha(x)\, ds.$$

This implies that $v \in \mathcal{D}_x$ with $\overline{\mathrm{d}\pi}(x)v = \alpha(x)$. We conclude that $v \in \mathcal{D}_{\mathfrak{g}}$ with $\alpha = \omega_v$, and hence that $\Gamma(\omega)$ is closed. □

Lemma 9.3. *For each $v \in \mathcal{D}^n_{\mathfrak{g}}$, the corresponding n-linear map*

$$\omega^n_v : \mathfrak{g}^n \to V, \quad (x_1, \ldots, x_n) \mapsto \overline{\mathrm{d}\pi}(x_1)\overline{\mathrm{d}\pi}(x_2)\cdots\overline{\mathrm{d}\pi}(x_n)v$$

is continuous.

Proof. First we observe that Theorem 8.2 implies that ω_v^n is n-linear. We argue by induction on n to show that it is continuous. For $n = 1$ this follows from Lemma 8.4. Now we assume $n > 1$ and that $\omega_v^{n-1} : \mathfrak{g}^{n-1} \to V$ is a continuous $(n-1)$-linear map. From $v \in \mathcal{D}_\mathfrak{g}^n$ we derive that $\operatorname{im}(\omega_v^{n-1}) \subseteq \mathcal{D}_\mathfrak{g}$. Since $\mathcal{D}_\mathfrak{g}$ is a Banach space with a continuous injection into V (Lemma 9.2), Lemma 9.1 implies that the $(n-1)$-linear map $\omega_v^{n-1} : \mathfrak{g}^{n-1} \to \mathcal{D}_\mathfrak{g}$ is continuous. Since the bilinear evaluation map $\mathfrak{g} \times \mathcal{D}_\mathfrak{g} \to V$, $(x, w) \mapsto \omega_w(x)$ is continuous, we see that

$$\omega_v^n(x_1, \ldots, x_n) = \omega_{\omega_v^{n-1}(x_2,\ldots,x_n)}(x_1)$$

is a continuous n-linear map. □

Theorem 9.4. *For a continuous representation of the Banach–Lie group G on the Banach space V, the space $\mathcal{D}_\mathfrak{g}^k$ coincides with the space V^k of C^k-vectors. This space is complete with respect to the norm*

$$\|v\|_k := \|v\| + \sum_{j=1}^{k} \|\omega_v^j\|, \tag{13}$$

hence a Banach space, and the bilinear map

$$\xi : \mathfrak{g} \times \mathcal{D}_\mathfrak{g}^{k+1} \to \mathcal{D}_\mathfrak{g}^k, \quad (x, v) \mapsto \overline{\mathtt{d}\pi}(x)v$$

is continuous with respect to the norms $\|\cdot\|_j$ on $\mathcal{D}_\mathfrak{g}^j$, $j = k, k+1$.

Proof. Combining the preceding lemma with the general Lemma 3.4, it follows that $V^k = \mathcal{D}_\mathfrak{g}^k$. To verify the completeness of $\mathcal{D}_\mathfrak{g}^k$, we argue by induction. For $k = 1$, this is Lemma 9.2. Assume that $k > 1$. We have to show that the graph of the linear map

$$\omega : \mathcal{D}_\mathfrak{g}^k \to \prod_{j=1}^{k} \operatorname{Mult}^j(\mathfrak{g}, V), \quad v \mapsto (\omega_v^1, \ldots, \omega_v^k)$$

is closed in $V \times \prod_{j=1}^k \operatorname{Mult}^j(\mathfrak{g}, V)$. Suppose that

$$(v_n, \omega_{v_n}^1, \ldots, \omega_{v_n}^k) \to (v, \alpha_1, \ldots, \alpha_k) \in V \times \prod_{j=1}^{k} \operatorname{Mult}^j(\mathfrak{g}, V).$$

Our induction hypothesis implies that $v \in \mathcal{D}_\mathfrak{g}^{k-1}$ with $\alpha_j = \omega_v^j$ for $j \leqslant k-1$.

For $x, x_2, \ldots, x_k \in \mathfrak{g}$ we further have

$$\overline{\mathtt{d}\pi(x)}\omega_{v_n}^{k-1}(x_2, \ldots, x_k) = \omega_{v_n}^k(x, x_2, \ldots, x_k) \to \alpha(x, x_2, \ldots, x_k)$$

and $\omega_{v_n}^{k-1}(x_2, \ldots, x_k) \to \omega_v^{k-1}(x_2, \ldots, x_k)$. Since the graph of $\overline{\mathtt{d}\pi}(x)$ is closed (apply Lemma 9.2 with $\mathfrak{g} = \mathbb{R}$), we obtain

$$\omega_v^{k-1}(x_2, \ldots, x_k) \in \mathcal{D}_x \quad \text{and} \quad \overline{\mathtt{d}\pi}(x)\omega_v^{k-1}(x_2, \ldots, x_k) = \alpha(x, x_2, \ldots, x_k).$$

This implies that $v \in \mathcal{D}_\mathfrak{g}^k$ with $\alpha = \omega_v^k$.

To show that the bilinear map ξ is continuous, we first note that, for $v \in \mathcal{D}_{\mathfrak{g}}^{k+1}$, we have for $j \leqslant k$ the estimate

$$\left\|\omega^j_{\xi(x,v)}\right\| = \left\|\omega^j_{\overline{\mathrm{d}\pi(x)v}}\right\| \leqslant \left\|\omega^{j+1}_v\right\| \|x\|,$$

and this implies that

$$\left\|\xi(x,v)\right\|_k = \sum_{j=0}^{k} \left\|\omega^j_{\xi(x,v)}\right\| \leqslant \sum_{j=1}^{k+1} \left\|\omega^j_v\right\| \|x\| \leqslant \|v\|_{k+1} \|x\|.$$

Therefore ξ is continuous. □

Remark 9.5. Note that we have for each n an isometric embedding

$$\mathcal{D}^n_{\mathfrak{g}} \hookrightarrow \mathcal{D}^{n-1}_{\mathfrak{g}} \times \mathrm{Mult}^n(\mathfrak{g}, V),$$

where the norm on the product space is $\|(v,\alpha)\| := \|v\|_{n-1} + \|\alpha\|$.

Remark 9.6. The preceding discussion leads in particular to a Fréchet structure on $\mathcal{D}^\infty_{\mathfrak{g}}$, considered as a subspace of $\prod_{n=0}^\infty \mathrm{Mult}^n(\mathfrak{g}, V)$. It coincides with the one from Definition 4.1, so that we obtain a second proof of Proposition 5.4.

Now that we know that each space $\mathcal{D}^k_{\mathfrak{g}}$ is complete, it is natural to ask for the extent to which the G-action on this space is continuous.

Proposition 9.7. *The representation π^k of G on the Banach space V^k of C^k-vectors has the following properties*:

(i) *π^k is locally bounded.*
(ii) *For an element $v \in V^k$, the following are equivalent*:
  (a) *The maps $G \to \mathrm{Mult}^j(\mathfrak{g}, V)$, $g \mapsto \pi(g) \circ \omega^j_v$ are continuous for $j \leqslant k$.*
  (b) *The maps $G \to \mathrm{Mult}^j(\mathfrak{g}, V)$, $g \mapsto \pi(g) \circ \omega^j_v \circ (\mathrm{Ad}(g)^{-1})^{\times j}$ are continuous for $j \leqslant k$.*
  (c) *$v \in (V^k)^0$, i.e., the orbit map $G \to V^k$, $g \mapsto \pi(g)v$ is continuous.*
  (d) *The orbit map $\pi^v : G \to V$ is Fréchet-C^k, i.e., $v \in FV^k$.*
(iii) *$V^{k+1} \subseteq FV^k$.*
(iv) *The subspace FV^k of V^k is closed and the G-action on FV^k is continuous. It is the maximal G-invariant subspace of V^k for which this is the case. In particular, the action of G on V^k is continuous if and only if $V^k = FV^k$.*

Proof. (i) The group G acts naturally by continuous linear operators on $\mathrm{Mult}^n(\mathfrak{g}, V)$ via

$$g.\omega := \pi(g) \circ \omega \circ \left(\mathrm{Ad}\left(g^{-1}\right) \times \cdots \times \mathrm{Ad}\left(g^{-1}\right)\right) = \pi(g) \circ \omega \circ \left(\mathrm{Ad}\left(g^{-1}\right)^{\times n}\right.,$$

and we have

$$\|g.\omega\| \leqslant \left\|\mathrm{Ad}\left(g^{-1}\right)\right\|^n \left\|\pi(g)\right\| \|\omega\|,$$

i.e., the corresponding representation is locally bounded. Since the topological embedding

$$\omega : V^k \to V \times \prod_{j=1}^{k} \operatorname{Mult}^j(\mathfrak{g}, V), \quad v \mapsto \left(v, \omega_v^1, \ldots, \omega_v^k\right) \tag{14}$$

is G-equivariant, the local boundedness of π^k follows.

(ii) The equivalence of (a) and (c) follows from the embedding (14).

Further, the equivalence of (a) and (b) follows from the fact that the action of G on each space $\operatorname{Mult}^j(\mathfrak{g}, V)$ by $g * \omega := \omega \circ (\operatorname{Ad}(g)^{-1})^{\times j}$ defines a morphism of Banach–Lie groups $G \to \operatorname{GL}(\operatorname{Mult}^j(\mathfrak{g}, V))$ (cf. [42, Exer. IV.6]).

For $v \in \mathcal{D}_\mathfrak{g}$, the orbit map $\pi^v : G \to V$ is C^1 with

$$\partial_{g.x}\left(\pi^v\right)(g) := T_g\left(\pi^v\right)(g.x) = \pi(g)\overline{\mathrm{d}\pi}(x)v = \pi(g)\omega_v(x),$$

and, by iterating this argument, we obtain for $v \in V^k$ and $j \leqslant k$:

$$\partial_{g.x_1} \cdots \partial_{g.x_j}\left(\pi^v\right)(g) = \pi(g)\omega_v^j(x_1, \ldots, x_j).$$

With similar arguments as in the proof of [38, Thm. I.7], we now see that the C^k-map π^v is C^k in the Fréchet sense if and only if, for $j \leqslant k$, the maps

$$G \to \operatorname{Mult}^j(\mathfrak{g}, V), \quad g \mapsto \left((x_1, \ldots, x_j) \mapsto \left(\partial_{g.x_1} \cdots \partial_{g.x_j}\pi^v\right)(g)\right)$$

are continuous. In view of the preceding calculations, this means that (a) is equivalent to (d).

(iii) Follows from the general fact that each C^{k+1}-map is C^k in the Fréchet sense (cf. [38, Thm. I.7(ii)], [13]).

(iv) Since the G-representation on $V^k \subseteq \prod_{j=0}^k \operatorname{Mult}^j(\mathfrak{g}, V)$ is locally bounded, the subspace of elements with continuous orbit maps is closed. As we have seen in (iii) above, this subspace coincides with FV^k. It is clearly invariant and Lemma 5.2 implies that G acts continuously on FV^k. □

Remark 9.8. If G is finite dimensional, then each C^k-map is also Fréchet-C^k, so that the action of G on $\mathcal{D}_\mathfrak{g}^k$ is again a continuous action (Remark 5.3). In sharp contrast to this situation is the fact that, for an infinite dimensional Banach–Lie group, $\mathcal{D}_\mathfrak{g}$ need not contain any non-zero continuous vector (cf. Remark 10.7 below).

Example 9.9. For the examples discussed in Section 10 below, it is shown that the action of G on $\mathcal{D}_\mathfrak{g}$ is continuous for $p \geqslant 4$, because in this case $\mathcal{D}_\mathfrak{g}^2$ contains the dense subspace $L^\infty([0, 1])$ (cf. Remarks 5.3 and 10.7).

Example 9.10. We consider the action of the one-dimensional Lie group $G = \mathbb{R}$ on the Banach space $V = C(\mathbb{R}, \mathbb{R})_{\mathrm{per}}$ of 1-periodic functions by $(\pi(g)f)(x) := f(g + x)$. Then the uniform continuity of each element of V implies that the orbit maps are continuous, and since G acts by isometries, Lemma 5.2 implies the continuity of the G-action on V.

Since the point evaluations are continuous, the space $\mathcal{D}^k_{\mathfrak{g}}$ of C^k-vectors consists of C^k-functions on $\mathbb{R}$. If, conversely, f is a C^1-function, then

$$\frac{f(x+h)-f(x)}{h} = \int_0^1 f'(x+sh)\,ds$$

converges uniformly to $f'(x)$ for $h \to 0$, so that f is a C^1-vector. A similar argument, based on the integral form of the remainder term in the Taylor formula implies that $\mathcal{D}^k_{\mathfrak{g}} = C^k(\mathbb{R},\mathbb{R})_{\mathrm{per}}$.

10. A family of interesting examples

We take a closer look at the unitary representation of the Banach–Lie group $G := (L^p([0,1],\mathbb{R}),+)$, $p \in [1,\infty[$, on the Hilbert space $\mathcal{H} = L^2([0,1],\mathbb{C})$ by $\pi(g)f := e^{ig}f$. In [3] it is shown that, for $p = 2$, this representation is continuous with $\mathcal{H}^\infty = \{0\}$. Here we shall see that it is always continuous and determine the space of C^k-vectors. In the following we write $\mathfrak{g} := L^p([0,1],\mathbb{R})$ for the Lie algebra of G and abbreviate $L^p([0,1]) := L^p([0,1],\mathbb{C})$.

We start with a general observation on the inclusions between L^p-spaces.

Remark 10.1. (a) If (X,μ) is a finite measure space, then

$$L^q(X,\mu) \subseteq L^p(X,\mu) \quad \text{for } 1 \leqslant p < q,$$

where the inclusion is continuous. In fact, for any measurable function $f : X \to \mathbb{C}$, we have

$$\int |f|^p = \int_{\{|f|\leqslant 1\}} |f|^p + \int_{\{|f|>1\}} |f|^p \leqslant \mu(X) + \int_{\{|f|>1\}} |f|^q.$$

This implies that $\|\cdot\|_p$ is bounded on the unit ball of $L^q(X,\mu)$.

(b) Assume that X has a decomposition into a sequence (X_n) of pairwise disjoint subsets with $0 < \mu(X_n) \leqslant 2^{-n}\mu(X)$. We claim that

$$L^q(X,\mu) \neq L^p(X,\mu) \quad \text{for } 1 \leqslant p < q.$$

We consider the function f whose value on X_n is constant $\mu(X_n)^{-1/q}$. Then $\|f\|_q^q = \sum_n \mu(X_n)^{-1}\mu(X_n) = \infty$ and

$$\|f\|_p^p = \sum_n \mu(X_n)^{1-p/q} \leqslant \mu(X)^{1-p/q} \sum_n \bigl(2^{p/q-1}\bigr)^n < \infty.$$

(c) If $X = [0,1]$ is the unit interval and μ is Lebesgue measure, then the property under (b) is satisfied for every subset $Y \subseteq X$ of positive measure.

Lemma 10.2. *The representation $(\pi,\mathcal{H})$ is continuous for every $p \geqslant 1$.*

Proof. (Cf. [3, Prop. 2.1] for $p = 2$.) Let (g_n) be a sequence in $G = L^p([0,1],\mathbb{R})$ converging to 0. Then there exists a subsequence $(g_{n_k})_{k\in\mathbb{N}}$ converging almost everywhere to 0. Then $|e^{ig_{n_k}} - 1|^2 \to 0$ almost everywhere, so that the Dominated Convergence Theorem implies that $\int_{[0,1]} |e^{ig_{n_k}} - 1|^2 \to 0$. For every $\xi \in L^\infty([0,1],\mathbb{C})$, we thus obtain

$$\|\pi(g_{n_k})\xi - \xi\|_2^2 = \int_{[0,1]} \left|e^{ig_{n_k}} - 1\right|^2 |\xi|^2 \leqslant \|\xi\|_\infty^2 \int_{[0,1]} \left|e^{ig_{n_k}} - 1\right|^2 \to 0.$$

We conclude that $\pi(g_{n_k}) \to \mathbf{1}$ with respect to the strong topology.

If π is not continuous, then there exists a sequence $h_n \to 0$ in G with $\pi(h_n) \not\to \pi(0)$. This means that there exist a $\mathbf{1}$-neighborhood U in $\mathrm{U}(\mathcal{H})$ with respect to the strong operator topology for which $\{n\colon \pi(h_n) \notin U\}$ is infinite. This leads to a sequence g_n in G with $g_n \to 0$ and $\pi(g_n) \notin U$ for every n, and we thus obtain a contradiction to the argument in the preceding paragraph. We conclude that π is continuous. □

Since the infinitesimal generator of the one-parameter group $t \mapsto \pi(tf)$ is the multiplication with the function f, it follows that

$$\mathcal{D}_\mathfrak{g} = \mathcal{D}_\mathfrak{g}^1 = \left\{\xi \in \mathcal{H}\colon \left(\forall f \in L^p\left([0,1],\mathbb{R}\right)\right)\ f\xi \in \mathcal{H}\right\}.$$

Lemma 10.3. *We have*

$$\mathcal{D}_\mathfrak{g} = \begin{cases} \{0\} & \text{for } p < 2, \\ L^\infty([0,1]) & \text{for } p = 2, \\ L^{\frac{2p}{p-2}}([0,1]) & \text{for } p > 2. \end{cases}$$

Proof. Let $\xi \in \mathcal{D}_\mathfrak{g}$. If $\xi \neq 0$, then there exists an $\varepsilon > 0$ for which $X_\varepsilon := \{|\xi| > \varepsilon\}$ has positive measure. Now $L^p([0,1]) \cdot \xi \subseteq L^2([0,1])$ implies that $L^p(X_\varepsilon) \subseteq L^2(X_\varepsilon)$, and in view of Remark 10.1(b/c), this leads to $p \geqslant 2$.

For $p = 2$, the multiplication with ξ defines a bounded operator on $L^2([0,1])$, so that $\xi \in L^\infty([0,1])$. This proves that $\mathcal{D}_\mathfrak{g} = L^\infty([0,1])$ in this case.

Now we assume that $p > 2$. The condition $\xi \cdot L^p([0,1]) \subseteq L^2([0,1])$ is equivalent to $\xi^2 \cdot L^{p/2}([0,1]) \subseteq L^1([0,1])$, which is equivalent to $\xi^2 \in L^q([0,1])$ for $\frac{1}{q} + \frac{2}{p} = 1$, i.e., $q = \frac{p}{p-2}$. This proves that $\mathcal{D}_\mathfrak{g} = L^{\frac{2p}{p-2}}([0,1])$. □

Next we note that $\operatorname{span}\{f^k\colon f \in \mathfrak{g}\} = \operatorname{span}\{f_1 \cdots f_k\colon f_1, \ldots, f_k \in \mathfrak{g}\}$ shows that

$$\mathcal{D}_\mathfrak{g}^k = \left\{\xi \in \mathcal{H}\colon \left(\forall f \in L^p\left([0,1],\mathbb{R}\right)\right)\ f^k\xi \in \mathcal{H}\right\}.$$

Now

$$\left\{f^k\colon f \in \mathfrak{g} = L^p\left([0,1]\right)\right\} = L^{p/k}\left([0,1]\right) \quad \text{for } p \geqslant k,$$

leads for $p \geqslant k$ to

$$\mathcal{D}_\mathfrak{g}^k = \left\{\xi \in \mathcal{H}\colon \left(\forall f \in L^{p/k}\left([0,1],\mathbb{R}\right)\right)\ f\xi \in \mathcal{H}\right\} = \mathcal{D}_{L^{p/k}}. \tag{15}$$

Proposition 10.4. *For $p \in \mathbb{N}$ and $k \leqslant p$ we have*

$$\mathcal{D}_{\mathfrak{g}}^k = \mathcal{D}_{L^{p/k}} = \begin{cases} \{0\} & \text{for } k > \frac{p}{2}, \\ L^\infty([0,1]) & \text{for } k = \frac{p}{2}, \\ L^{\frac{2p}{p-2k}}([0,1]) & \text{for } k < \frac{p}{2}. \end{cases}$$

For $p = 2$ we obtain in particular $\mathcal{D}_{\mathfrak{g}}^2 = \{0\}$ and $\mathcal{D}_{\mathfrak{g}}^1 = L^\infty([0,1])$ which refines the observations in [3].

Remark 10.5. The preceding proposition also shows that there exists for every $n \in \mathbb{N}$ a continuous unitary representation $(\pi, \mathcal{H})$ of a Lie group $G = (L^{2n}([0,1], \mathbb{R}), +)$ with $\mathcal{D}_{\mathfrak{g}}^{n+1} = \{0\}$ and $\mathcal{D}_{\mathfrak{g}}^n \neq \{0\}$.

Remark 10.6. It is easy to see that the norm $\|\cdot\|_k$ on the space $\mathcal{D}_{\mathfrak{g}}^k$ is equivalent to the natural norm suggested by Proposition 10.4.

For $p = 4$ and $\mathcal{D}_{\mathfrak{g}} = L^4([0,1])$, the density of $\mathcal{D}^2 = L^\infty([0,1])$ in $\mathcal{D}_{\mathfrak{g}}$ implies the continuity of the isometric action of G on $\mathcal{D}_{\mathfrak{g}}$ (cf. Remark 5.3).

Remark 10.7. For maps between Fréchet spaces, we also have the stronger notion of C^k-maps in the Fréchet sense. If $\xi \in \mathcal{H} = L^2([0,1], \mathbb{C})$ is a C^1-vector for $G = (L^p([0,1], \mathbb{R}), +)$, then the orbit map $\pi^\xi(g) = e^{ig}\xi$ has in g the differential

$$T_g(\pi^\xi)f = e^{ig} f\xi.$$

Therefore ξ is a Fréchet-C^1-vector if and only if the map

$$F : G \to \mathcal{L}(\mathfrak{g}, \mathcal{H}), \quad g \mapsto M_{e^{ig}\xi}, \quad M_h f = hf,$$

is continuous.

We consider the case $p = 2$. Then $\mathcal{L}(\mathfrak{g}, \mathcal{H}) \cong \mathcal{L}(\mathcal{H})$, so that we may consider F as a map

$$F : G \to L^\infty\big([0,1], \mathbb{C}\big), \quad g \mapsto e^{ig}\xi,$$

and the question is when this map is continuous.

First we consider the case $\xi = 1$. Then F is a homomorphism of Banach–Lie groups. If it is continuous, it is smooth, which implies that $\mathbf{L}(F) : \mathfrak{g} \to L^\infty([0,1], \mathbb{C})$, $g \mapsto ig$, is a continuous liner map, which is not the case.

For a general $\xi \in L^\infty([0,1], \mathbb{C})$, we consider for $\varepsilon > 0$ the subsets $X_\varepsilon := \{|\xi| \geqslant \varepsilon\}$. On each of these sets, $\xi|_{X_\varepsilon}$ is invertible in the Banach algebra $L^\infty([0,1], \mathbb{C})$, so that multiplication with ξ^{-1} leads to the situation of the previous paragraph. Therefore X_ε has measure zero, and since $\varepsilon > 0$ was arbitrary, $\xi = 0$. Hence all Fréchet-C^1-vectors for $G = L^2([0,1], \mathbb{R})$ are trivial. Another way to put this is to say that the action of G on the Banach space $\mathcal{D}_{\mathfrak{g}} = L^\infty([0,1], \mathbb{C})$ has no non-zero continuous vector.

On the other hand, we know that every C^2-vector is a Fréchet-C^1-vector [38, Thm. I.7], so that we obtain non-trivial Fréchet-C^1-vectors for $G = L^4([0,1], \mathbb{R})$.

11. Smooth vectors for direct limits

Definition 11.1. If $(G_n)_{n\in\mathbb{N}}$ is a sequence of finite dimensional Lie groups with homomorphisms $\varphi_n : G_n \to G_{n+1}$, then Glöckner has shown in [11] that the corresponding direct limit group $G := \varinjlim G_n$, endowed with the direct limit topology carries a compatible Lie group structure with Lie algebra $\mathfrak{g} := \varinjlim \mathbf{L}(G_n)$, endowed with the direct limit topology. We call G a *direct limit Lie group*.

Remark 11.2. The Lie groups obtained by this construction are precisely the Lie groups G with a smooth exponential function whose Lie algebra $\mathfrak{g}$ is a countable union of finite dimensional subalgebras, endowed with the direct limit topology. In fact, writing $\mathfrak{g} = \bigcup_{n\in\mathbb{N}} \mathfrak{g}_n$ with finite dimensional Lie algebras $\mathfrak{g}_n \subseteq \mathfrak{g}_{n+1}$, we obtain a corresponding sequence of Lie groups G_n injecting into G, and then it is not hard to verify that the corresponding morphism of Lie groups $\varinjlim G_n \to G$ is an isomorphism (cf. [13]).

Theorem 11.3. *For each continuous unitary representation $(\pi, \mathcal{H})$ of a direct limit Lie group G, the space of smooth vectors is dense.*

Proof. Since $\mathcal{H}$ is a direct sum of subspaces on which the representation if cyclic, we may w.l.o.g. assume that the representation is cyclic. Since smoothness of a vector in $\mathcal{H}$ is equivalent to smoothness for the identity component, we may also assume that G is connected. Then G is a countable direct limit of connected finite dimensional Lie groups, hence separable, and therefore the cyclicity of $(\pi, \mathcal{H})$ implies that $\mathcal{H}$ is separable.

Danilenko shows in [7] that there exists a dense subspace $\mathcal{D} \subseteq \mathcal{H}$ satisfying

(a) $\mathcal{D}$ is $\pi(G)$-invariant.
(b) $\mathcal{D} \subseteq \mathcal{D}_{\mathfrak{g}}$.
(c) $\overline{\mathrm{d}\pi(x)\mathcal{D}} \subseteq \mathcal{D}$ for every $x \in \mathfrak{g}$.

Since each group G_n is in particular a Banach–Lie group, conditions (b) and (c) imply that $\mathcal{D} \subseteq \mathcal{D}^\infty_{\mathfrak{g}_n}$, so that Lemma 3.4 implies that $\mathcal{D}$ consists of smooth vectors for each G_n. Since G is also the direct limit of the G_n in the category of smooth manifolds [11], $\mathcal{D} \subseteq \mathcal{H}^\infty$ consists of smooth vectors for G. □

Remark 11.4. Typical examples of Lie algebras $\mathfrak{g}$ which are locally finite in the sense that every finite subset generates a finite dimensional subalgebra are nilpotent Lie algebras. These Lie algebras are countable direct limits of finite dimensional ones if they are countably dimensional. This condition is very restrictive, so that one is also interested in situations, where the Lie algebra $\mathfrak{g}$ is not of countable dimension but carries a locally convex topology which is coarser than the direct limit topology.

An important class of corresponding groups are the Heisenberg groups $\mathrm{Heis}(V)$ of a locally convex space V, endowed with a continuous scalar product (a locally convex Euclidean space). More precisely, we have

$$\mathrm{Heis}(V) = \mathbb{R} \times V \times V,$$
$$(z, v, w)(z', v', w') := \bigl(z + z' + 1/2\bigl(\langle v, w'\rangle - \langle v', w\rangle\bigr), v + v', w + w'\bigr).$$

A unitary representation $(\pi, \mathcal{H})$ of this group with $\pi(z, 0, 0) = e^{iz}$ for $z \in \mathbb{R}$ provides a representation of the *canonical commutation relations* in the sense that the restriction to the subgroups $\{0\} \times V \times \{0\}$ and $\{(0, 0)\} \times V$ defines a unitary representation, and for $v, w \in V$ we have

$$\pi(0, v, 0)\pi(0, 0, w) = e^{i\langle v,w\rangle}\pi(0, v, w) = e^{2i\langle v,w\rangle}\pi(0, 0, w)\pi(0, v, 0).$$

In [18] Hegerfeldt shows that, if V is separable, barreled and nuclear, for any continuous unitary representation of $G = \mathrm{Heis}(V)$ there exists a dense subspace $\mathcal{D}$ with the following properties:

(a) $\mathcal{D}$ is $\pi(G)$-invariant.
(b) $\mathcal{D} \subseteq \mathcal{D}_{\mathfrak{g}}$.
(c) $\overline{\mathrm{d}\pi}(x)\mathcal{D} \subseteq \mathcal{D}$ for every $x \in \mathfrak{g}$, in particular $\mathcal{D} \subseteq \mathcal{D}_{\mathfrak{g}}^{\infty}$.
(d) For each $v \in \mathcal{D}$ the map $\omega_v^n : \mathfrak{g}^n \to \mathcal{H}$ is continuous.

In view of Lemma 3.4 and the local exponentiality of G, (c) and (d) imply that $\mathcal{D}$ consists of smooth vectors for G.

Actually Hegerfeldt shows that the elements of $\mathcal{D}_{\mathfrak{g}}$ even have analyticity properties that lead to holomorphic extensions of their orbit maps to the complexified group $G_{\mathbb{C}}$. For a detailed discussion of this aspect we refer to [44].

12. Smooth vectors for projective limits

Structurally direct limits of finite dimensional Lie groups are groups with a relatively simple structure, but they have many continuous unitary representation because they carry a very fine topology. This situation is the opposite of what we find for projective limits of finite dimensional Lie groups. These groups are also called *pro-Lie groups*, and the Lie groups among the pro–Lie groups have been characterized recently in [19]. For any such Lie group $G = \varprojlim G_j$, it makes sense to ask for smooth vectors in unitary representations. As we shall see in this section, for this class of Lie groups the space $\mathcal{H}^{\infty}$ is always dense. Here the main point is a quite general argument concerning unitary representations of projective limits of topological groups.

Let G be a topological group and $\mathcal{N} := (N_i)_{i \in I}$ be a filter basis of closed normal subgroups $N_i \trianglelefteq G$ with $\lim \mathcal{N} = \{\mathbf{1}\}$, i.e., for each $\mathbf{1}$-neighborhood U in G there exists some $i \in I$ with $N_i \subseteq U$.

Lemma 12.1. *Let $(\pi, \mathcal{H})$ be a continuous unitary representation of G. Then the union $\bigcup_{i \in I} \mathcal{H}^{N_i}$ of the closed invariant subspaces $\mathcal{H}^{N_i}$ of N_i-fixed vectors is dense in $\mathcal{H}$.*

Proof. Let $v \in \mathcal{H}$ and pick $\varepsilon > 0$. Then there exists some i with $\pi(N_i)v \subseteq B_{\varepsilon}(v)$. Then $\overline{\mathrm{conv}(N_i.v)}$ is a bounded closed convex N_i-invariant subset of $\mathcal{H}$, hence contains a N_i-fixed point by the Bruhat–Tits Fixed Point Theorem [30]. Therefore $\mathrm{dist}(v, \mathcal{H}^{N_i}) \leqslant \varepsilon$. □

Theorem 12.2. *Any continuous representation $(\pi, \mathcal{H})$ of G is a direct sum of representations on which some N_i acts trivially.*

Proof. Let $\mathcal{F}$ be the set of all sets F of pairwise orthogonal G-invariant closed subspaces of $\mathcal{H}$ on which some N_i acts trivially. We order $\mathcal{F}$ by set inclusion. Then $\mathcal{F}$ is inductively ordered, so that Zorn's Lemma implies the existence of a maximal element F_m. Then

$$\mathcal{K} := \overline{\bigoplus_{\mathcal{H}_j \in F_m} \mathcal{H}_j}$$

is a closed G-invariant subspace of $\mathcal{H}$. We claim that $\mathcal{K} = \mathcal{H}$, which implies the assertion. If this is not the case, $\mathcal{K}^\perp$ is non-zero, and Lemma 12.1 implies that for some i the set $(\mathcal{K}^\perp)^{N_i}$ is non-zero, contradicting the maximality of F_m. □

Corollary 12.3. *Each irreducible continuous unitary representation of G factors through some G/N_i, i.e., the set $\widehat{G}$ of equivalence classes of irreducible continuous unitary representations satisfies $\widehat{G} = \bigcup_{i \in I} (G/N_i)\widehat{\ }$.*

Theorem 12.4. *If $G = \varprojlim G_j$ is a Lie group which, as a topological group, is a projective limit of finite dimensional Lie groups G_j, then for each continuous unitary representation $(\pi, \mathcal{H})$ of G the space $\mathcal{H}^\infty$ of smooth vectors is dense.*

Proof. Let $q_j : G \to G_j$ be the natural projections and apply Theorem 12.2 to the family $N_j = \ker q_j$. This reduces the problem to the case where some N_j acts trivially on $\mathcal{H}$, so that we actually have a representation of the finite dimensional quotient Lie group $G_j \cong G/N_j$. Now the assertion follows from the density of smooth vectors for G_j in $\mathcal{H}^{N_j}$ [10]. □

Remark 12.5. The group $G = \mathbb{R}^{\mathbb{N}}$ is a projective limit of the Lie group $G_n = \mathbb{R}^n$, where q_n: $G \to G_n$ is the projection onto the first n factors. In this sense Example 4.8 is a continuous unitary representation of a pro-Lie group.

Remark 12.6. Let $G = (V, +)$ be the additive group of a locally convex space V. For each continuous seminorm $p \in \mathcal{P}(V)$, we have a closed subspace $N_p := p^{-1}(0)$ for which p induces a norm on the quotient space V/N_p. Now $\mathcal{N} = \{N_p\colon\ p \in \mathcal{P}(V)\}$ is a filter basis of closed subgroups with $\lim \mathcal{N} = \{\mathbf{1}\}$, so that Theorem 12.2 applies. We conclude that every continuous unitary representation $(\pi, \mathcal{H})$ of V is a direct sum of representations $(\pi, \mathcal{H}_i)$ on which some N_p acts trivially, so that the representation π_i factors through a representation of the normed space V/N_p.

13. Perspectives

There are several interesting problems concerning representations of infinite dimensional Lie groups G on Banach spaces V.

Problem 13.1 *(Integrability).* Suppose that V is a locally convex space, $\mathcal{D} \subseteq V$ a dense subspace and $\rho : \mathfrak{g} \to \operatorname{End}(\mathcal{D})$ a representation of the Lie algebra $\mathfrak{g}$ on $\mathcal{D}$.

We thus obtain for each $x \in \mathfrak{g}$ an unbounded operator $\rho(x)$ on V. We assume that all these operators are closable and that each closure $\overline{\rho(x)}$ generates a strongly continuous one-parameter group. A characterization of such operators on Banach spaces is given by the Hille–Yoshida Theorem, and [24] contains some generalization to locally convex spaces.

Suppose that $\mathfrak{g}$ is a Banach–Lie algebra. Then we obtain a map

$$F : \mathfrak{g} \to \mathrm{GL}(V), \quad x \mapsto e^{\overline{\rho(x)}}.$$

When does this map define a local group homomorphism, hence a representation of any corresponding simply connected Lie group G? More precisely, under which conditions on ρ do we have

$$F(x * y) = F(x)F(y)$$

for x and y in a small ball centered in 0?

For finite dimensional Lie algebras, problems of this kind are studied in [20, Thms. 1.1] and Chapter 8 of [23]. Maybe some of these results, such as Theorem A.1–3 in [21] can be extended to Banach–Lie algebras. It would also be interesting to have a version of the integrability result [23, Thm. 8.1] for representations on locally convex space or [23, Thm. 8.6] for representations on Banach spaces (cf. also [34]). A first step in this direction is taken by S. Merigon in [31], where he obtains such a result for representation on Hilbert spaces.

Problem 13.2 *(Smoothness).* We have seen above that for every continuous representation (π, V) of the Banach–Lie group G on the Banach space V, we obtain a sequence $(V^k, \|\cdot\|_k)$ of Banach spaces, where $V^k = \mathcal{D}^k_{\mathfrak{g}}$ is the space of C^k-vectors, endowed with its natural norm (Theorem 9.4), and in this picture V^∞ is the projective limit of the Banach spaces V^k (in the category of locally convex spaces) (cf. [23, p. 20] for finite dimensional Lie algebras).

The C^k-variant of the derived action is given by the sequence

$$\mathfrak{g} \times V^k \to V^{k-1}, \quad (x, v) \mapsto \overline{\mathrm{d}\pi}(x)v, \quad k \in \mathbb{N}, \tag{16}$$

of continuous bilinear maps. Interesting questions in this context are:

(a) Does the density of V^∞ in V imply the density in each of the Banach spaces V^k?
(b) In [23] the continuity of the action of G on the Banach space $(\mathcal{D}_{\mathfrak{g}}, \|\cdot\|_1)$ (graph density) plays an important role. As follows from Proposition 9.7 and Remark 5.3, this is equivalent to the density of the continuous vectors in $\mathcal{D}_{\mathfrak{g}}$, which in turn follows from the density of $\mathcal{D}^2_{\mathfrak{g}}$ in $\mathcal{D}^1_{\mathfrak{g}}$. Maybe these conditions can also be exploited for Banach–Lie groups.
(c) Suppose that we are given a Lie algebra representation $\rho : \mathfrak{g} \to \mathrm{End}(\mathcal{D})$, where $\mathcal{D} \subseteq V$ is a dense subspace. Suppose that all the k-linear maps $\omega^k_v : \mathfrak{g}^k \to V$ are continuous and write V^k for the completion of $\mathcal{D}$ with respect to the norm

$$\|v\|_k := \|v\| + \sum_{j=1}^{k} \|\omega^j_v\|.$$

Then V^k can be realized as subspaces of V. Is it possible to characterize in this context representations which are integrable to continuous representations of G on V (cf. [23] for a discussion of similar problems for finite dimensional Lie algebras). Natural assumptions in this context are that the closures of the operators $\rho(x)$ generate one-parameter groups preserving $\mathcal{D}$ (and all the spaces V^k).

Problem 13.3. We have seen in Sections 4 and 5 that for every unitary representation $(\pi, \mathcal{H})$ of a Banach–Lie group G, the space $\mathcal{H}^\infty$ carries a natural Fréchet topology with respect to which G acts smoothly. Can this information be used to show in certain situations that a direct integral decomposition

$$(\pi, \mathcal{H}) = \int_X^{\oplus} (\pi_x, \mathcal{H}_x)\, d\mu(x)$$

also yields a "direct integral decomposition"

$$\mathcal{H}^\infty = \int_X^{\oplus} \mathcal{H}_x^\infty\, d\mu(x)?$$

This would be extremely useful for the analysis of smooth unitary representations. For results of this type for finite dimensional groups we refer to [2].

Problem 13.4. The argument in the proof of Theorem 8.2 touches on an interesting question concerning the differentiability of functions $f : G \to \mathbb{R}$ on a Banach–Lie group. Suppose that for every $g \in G$ and $x \in \mathfrak{g}$ the derivative

$$\mathtt{d}f(g)(g.x) := \frac{d}{dt}\bigg|_{t=0} f\big(g \exp_G(tx)\big)$$

exists. When are the maps $\mathtt{d}f(g) : \mathfrak{g} \to \mathbb{R}$ linear? They clearly satisfy $\mathtt{d}f(g)(\lambda x) = \lambda \mathtt{d}f(g)x$ for $\lambda \in \mathbb{R}$, so that the additivity is the crucial issue.

The connection to Theorem 8.2 is given by functions of the form $f(g) = \alpha(\pi(g)v)$ with $\alpha \in V'$ and $v \in \mathcal{D}_\mathfrak{g}$, because in this case we have $\mathtt{d}f(\mathbf{1})(x) = \alpha(\overline{\mathtt{d}\pi}(x)v) = \alpha(\omega_v(x))$, and the additivity of every α is equivalent to the additivity of ω_v.

In Lemma 3.3 we have already seen that, for $v \in \mathcal{D}_\mathfrak{g}$, the continuity of the map $\omega_v : \mathfrak{g} \to V$ implies its linearity. Any more direct proof of the continuity of ω_v for $v \in \mathcal{D}_\mathfrak{g}$ would therefore lead to a more direct proof of Theorem 8.2.

Problem 13.5. For "selfadjoint" representations of the complex enveloping algebra $\mathcal{U}(\mathfrak{g})_\mathbb{C}$ of a finite dimensional Lie algebra $\mathfrak{g}$ on the dense subspace $\mathcal{D}$ of the Hilbert space $\mathcal{H}$, there exists an integrability criterion due to Powers [59, Thm. 4.5]. The requirement is that the map $\pi : \mathcal{U}(\mathfrak{g})_\mathbb{C} \to \mathrm{End}(\mathcal{D})$ is "completely strongly positive" with respect to a certain convex cone $Q \subseteq \mathcal{U}(\mathfrak{g})_\mathbb{C}$. It would be very interesting to see if this result extends to Banach–Lie groups.

Problem 13.6. As we have seen in Corollary 7.4, a positive definite function on a Lie group G is smooth if it is smooth in some identity neighborhood. In some case one may even expect that the whole function can be reconstructed from the restriction to some identity neighborhood, even if it is not analytic. Theorem 2.1 in [22] contains a criterion for the extendability of a "local" positive definite function to the whole group for finite dimensional unimodular Lie groups. It would be very interesting to understand if there are variants of this result for more general topological groups and in particular for infinite dimensional Lie groups. Here the key point is to find appropriate positivity conditions, such as the *complete strong positivity* used in [22, Cor. 4.1].

Acknowledgments

We thank Stéphane Merigon and Christoph Zellner for a careful reading of this paper and for several discussions on its subject matter.

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