Generalizations of Bose's Equivalence between Complete Sets of Mutually Orthogonal Latin Squares and Affine Planes

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Using affine resolvable designs and complete sets of mutually orthogonal frequency squares and hypercubes, we provide several generalizations of Bose's equivalence between affine planes of order $n$ and complete sets of mutually orthogonal latin squares of order $n$. We also characterize those complete sets of orthogonal frequency squares and hypercubes which are equivalent to affine geometries. © 1992 Academic Press, Inc.

1. Introduction

One of the most often quoted results in combinatorics is that first proved by Bose [3] in 1938.

Theorem 1.1. There exists an affine plane of order $n$ if and only if there exists a complete set of $n-1$ mutually orthogonal latin squares of order $n$.

In this paper we prove several generalizations of this result. In attempting to generalize Bose's result, one might proceed in several

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directions. Affine geometries of dimension $d \geq 2$ are, of course, natural generalizations of affine planes. Moreover, frequency squares in which one allows repetitions of elements in each row and column are natural generalizations of Latin squares. Consequently one might try to generalize Bose’s result by proving the equivalence of the existence of affine geometries of the appropriate dimension and complete sets of orthogonal frequency squares. However, as indicated by Laywine [13], not all complete sets of orthogonal frequency squares can be reduced to an affine geometry. In Section 5 we characterize those frequency squares that are equivalent to affine geometries and in doing so explain the example of [13].

We would like to express our very sincere appreciation to Geoff Whittle of the University of Tasmania, not only for his numerous helpful comments, but more importantly, for his insight that led us to consider a number of these generalizations. Without his assistance, this project might well have not been undertaken.

It is well known that any balanced incomplete block design with the parameters of an affine plane is an affine resolvable design, see Street and Street [16, Theorem 12, p. 172]. Generalizing in this direction through the use of affine resolvable designs, we are able in Theorem 3.1 to generalize Bose’s equivalence by showing that the existence of certain affine resolvable balanced incomplete block designs is equivalent to the existence of certain complete sets of mutually orthogonal frequency squares. Bose’s equivalence of affine planes and complete sets of orthogonal Latin squares then becomes a corollary of our Theorem 3.1.

In Section 4 we generalize Bose’s equivalence even further by considering more general affine resolvable designs and complete sets of mutually orthogonal Latin and frequency Youden hypercubes of dimension $d \geq 2$. In Section 5 we prove an equivalence between affine geometries of even dimension and certain complete sets of orthogonal frequency squares and more generally between certain geometries whose dimension is a multiple of $d$ and certain sets of $d$-dimensional frequency Youden hypercubes.

2. Notation and Terminology

Throughout this paper we let $n = m^h$, where $m$ and $h$ are positive integers with $m \geq 2$. As considered by Cheng [4, 5], a $d \geq 2$-dimensional Youden frequency hypercube means an $n \times \cdots \times n$ array of dimension $d$ based upon $m$ distinct symbols with the property that for each $i = 1, \ldots, m$, the $i$th symbol occurs exactly $n^{d-1}/m$ times in each subarray $A_i(1), \ldots, A_i(n)$, where $A_i(k)$ consists of all cells $(k_1, \ldots, k_d)$ with $k_i = k$ and the remaining $1 \leq k_j \leq n$ for $j \neq i$. Following Cheng’s notation, such a hypercube will be
denoted by YFHC(m; n^d). For d > 3 such hypercubes were shown by Cheng [4] to be both D-optimal and A-optimal designs and hence are very useful in experimental design theory. Thus YFHC(n; n^2) hypercubes are Latin squares of order n and YFHC(n; n^d) hypercubes are d-dimensional Latin hypercubes considered by Kishen [11]. Moreover, YFHC(m; n^2) hypercubes are \( F(m^n; m^{n-1}; \ldots, m^1) = F(m^n; m^{n-1}) \) frequency squares considered by Hedayat, Raghavarao, and Seiden [9] and Mullen [14].

Two such hypercubes are orthogonal, if upon superposition, every ordered pair appears exactly \( n^d/m^2 = m^{d-2} \) times. Cheng [5] proved the following bound for \( t \) the maximal number of mutually orthogonal Youden hypercubes (MOYFHC) \( F_1, \ldots, F_t \), where each \( F_i \) has type YFHC(m; n^d):

\[
t \leq (n^d - dn + d - 1)/(m - 1).
\]

A set is said to be complete if equality holds in (1).

Thus there are \( n-1 \) mutually orthogonal Latin squares (MOLS) in a complete set of Latin squares of order n; \( (n-1)^2/(m-1) \) mutually orthogonal frequency squares (MOFS) in a complete set of \( F(n; n/m) \) frequency squares; and \( (n^d - dn + d - 1)/(n - 1) \) elements in a complete set of Latin hypercubes as considered by Kishen [11]. For an excellent survey on Latin squares, their generalizations and applications, see Dénes and Keedwell [7].

Cheng [5] constructed complete sets of Youden hypercubes, in fact more generally, he constructed complete sets of Youden hyperrectangles in the case when \( m \) is a prime power by using affine geometries. More recently, Suchower [17] considered even more general hyperrectangles and in [18] he constructed complete sets using a polynomial representation technique over finite fields, extending Mullen’s technique [14] for MOFS. We shall, however, restrict our attention to the cases with constant frequency vectors.

A balanced incomplete block (BIB) design with parallelism is said to be affine resolvable if there exists an integer \( \mu \) such that any two non-parallel blocks intersect in exactly \( \mu \) elements and it is called \( \alpha \)-resolvable if the set of blocks can be partitioned into parallel classes so that each element appears \( \alpha \) times in each parallel class. Since \( \alpha = 1 \) in all of our designs, we will simply use the term affine resolvable. In an affine resolvable BIB design (ARBIBD), as shown by Shrikhande [15], the parameters \( v, b, r, k, \) and \( \lambda \) can be given in terms of two parameters \( n \geq 2 \) and \( t \geq 0 \) so that

\[
\begin{align*}
v &= nk = n^2[(n-1)t + 1] \\
b &= nr = n(n^2t + n + 1) \\
\lambda &= nt + 1, \quad \mu = (n-1)t + 1.
\end{align*}
\]
Moreover, \( b = v + r - 1 \), \( \mu = k^2/v \), and if each class contains \( \beta \) blocks, then \( b = \beta c \), where \( c \) is the number of parallel classes.

As in Shrikhande [15], we let \( AD(n, r) \) denote an ARBIBD with parameters given by (2) so that \( AG(2, n) \), an affine plane of order \( n \), is an \( AD(n, 0) \) and, more generally, the design formed by taking points and hyperplanes of \( AG(d, q) \), the \( d \)-dimensional affine geometry over the finite field \( GF(q) \), is denoted by \( AD(q, (q^d - 2 - 1)/(q - 1)) \). For many more properties of affine resolvable designs, the reader is referred to the survey paper [15] of Shrikhande.

A quasi-Youden frequency hypercube (QYFHC) of dimension \( d \), frequency \( n^{d-1}/m \), based on \( m \) distinct symbols is an \( n \times \cdots \times n \) array of dimension \( d \), where \( n = \mu m \), with the property that the \( i \)th symbol occurs exactly \( n^{d-2}/m^2 \) times in each subarray \( A_i(1), ..., A_i(n) \), where \( A_i(k) \) consists of all \( \mu n^{d-1} \) cells \( (k_1, ..., k_d) \) with \( k_i = \mu \) and the remaining \( 1 \leq k_j \leq n \) for \( j \neq i \). Thus every YFHC is a QYFHC, but not conversely. Orthogonality between QYFHCs is defined as before for YFHCs so that each ordered pair \((i, j)\) occurs exactly \( n^{d}/m^2 \) times when one QYFHC is superimposed on the other. Two-dimensional quasi-frequency squares (QFS) were considered by Huang and Laurent [10, Section III].

A finite incidence structure \( \pi = (\mathcal{P}, \mathcal{B}) \) is called an \((m, r; \mu)\)-net of multiplicity \( \mu \) if the block set \( \mathcal{B} \) can be partitioned into \( r \geq 3 \) parallel classes \( \beta_1, ..., \beta_r \) such that

1. the blocks of each parallel class \( \beta_i \) form a partition of \( \mathcal{P} \),
2. any two blocks from distinct parallel classes intersect in exactly \( \mu \) points,
3. each parallel class consists of \( m \) blocks.

Hence each block consists of \( m\mu \) points, each point is in \( r \) blocks and \( |\mathcal{P}| = m^2 \mu \). It is known [2, Theorem 8.8] that \( r \leq (m^2 \mu - 1)/(m - 1) \) and equality holds if and only if \( \pi \) is an affine 2-design, i.e., an affine BIB design, in which any two distinct points occur in exactly \( \lambda = (m\mu - 1)/(m - 1) \) blocks. In general, an \((m, r; \mu)\)-net is equivalent, in the notation of [2], to an affine \( S_r(1, m\mu; m^2 \mu) \) design, see [2, Corollary 8.3]. We note that an \( S_r(1, k; v) \) design is simply a tactical configuration, see [2, Definition 3.5]. For further details regarding nets and affine designs, the reader should see [2].

3. MOFS AND AFFINE RESOLVABLE DESIGNS

In this section we consider \( F(m^h; m^h - 1) \) frequency squares. It is quite natural to study frequency squares by making substitutions on the symbols
of Latin squares. This idea was exploited by Laywine [12], where he constructed complete sets of $F(m^h; m^h-1)$ frequency squares by making $(m^h-1)/(m-1)$ substitutions on the $m^h$ symbols of a complete set of $m^h-1$ MOLS of order $m^h$. The substitutions in that paper turn out to be determined by the parallel classes of $AD(m, (m^h-2-1)/(m-1))$ with $h > 1$. Since this idea is crucial to our later development, we include the following example.

Consider the three MOLS of order 4 given below. With $m = h = 2$, we view the three parallel classes of $AD(2, 0)$ as

$$
\begin{align*}
1, 2 &\rightarrow 0 \\
1, 3 &\rightarrow 0 \\
1, 4 &\rightarrow 0 \\
P_1: \\
3, 4 &\rightarrow 1 \\
2, 4 &\rightarrow 1 \\
2, 3 &\rightarrow 1 \\
P_2: \\
P_3:
\end{align*}
$$

Each parallel class determines a substitution as shown above. Each parallel class and hence each substitution converts a Latin square of order 4 into an $F(4; 2)$ frequency square. We thus obtain a complete set of nine $F(4; 2)$ squares which, as shown in Laywine [12], are mutually orthogonal (see Fig. 3.1).

The following result indicates why affine resolvable designs provide a natural setting in which to generalize Bose's equivalence between affine planes and MOLS. In Street and Street [16, Theorem 12, p. 172] it is shown that any BIB with the parameters of an affine plane is an affine resolvable BIB. Hence in our notation, the existence of an affine plane

<table>
<thead>
<tr>
<th>Latin Square</th>
<th>Derived Frequency Squares</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>$P_1$</td>
</tr>
<tr>
<td>1 2 3 4</td>
<td>0 0 1 1</td>
</tr>
<tr>
<td>2 1 4 3</td>
<td>0 0 1 1</td>
</tr>
<tr>
<td>3 4 1 2</td>
<td>1 1 0 0</td>
</tr>
<tr>
<td>4 3 2 1</td>
<td>1 1 0 0</td>
</tr>
<tr>
<td>1 2 3 4</td>
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<tr>
<td>3 4 1 2</td>
<td>1 1 0 0</td>
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<td>2 1 4 3</td>
<td>0 0 1 1</td>
</tr>
<tr>
<td>3 4 1 2</td>
<td>1 1 0 0</td>
</tr>
</tbody>
</table>

**Figure 3.1**
$AG(2, n)$ is equivalent to the existence of an $AD(n, 0)$ design. We are now ready to prove

**Theorem 3.1.** There exists an $AD(m, (m^{2h-2} - 1)/(m-1))$ design whose blocks can be simultaneously partitioned into $(m^h - 1)/(m^{h-1}(m-1))$ copies of $AG(2, m^h)$ and $m^h + 1$ copies of an $AD(m, (m^{h-2} - 1)/(m-1))$ design if and only if there exists a complete set of $(m^h - 1)^2/(m-1)F(m^h; m^{h-1})$ MOFS derived from a complete set of $m^h - 1$ MOLS of order $m^h$ by substitutions determined by the parallel classes of an $AD(m, (m^{h-2} - 1)/(m-1))$ design.

**Proof.** Let $D = AD(m, (m^{h-2} - 1)/(m-1))$ so that $D$ has $(m^{h-1})/(m-1)$ parallel classes where $h \geq 2$. If $h = 1$ the result reduces to Bose's equivalence since $D$ can be viewed as a trivial design whose blocks consist of single points and so the substitutions will be just the identity map. Hence we may assume $h \geq 2.$

For sufficiency, we construct blocks from the frequency squares by generalizing the usual way lines are constructed in an affine plane from a complete set of MOLS. Consider the $m^h(m^h + 1)$ blocks of $m^h$ points arising from the $m^h + 1$ parallel classes of $AD(m^h, 0) = AG(2, m^h)$. We label the blocks within each parallel class by the $m^h$ points of $D$. Within each parallel class of $AD(m^h, 0)$, we now form blocks of a new structure by taking the union of those blocks of $AD(m^h, 0)$ which are labelled by the elements of a given block of $D$. A block in the new structure thus has $m^{h-1} \cdot m^h = m^{2h-1}$ points. Doing this for each parallel class of $D$ yields a collection of $m(m^h - 1)/(m-1)$ blocks for each parallel class of $AD(m^h, 0)$ and hence a total of $b = m(m^{2h} - 1)/(m-1)$ blocks.

Since the MOFS were formed from MOLS by substitutions, the set of new blocks are precisely those obtained by extending the usual method of obtaining the $n(n + 1)$ lines of $AG(2, n)$ from where $n(n - 1)$ lines arise from the MOLS and $2n$ from the rows and columns. Of the $b$ blocks so constructed, $m(m^h - 1)^2/(m-1)$ arise from the complete set of MOFS and $2m(m^h - 1)/(m-1)$ arise from the rows and columns.

Call this incidence structure $I$. To show that $I$ is a BIB design, we must show that any two distinct points occur in the same number $\lambda$ of blocks of $I$. We will show that $\lambda = (m^{2h-1} - 1)/(m-1)$. Let $\bar{X}$ and $\bar{Y}$ be two distinct points of $I$. In $AD(m^h, 0)$ $\bar{X}$ and $\bar{Y}$ occur together in exactly one block. Within this parallel class of $AD(m^h, 0)$, $\bar{X}$ and $\bar{Y}$ will occur together in any of the $(m^h - 1)/(m-1)$ blocks of $I$. Now consider any of the other $m^h$ parallel classes of $AD(m^h, 0)$. In any such parallel class, suppose $\bar{X}$ is in a block of $D$ labelled by $\alpha$. Similarly suppose $\bar{Y}$ is in a block of $D$ labelled by $\beta$, where $\alpha \neq \beta$. Thus when taking unions to form the blocks of $I$, $\bar{X}$ and $\bar{Y}$ will occur together exactly the same number of times that $\alpha$ and $\beta$ occur.
together in blocks of $D$, which is $(m^{h-1} - 1)/(m - 1)$. Consequently in total, $\bar{X}$ and $\bar{Y}$ will occur together in $I$ exactly

$$\frac{m^h - 1}{m - 1} + \frac{m^h(m^h - 1)}{m - 1} = \frac{m^{2h-1} - 1}{m - 1}$$

times as desired. Since $\bar{X}$ and $\bar{Y}$ were arbitrary, our incidence structure $I$ is a BIB design.

Since $AD(m^h, 0)$ is resolvable and $D$ is resolvable, by construction $I$ is resolvable. Moreover, in $I$, $v = m^{2h}$ and $r = (m^{2h} - 1)/(m - 1)$ so that $b = v + r - 1$, and hence by Shrikhande [15, Theorem 1.3], $I$ is an affine design. By construction the blocks of $I$ have the correct number of copies of $AG(2, m^h)$ and $D$. This completes the proof of sufficiency.

For necessity let $L$ be one of the $m^h - 1$ MOLS of order $m^h$ constructed from $AG(2, m^h)$. Since $D$ has $m^h$ points, we may view the elements of $L$ as the elements of $D$. Each block of $D$ contains $m^{h-1}$ elements and so when a substitution is made on the elements of $L$, we clearly obtain an $F(m^h; m^{h-1})$ frequency square. Moreover, since there are $(m^h - 1)/(m - 1)$ parallel classes of $D$ and the same number of distinct substitutions, we will obtain $(m^h - 1)^2/(m - 1)$ frequency squares each of type $F(m^h; m^{h-1})$. It remains to show that these frequency squares are mutually orthogonal.

Suppose $L_k$ and $L_i$ are distinct orthogonal Latin squares. Upon superposition of $L_k$ and $L_i$ each of the $m^{2h}$ ordered pairs occurs exactly once. Consequently after any substitution is applied, each of the $m^2$ ordered pairs will occur $m^{2h-2}$ times so the resulting frequency squares are orthogonal.

Now suppose two distinct substitutions are applied to a single Latin square. Since $D$ is affine resolvable, these substitutions will intersect in $\mu = m^{h-2}$ points. Thus in each row, each of the $m^2$ pairs will be obtained exactly $m^{h-2}$ times. But there are $m^h$ rows, so each of the $m^2$ pairs will occur, upon superposition of the frequency squares, exactly $m^h \cdot m^{h-2} = m^{2h-2}$ times. Hence in either case, the resulting frequency squares are orthogonal so the proof of necessity, and hence of the theorem, is complete.

As indicated in [1, p. 137] an affine resolvable design is precisely a strongly resolvable 2-design. Hence as indicated from [1, Section 7], the design $AD(m, (m^{2h-2} - 1)/(m - 1))$ that we constructed is a strongly resolvable $2 - (m^{2h}, m^{2h-1}, (m^{2h-1} - 1)/(m - 1))$ design with a 0-decomposition and block intersection numbers 0 and $\rho_{ij} = m^{2h-2}$ for $i \neq j$.

**Corollary 3.2** (Bose). $AG(2, n)$ exists if and only if there exist $n - 1$ MOLS order $n$.

**Proof.** This is the $h = 1$ case of the theorem.

From Shrikhande [15, Theorem 3.2] we may state
COROLLARY 3.3. There exists a complete orthogonal array \( A[m, (m^{2h-1})/(m-1), m^{2h-2}] \) if and only if there exists a complete set of \( F(m^h; m^{h-1}) \) MOFS derived from a complete set of MOLS of order \( m^h \) by substitution from parallel classes of an \( AD(m, (m^{h-2} - 1)/(m-1)) \) design.

4. MOYFHCs and Affine Resolvable Designs

In this section we generalize the results of Section 3 to \( d \geq 2 \) dimensional hypercubes. Kishen [11] showed that given \( AG(d, n) \), one can construct a complete set of \( (n^d - dn + d - 1)/(n - 1) \) mutually orthogonal \( YFHC(n; n^d) \) hypercubes. However, as pointed out by Mann in his review [MR 11(1950), p. 637] of Kishen's paper, "it is important to note that for \( d > 2 \) a finite geometry cannot necessarily be constructed from the configuration." The reason for this is that the affine geometry structure should be replaced by the weaker structure of an affine resolvable design.

THEOREM 4.1. There exists an \( AD(m, (m^{dh-2} - 1)/(m-1)) \) design whose blocks can be simultaneously partitioned into \( (m^h - 1)/(m^{h-1}(m-1)) \) copies of \( AD(m^h, (m^{dh-2} - 1)/(m^{h-1})) \) and \( (m^{dh-1} - 1)/(m^{h-1}) \) copies of an \( AD(m, (m^{h-2} - 1)/(m-1)) \) design if and only if there exists a complete set of \( (n^d - dn + d - 1)/(m - 1) \) mutually orthogonal \( YFHC(m; n^d) \) hypercubes derived from a complete set of \( YFHC(n; n^d) \) hypercubes by substitutions from the parallel classes of an \( AD(m, (m^{h-2} - 1)/(m-1)) \) design.

Proof. It is easy to see that the proof of necessity in Theorem 3.1 can be extended to the \( d \geq 2 \) dimensional setting. As in the proof of Theorem 3.1 we take unions of blocks from the design \( K = AD(n, (n^{d-2} - 1)/(n - 1)) \) formed from the \( YFHC(n; n^d) \) hypercubes. Let \( D = AD(m, (m^{h-2} - 1)/(m-1)) \) and consider the \( n(n^d - 1)/(n - 1) \) blocks from \( K \). Each block has \( n^{d-1} \) points, there are \( (n^d - 1)/(n - 1) \) parallel classes, and each parallel class contains \( n \) blocks. We label the blocks within each parallel class by the \( m^h \) points of \( D \). We now form blocks of a new structure \( I \) by taking, within each parallel class of \( K \), the union of those blocks of \( K \) which are labelled by the elements of a given block of \( D \). Each block in the new structure thus has \( m^{h-1}n \) points. Doing this for each parallel class of \( D \) yields a collection of \( m(n - 1)/(m - 1) \) blocks for each parallel class of \( K \) and hence a total of \( b = m(m^{dh-1} - 1)/(m-1) \) blocks.

Let \( \bar{X} \) and \( \bar{Y} \) be two distinct points in \( I \). Generalizing the proof from Theorem 3.1, we count the number of blocks containing \( \bar{X} \) and \( \bar{Y} \) as follows. In the design \( K \) there are \( (n^{d-1} - 1)/(n - 1) \) blocks containing \( \bar{X} \) and \( \bar{Y} \). Hence in the larger blocks of \( I \) formed by taking unions of blocks
of $K$, $\bar{X}$ and $\bar{Y}$ will remain together. Moreover, there are $(m^h - 1)/(m - 1)$ substitutions from the distinct parallel classes of $D$.

In $K$ there are $n^{d-1}$ parallel classes in which $X$ and $Y$ occur in different blocks. Within such a parallel class, suppose the block containing $X$ is labelled by $\alpha$ and that containing $Y$ is labelled by $\beta$ with $\alpha, \beta \in D$. In each parallel class, $X$ and $Y$ will end up together in the same block of $I$ exactly the same number of times that $\alpha$ and $\beta$ occur together in blocks of the substitution design $D$, which is $(m^{h-1} - 1)/(m - 1)$.

Hence, in total, $X$ and $Y$ will occur together in $I$ in exactly

$$\frac{n-1}{m-1} \cdot \frac{n^{d-1} - 1}{m-1} + \frac{n^{d-1}(m^{h-1} - 1)}{m-1} = \frac{m^{dh-1} - 1}{m-1}$$

blocks. Since $X$ and $Y$ were arbitrary, our incidence structure $I$ is a BIB design. Generalizing arguments from the proof of Theorem 3.1, it is easy to check that $I$ is an affine resolvable design and that it has the desired block structure.

We note that the design $AD(m, (m^{dh-2} - 1)/(m-1))$ is a strongly resolvable $2 - (m^{dh}, m^{dh-1}, (m^{dh-1} - 1)/(m - 1))$ design with a $0$-decomposition and block intersection numbers $0$ and $p_{ij} = m^{dh-2}$ for $i \neq j$. The following corollary is the equivalence that eluded Kishen [11].

**Corollary 4.2.** There exists an $AD(n, (n^{d-2} - 1)/(n-1))$ design if and only if there exists a complete set of $(n^{d-dn + d-1} - 1)/(n-1)$ mutually orthogonal YFHC($n; n^d$) hypercubes.

**Corollary 4.3 (Bose).** $AG(2, n)$ exists if and only if there exist $n-1$ MOLS of order $n$.

5. MOFS AND AFFINE GEOMETRIES

A natural generalization of an affine plane $AG(2, n)$ is an affine geometry $AG(d, n)$ of dimension $d \geq 2$. In [14] Mullen used linear equations over the finite field $GF(m)$ to construct a complete set of $F(m^h; m^{h-1})$ MOFS. His equations are, of course, exactly the equations normally used to define the non-row and column hyperplanes of $AG(d, m)$ over $GF(m)$. Consequently it is very natural to hope that one can generalize Bose's equivalence from affine planes and MOLS to affine geometries and MOFS.

Given a complete set of $F(m^h; m^{h-1})$ MOFS we can construct blocks in a natural way as in Section 3. Let $\bar{X}$ and $\bar{Y}$ be two distinct points of the corresponding incidence structure $I$. If $X$ and $Y$ occur in the same number
λ of blocks we of course, have a BIB design. Moreover, if the intersection of all blocks containing \( \bar{X} \) and \( \bar{Y} \) has exactly \( m \) points, then our design is an affine geometry. (See Theorem 12 of Dembowski [6].)

Not every complete set of \( F(m^h; m^{h-1}) \) MOFS satisfies this last property. Theorem 5.1 which follows shows that the equivalence of a complete set of MOFS and an affine geometry depends on both the design from which the substitutions are derived and the nature of the MOLS from which the MOFS are obtained. Examples, which follow, illustrate the procedure for constructing MOFS which fail to give an affine geometry in each case.

**Definition 5.1.** A complete set of MOLS of order \( m^h \) is said to be \( m^{hi} \)-partitionable, where \( i \) divides \( h \) if for any pair of distinct points \( \bar{X} \) and \( \bar{Y} \), the line \( l \) joining them in the associated affine plane \( P \), can be partitioned into equal segments of size \( m^{hi} \) such that \( \bar{X} \) and \( \bar{Y} \) have a common segment, and in any parallel class of \( P \) other than that containing \( l \), the symbols attached to the points of every segment form a line in \( AG(\lambda, m^{hi}) \).

In order that the symbols of the MOLS identify with the points of \( AG(\lambda, m) \), we label those points from 0 to \( m^h - 1 \).

Observe first that every complete set of MOLS of order \( m^h \) is \( m^h \)-partitionable. In this case every line is degenerately partitioned into a single segment.

Figures 5.1 and 5.2 show respectively the desarguesian plane of order 9 which is 3-partitionable and the Hughes plane of order 9 which does not have this property. In the former case, consider the 0-line in square \( L_4 \) identified in Fig. 5.1, consisting of all elements on the main diagonal. The partition imposed by choosing \( \bar{X} = (0, 0) \) and \( \bar{Y} = (1, 1) \) consists of the segments \{ \( (0, 0), (1, 1), (2, 2) \}, \{ (3, 3), (4, 4), (5, 5) \}, and \{ (6, 6), (7, 7), (8, 8) \}. The symbols associated with these segments in squares \( L_i, i \neq 4 \), are consistent with lines of \( AG(2, 3) \). This property applies to row and column lines as well. For example, column zero can be partitioned into segments \{ \( (0, 0), (0, 1), (0, 2) \}, \{ (0, 3), (0, 4), (0, 5) \}, \{ (0, 6), (0, 7), (0, 8) \}, whose symbols agree with those of the 0-line of \( L_4 \). In Theorem 5.5 we show that this property applies generally for all desarguesian planes. Figure 5.2 gives the MOLS corresponding to the Hughes plane of order 9.

Consider the 0-line of \( L_8 \) consisting of points \( (0, 0), (1, 1), (2, 7), (3, 5), (4, 6), (5, 4), (6, 2), (7, 8), \) and \( (8, 3) \). Any segment containing \( (0, 0) \) and \( (1, 1) \) must have a third point lying on the 0-line of \( L_8 \). That point cannot be \( (2, 7) \), since the labels attached to \( (0, 0), (1, 1), \) and \( (2, 7) \) are \{ \( 0, 4, 3 \} \) in \( L_1 \) and \( \{ 0, 2, 4 \} \) in \( L_6 \). Both cannot be lines in \( AG(2, 3) \), since \{ \( 0, 4, 3 \} \cap \{ 0, 2, 4 \} \} = 2 \). In the same way, each of the points \( 3, 5), (4, 6), (5, 4) \), \( 6, 2), (7, 8), \) and \( 8, 3 \) can be excluded from completing the segment containing \( (0, 0) \) and \( (1, 1) \).
### GENERALIZATIONS OF BOSE’S EQUIVALENCE

\[
\begin{array}{ccc}
L_1 & L_2 & L_3 \\
0 1 2 3 4 5 6 7 8 & 0 1 2 3 4 5 6 7 8 & 0 1 2 3 4 5 6 7 8 \\
5 3 4 8 6 7 2 0 1 & 1 2 0 4 5 3 7 8 6 & 6 7 8 0 1 2 3 4 5 \\
7 8 6 1 2 0 4 5 3 & 2 0 1 5 3 4 8 6 7 & 3 4 5 6 7 8 0 1 2 \\
4 5 3 7 8 6 1 2 0 & 3 4 5 6 7 8 0 1 2 & 5 3 4 8 6 7 2 0 1 \\
0 7 8 0 1 2 3 4 5 & 4 5 3 7 8 6 0 1 2 & 2 0 1 5 3 4 8 6 7 \\
2 0 1 5 3 4 8 6 7 & 5 3 4 8 6 7 2 0 1 & 8 6 7 2 0 1 5 3 4 \\
8 6 7 2 0 1 5 3 4 & 6 7 8 0 1 2 3 4 5 & 7 8 6 1 2 0 4 5 3 \\
1 2 0 4 5 3 7 8 6 & 7 8 6 1 2 0 4 5 3 & 4 5 3 7 8 6 1 2 0 \\
3 4 5 6 7 8 0 1 2 & 8 6 7 2 0 1 5 3 4 & 1 2 0 4 5 3 7 8 6 \\
\end{array}
\]

\[
\begin{array}{ccc}
L_4 & L_5 & L_6 \\
0 1 2 3 4 5 6 7 8 & 0 1 2 3 4 5 6 7 8 & 0 1 2 3 4 5 6 7 8 \\
2 0 1 5 3 4 8 6 7 & 7 8 6 1 2 0 4 5 3 & 3 4 5 6 7 8 0 1 2 \\
1 2 0 4 5 3 7 8 6 & 5 3 4 8 6 7 2 0 1 & 6 7 8 0 1 2 3 4 5 \\
6 7 8 0 1 2 3 4 5 & 8 6 7 2 0 1 5 3 4 & 7 8 6 1 2 0 4 5 3 \\
8 6 7 2 0 1 5 3 4 & 3 4 5 6 7 8 0 1 2 & 1 2 0 4 5 3 7 8 6 \\
7 8 6 1 2 0 4 5 3 & 1 2 0 4 5 3 7 8 6 & 4 5 3 7 8 6 1 2 0 \\
3 4 5 6 7 8 0 1 2 & 4 5 3 7 8 6 1 2 0 & 5 3 4 8 6 7 2 0 1 \\
5 3 4 8 6 7 2 0 1 & 2 0 1 5 3 4 8 6 7 & 8 6 7 2 0 1 5 3 4 \\
4 5 3 7 8 6 1 2 0 & 6 7 8 0 1 2 3 4 5 & 2 0 1 5 3 4 8 6 7 \\
\end{array}
\]

\[
\begin{array}{ccc}
L_7 & L_8 \\
0 1 2 3 4 5 6 7 8 & 0 1 2 3 4 5 6 7 8 \\
8 6 7 2 0 1 5 3 4 & 4 5 3 7 8 6 1 2 0 \\
4 5 3 7 8 6 1 2 0 & 8 6 7 2 0 1 5 3 4 \\
2 0 1 5 3 4 8 6 7 & 1 2 0 4 5 3 7 8 6 \\
7 5 6 1 2 0 4 5 3 & 5 3 4 8 6 7 2 0 1 \\
3 4 5 6 7 8 0 1 2 & 6 7 8 0 1 2 3 4 5 \\
1 2 0 4 5 3 7 8 6 & 2 0 1 5 3 4 8 6 7 \\
6 7 8 0 1 2 3 4 5 & 3 4 5 6 7 8 0 1 2 \\
5 3 4 8 6 7 2 0 1 & 7 8 6 1 2 0 4 5 3 \\
\end{array}
\]

**Figure 5.1**

**Theorem 5.1.** A complete set of MOFS of type $F(m^h; m^{h-1})$ derived from a complete set of MOLS of order $m^h$, is equivalent to the affine geometry $AG(2h, m)$ if and only if the substitutions correspond to the hyperplanes of $AG(h, m)$ and the MOLS are $m$-partitionable. In this case, the segments of the MOLS identify with the lines of $AG(2h, m)$.

**Proof.** To establish the equivalence, it remains only to show that the intersection of blocks containing any two points $\bar{X}$ and $\bar{Y}$ has exactly $m$ points. If this is the case, then the blocks become the hyperplanes of $AG(2h, m)$ and the intersection of these blocks is the line joining $\bar{X}$ and $\bar{Y}$. 
Consider the collection of all blocks containing $\bar{X}$ and $\bar{Y}$ in the design obtained from the MOFS when the substitutions are determined by the hyperplanes in $AG(h, m)$ and the MOLS are $m$-partitionable. In what follows we use the term symbol for the elements of the derived frequency squares and the term label for the elements of the Latin squares.

These blocks consist of row, column, and symbol blocks. The symbol blocks originate from (1) all frequency squares derived from the Latin square containing line $l$ joining $\bar{X}$ and $\bar{Y}$ and (2) those frequency squares derived by substitutions from Latin squares in which $\alpha$ and $\beta$, the labels associated with $\bar{X}$ and $\bar{Y}$, belong to the same hyperplane of $AG(h, m)$.
Similarly, if the rows (columns) are each labelled 0, 1, ..., \(m^h - 1\), the row (column) blocks containing \(\mathcal{X}\) and \(\mathcal{Y}\) are those row (column) blocks in which the row (column) labels of \(\mathcal{X}\) and \(\mathcal{Y}\) are in the same hyperplane of \(AG(h, m)\). Since the MOLS are \(m\)-partitionable the intersection of all such blocks is exactly the segment containing both \(\mathcal{X}\) and \(\mathcal{Y}\).

The argument works in exactly the same way regardless of whether the line \(l\) belongs to the row or column class of the affine plane or, as assumed above, it is a line associated with a label of one of the MOLS.

Since the blocks containing \(X\) and \(Y\) are hyperplanes in this case, the intersection of all of them, i.e., their common segment, defines the line joining them in \(AG(2h, m)\).

Now, let us consider this same collection of blocks when either the MOLS are not \(m\)-partitionable or the substitutions are not determined by hyperplanes of \(AG(h, m)\). Assuming the former, let \(X_1, X_2, ..., X_m\) be \(m\) collinear points in \(P\) carrying labels \(x_1, x_2, ..., x_m\) and \(\beta_1, \beta_2, ..., \beta_m\), respectively, in two distinct parallel classes, where \(x_1, x_2, ..., x_m\) but not \(\beta_1, \beta_2, ..., \beta_m\) form a line in \(AG(h, m)\). The intersection of blocks containing points \(X_1\) and \(X_2\) is a proper subset of \(X_1, X_2, ..., X_m\), since the line of \(AG(h, m)\) containing \(\beta_1\) and \(\beta_2\) does not contain all of \(\beta_3, \beta_4, ..., \beta_m\).

Similarly, if the substitutions are not determined by the hyperplanes of \(AG(h, m)\), the intersection of all substitution blocks containing some pair of labels \(\beta_1\) and \(\beta_2\), will be less than \(m\). Accordingly, if points \(X_1\) and \(X_2\) carry these labels in some parallel class of \(P\), the intersection of blocks containing \(X_1\) and \(X_2\) will be less than \(m\).

The cases where \(h = 1\) or \(2\) are special as shown in the following corollaries.

**Corollary 5.2 (Bose).** \(AG(2, n)\) exists if and only if a complete set of MOLS of order \(n\) exists.

*Proof.* Set \(h = 1\) and interpret the substitutions determined by \(AG(1, m)\) as the identity map. Since every complete set of MOLS of order \(m^h\) is \(m^h\)-partitionable, the result follows.

**Corollary 5.3.** Every complete set of MOFS of type \(F(m^2; m)\) derived from a complete set of \(m\)-partitionable MOLS of order \(m^2\) is equivalent to the affine geometry \(AG(4, m)\).

*Proof.* If \(h = 2\) in Theorem 3.1, the design used to determine the substitutions is the affine plane \(AG(2, m)\). The result follows using Theorem 5.1.

**Corollary 5.4.** If \(m\) is not a prime power, then a complete set of MOLS of order \(m\) and a complete set of \(m\)-partitionable MOLS of order \(m^2\) cannot both exist.
\textbf{Proof.} If both existed, they would imply the existence of }AG(4, m)\text{. But }AG(4, m)\text{ exists only for }m\text{ a prime power.}

Laywine \cite{13} constructed a set MOFS of type }F(125; 25)\text{ to demonstrate that Bose's result could not be generalized to give an equivalence between an arbitrary complete set of MOFS and an affine geometry. His example can be explained by the above results. By Theorem 5.1, the MOFS of the example do not form AG(6, 5) because the substitutions used to construct them form the blocks of an affine resolvable design, but not the hyperplanes of AG(3, 5). The choice of }h=3\text{ in his example is the smallest possible as shown by Corollary 5.3. The choice of }m=5\text{ was dictated by the method of construction which was based on the earlier paper \cite{12}.

Let }GF(m)\text{ denote the finite field of order }m\text{ and suppose }X=(p, q)\text{ is a point in }AG(2, m^h)\text{, where }p\text{ and }q\text{ are both }h\text{-tuples over }GF(m)\text{. Let }X^*=(p \cdot q)\text{ represent the }2h\text{-tuple formed by concatenating }p\text{ and }q\text{. Then if }X^*\text{ is taken to be a point in }AG(2h, m)\text{ we say that }\overline{X}\text{ and }X^*\text{ are matching points.}

\textbf{Theorem 5.5.} For }m\text{ a prime power, the desarguesian plane of order }m^h\text{ is }m\text{-partitionable.}

\textbf{Proof.} We begin with collinear points in }AG(2h, m)\text{ and show that the matching points are collinear when }AG(2, m^h)\text{ is taken to be desarguesian. Then we show that a collection of }m\text{ such points in }AG(2, m^h)\text{, derived from all the points of a line in }AG(2h, m)\text{, form a segment.}

The line in }AG(2h, m)\text{ joining }X^*_1=(p_1, q_1)\text{ and }X^*_2=(p_2, q_2)\text{ is given by }Z^*=\alpha(X^*_1 - X^*_2) + X^*_2, \text{ where }\alpha\text{ is in }GF(m)\text{. Take }\alpha \neq 0, 1\text{ (the excluded cases just give }Z^*=X^*_2\text{ or }Z^*=X^*_1)\text{. Then}

\begin{align*}
Z^* &= \alpha X^*_1 + (1 - \alpha) X^*_2 = \alpha(p_1 q_1) + (1 - \alpha)(p_2 q_2) \\
&= ([\alpha p_1 + (1 - \alpha) p_2] + [\alpha q_1 + (1 - \alpha) q_2]) = (u \cdot v),
\end{align*}

where }u=\alpha p_1 + (1 - \alpha) p_2\text{ and }v=\alpha q_1 + (1 - \alpha) q_2\text{. Consider the line }l\text{ joining }\overline{X}_1=(p_1, q_1)\text{ and }\overline{X}_2=(p_2, q_2)\text{ in }AG(2, m^h)\text{. If }\overline{X}_1\text{ and }\overline{X}_2\text{ have a common label }c\text{ on this line, then there exists a }\beta_k\text{ in }GF(m^h)\text{ such that }p_1 + \beta_k q_1 = p_2 + \beta_k q_2 = c\text{. It follows that }\overline{Z}=(u, v)\text{ lies on line }l\text{ as well, since}

\begin{align*}
(u + \beta_k v) &= \alpha p_1 + (1 - \alpha) p_2 + \beta_k [\alpha q_1 + (1 - \alpha) q_2] \\
&= \alpha(p_1 + \beta_k q_1) + (1 - \alpha)(p_2 + \beta_k q_2) = c.
\end{align*}

But }\overline{Z}\text{ is any one of the remaining points \{ }\overline{X}_3, \overline{X}_4, \ldots, \overline{X}_m\text{ \} of }l\text{ depending on the choice of }\alpha.
Now consider any parallel class, other than the one containing \( l \). Suppose \( \tilde{X}_1 \) has label \( a_1 = p_1 + \beta_1 q_1 \) and \( \tilde{X}_2 \) has label \( a_2 = p_2 + \beta_2 q_2 \) in this class. The labels \( a_1 \) and \( a_2 \) can be taken as \( h \) tuples over \( GF(m) \) and interpreted as points in \( AG(h, m) \), the space determining the substitutions.

In this space they lie on the line \( b = a_2 + \alpha(a_1 - a_2) \), where \( \alpha \) is in \( GF(m) \). Then any point \( b \) on this line is given by

\[
b = \alpha a_1 + (1 - \alpha)a_2 = \alpha(p_1 + \beta_1 q_1) + (1 - \alpha)(p_2 + \beta_2 q_2)
\]

\[= u + \beta v.
\]

This means that the label \( b \) attached to the point \( \tilde{Z} = (u, v) \) on line \( l \) in \( AG(2, m^h) \) is on the line containing \( a_1 \) and \( a_2 \) in the substitution space. But \( \tilde{Z} \) is any one of \( \{ \tilde{X}_3, \tilde{X}_4, \ldots, \tilde{X}_m \} \), so that \( \{ \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_m \} \) is a segment of line \( l \) in \( AG(2, m^h) \).

The partitionable property is not restricted to desarguesian planes. Next, we show that it applies to those planes constructed using M. Hall's example of a Veblen–Wedderburn system or right quasifield \([7, \text{Theorem 8.4.1}] \) or \([8, \text{Section 12.4}] \). To maintain the flavor of the previous proof, and in particular the idea of matching points, we will represent the set of elements in the system as \( X = (p, q) \), where \( p \) and \( q \) are elements in a field, rather than the more conventional form \( X = p + uq \), where \( u \) is an indeterminate. The details giving a projective plane from a right quasifield may be found in either of the references cited above.

**Theorem 5.6.** For \( m \) a prime power, let \( P \) be the projective plane of order \( m^h \) derived from the right quasifield of M. Hall. Then the MOLS associated with \( P \) in the usual way are \( m \)-partitionable.

**Proof.** As in Theorem 5.2, if \( Z^* = (s \cdot t) \) is a point on the line joining \( X_1^* = (p_1 \cdot q_1) \) and \( X_2^* = (p_2 \cdot q_2) \) in \( AG(2h, m) \), where \( p_1, q_1, p_2, q_2, s, t \) are in \( GF(m^h) \); then

\[
Z^* = \alpha X_1^* + (1 - \alpha)X_2^* = ([\alpha p_1 + (1 - \alpha)p_2] \cdot [\alpha q_1 + (1 - \alpha)q_2]),
\]

\[= (s \cdot t)
\]

where \( \alpha \) is in \( GF(m) \). We need to show

1. the points in \( P \), matching those on a line in \( AG(2h, m) \), are collinear.
2. \( m \) such points form a segment because their labels are always consistent with a line of \( AG(h, m) \).

Let \( \tilde{X}_1 = (p_1, q_1) \), \( \tilde{X}_2 = (p_2, q_2) \), and \( Z = (s, t) \) be points in \( P \), where \( p_1, q_1, p_2, q_2, s, t \) are in \( GF(m^h) \). The line \( l \), containing \( \tilde{X}_1 \) and \( \tilde{X}_2 \) is of
the form \( y = x \wedge a + b \), where \( \wedge \) is the multiplication operation of the quasifield. The parameter \( a \) satisfies \((p_1 - p_2) \wedge a = (q_1 - q_2)\) and \( b = q_2 - p_2 \wedge a \).

If \( l' \), joining \( X_2 \) and \( Z \), is given by \( y = x \wedge a' + b' \) then \( a' \) satisfies \([ap_1 + (1 - x)p_2 - p_2] \wedge a' = aq_1 + (1 - x)q_2 - q_2\). Using field algebra on the field elements on both sides of the equal sign, the equation reduces to

\[ a(p_1 - p_2) \wedge a' = a(q_1 - q_2). \]

By multiplication rule (2) of Dènes and Keedwell [7, Theorem 8.4.1] it follows that

\[ (p_1 - p_2) \wedge a' = q_1 - q_2 \]

so that \( a' = a \). Since \( X_2 \) lies on both \( l \) and \( l' \), \( b = b' \) and \( l = l' \). In general, it follows that the \( m \) points in \( P \), matching any line in \( AG(2h, m) \) are themselves collinear.

Consider, now, the labels \( b_1, b_2, \) and \( b_3 \) associated with \( X_1, X_2, \) and \( Z \) in some parallel class of \( P \), other than that containing \( l \). Then for some \( a \)

\[
q_1 = p_1 \wedge a + b_1 \\
q_2 = p_2 \wedge a + b_2 \\
t = s \wedge a + b_3.
\]

If \( b_1 \) and \( b_2 \) are considered to be points in \( AG(h, m) \), the line joining them is given by

\[
c = \alpha b_1 + (1 - \alpha) b_2, \quad \alpha \text{ in } GF(m) \\
= \alpha(q_1 - p_1 \wedge a) + (1 - \alpha)(q_2 - p_2 \wedge a) \\
= \alpha q_1 + (1 - \alpha)q_2 - (ap_1 + (1 - \alpha)p_2) \wedge a \\
= t - s \wedge a = b_3
\]

by the algebra of Hall's system. Hence, \( P \) is \( m \)-partitionable.

It remains open whether any projective plane of order \( m^h \) constructed using a quasifield is \( m \)-partitionable. The earlier example of the Hughes plane of order 9, used to demonstrate the lack of this property, shows that not all projective planes are partitionable. For completeness, that result is stated next.

**Theorem 5.7.** The Hughes plane of order 9 is not 3-partitionable.
In Section 4 the relationship to affine designs, was extended from complete sets of MOFS to complete sets of hypercubes. Analogously, we now extend the relationship between even dimensional affine geometries and MOFS to affine geometries whose dimension is a multiple of $d$ and $d$-dimensional hypercubes.

We begin by considering complete sets of $\frac{(n^d - dn + d - 1)}{(n - 1)}$ mutually orthogonal YFHC($n; n^d$) hypercubes. Kishen [11] constructed such sets from the geometry $AG(d, n)$, refering to them as orthogonal Latin hypercubes of dimension $d$ and order $n$. We will characterize those sets which give an affine geometry, thus addressing the question raised by Mann and discussed in Section 4 and providing a further generalization of Bose's result.

From Theorem 4.1 and Corollary 4.2, complete sets of YFHC($n; n^d$) hypercubes are equivalent to specified affine designs in which a parallel class is obtained from each dimension and from the symbols in each hypercube. The first case generalizes the row and column classes in MOLS and the latter case, the parallel lines obtained from the symbols in each Latin square. To restrict the affine design of Corollary 4.2 to the geometry $AG(d, n)$ we must show that the intersection of all blocks containing two arbitrary points consists of exactly $n$ points. For this purpose, we extend the concept of partitionability to hypercubes. In the case $d=2$, the following definition reduces to Definition 5.1.

**Definition 5.2.** Suppose for $n=mh$ $B_1, B_2, ..., B_t$ are the blocks of a complete set of YFHC($n; n^d$) hypercubes containing an arbitrary pair of points $X$ and $Y$. Then the set of hypercubes is said to be $m^{hi}$-partitionable, where $i$ divides $h$, if each of $B_1, B_2, ..., B_t$ can be partitioned into segments of equal size $m^{hi}$ such that there exists a segment $S$ containing both $X$ and $Y$ common to the partition of each $B_j, j=1, 2, ..., t$; and in every parallel class of the hypercubes the set of labels associated with the points of any segment are either identical or belong to a line in the geometry $AG(i, m^{hi})$.

The following theorem which applies to Latin hypercubes requires only the case $i=1$. In this case, if the labels on the $mh$ points of some segment are not identical within a parallel class they form a line in $AG(1, mh)$. In other words all $n$ labels occur exactly once each in the segment.

As an example consider the complete set of 10 YFHC(3; 27) hypercubes in Fig. 5.3. The 27 points within a given cube are specified by a triple $(i, j, k)$, $i, j, k = 0, 1, 2$, as shown in the figure. Suppose $X = (0, 0, 0)$ and $Y = (1, 1, 1)$. Then $t=4$ and blocks $B_1, B_2, B_3, B_4$ are the 0-blocks of cubes 2, 4, 6, and 7, respectively. By inspection, the segment $S$ consists of $X$ and $Y$, and the additional point $Z = (2, 2, 2)$, since $Z$ carries label 0 in cubes 2, 4, 6, and 7, and $X$, $Y$, and $Z$ have labels 0, 1, 2 in some order in
the remaining cubes. This choice of $\overline{X}$ and $\overline{Y}$ imposes the partition of the 0-block of cube 2 into three segments $S_1$, $S_2$, $S_3$, where

$$S_1 = S = \{(0, 0, 0), (1, 1, 1), (2, 2, 2)\}$$

$$S_2 = \{(0, 1, 1), (1, 2, 2), (2, 0, 0)\}$$

$$S_3 = \{(0, 2, 2), (1, 0, 0), (2, 1, 1)\}.$$ 

Similar partitions, all containing $S$, are imposed on the 0-blocks of cubes 4, 6, and 7.

**Theorem 5.8.** A complete set of $YFHC(n; nd)$ hypercubes is equivalent to the affine geometry $AG(d, n)$ if and only if the set is $n$-partitionable.

**Proof.** Given $AG(d, n)$ Kishen constructed a complete set of $YFHC(n; nd)$ hypercubes. For this part of the proof we need only show that such a set is $n$-partitionable. After removal of $d$ parallel classes for the $d$ dimensions, Kishen's construction converts each parallel class of hyperplanes from $AG(d, n)$ to a hypercube. For two arbitrary points $\overline{X}$ and $\overline{Y}$ the intersection of all hyperplanes containing both is the line joining them. For a given line and an arbitrary parallel class of hyperplanes, either the line lies entirely within a single hyperplane of the class, or it intersects each hyperplane at exactly one point. In the first case, the $n$ points on the line all carry the same label in the hypercube derived from that parallel class. In the second case the $n$ points will have $n$ distinct labels. Just as the hyperplanes of $AG(d, n)$ give the blocks of the hypercubes, the lines of $AG(d, n)$ identify with the segments within the blocks.

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<td>120</td>
<td>102</td>
<td>012</td>
<td>012</td>
</tr>
</tbody>
</table>

$k = 0, 1, 2$

*These cubes are corrected versions of those listed in Dénes and Keedwell [7, p. 188].

**Figure 5.3**
Conversely, suppose we start with a complete set of \( n \)-partitionable hypercubes. Since the hypercubes form a design, there exists at least one block containing two arbitrary points \( \overline{X} \) and \( \overline{Y} \).

Furthermore, \( \overline{X} \) and \( \overline{Y} \) belong to a unique segment \( S \), common to all blocks containing these two points. It follows that the segment \( S \) represents the intersection of all blocks containing \( X \) and \( Y \), and since \( |S| = n \) the affine design in Theorem 4.1 becomes, in this case, \( AG(d, n) \).

**Corollary 5.9 (Bose).** \( AG(2, n) \) exists if and only if there exist \( n - 1 \) MOLS of order \( n \).

**Proof.** When \( d = 2 \), the blocks become lines of \( n \) points so that each block consists of a single segment.

As a last generalization, for \( n = m^h \) we consider the case of \( YFHC(m; n^d) \) hypercubes, the \( d \)-dimensional analogue of frequency squares. For these hypercubes to be equivalent to \( AG(hd, m) \), two conditions that parallel those of Theorem 5.1 are required. First, the \( YFHC(m; n^d) \) hypercubes must be derived from \( m \)-partitionable \( YFHC(n; n^d) \) hypercubes, and, second, the substitutions themselves must correspond to the hyperplanes of \( AG(h, m) \).

The \( m \)-partitionability comes from the case \( i = h \) in Definition 5.2; the segments consist of \( m \) points and the labels on the \( m \) points, when not identical, correspond to lines in \( AG(h, m) \).

We begin by showing that Kishen’s construction gives \( m \)-partitionable \( YFHC(n; n^d) \) hypercubes. Since his construction generalizes the common algebraic derivation of MOLS from the desarguesian plane, a similar generalization to Theorem 5.5 gives the required result. To this end, we extend the idea of matching points to \( d \) dimensions by identifying \( \overline{X} = (p_1, p_2, \ldots, p_d) \) and \( \overline{X}^* = (p_1 \cdot p_2 \cdot \ldots \cdot p_d) \) as matching points where each \( p_j, j = 1, 2, \ldots, d \), is an \( h \)-tuple over \( GF(m) \), \( \overline{X} \) is a position in the \( YFHC(n; n^d) \) hypercubes, and \( \overline{X}^* \) is a point in \( AG(dh, m) \).

**Theorem 5.10.** For \( n = m^h \), a complete set of \( YFHC(n; n^d) \) hypercubes derived by Kishen’s construction is \( m \)-partitionable.

**Proof.** Suppose \( Z^* = (r_1 \cdot r_2 \cdot \ldots \cdot r_d) \) is a point on the line joining \( X^* = (p_1 \cdot p_2 \cdot \ldots \cdot p_d) \) and \( X^*_2 = (q_1 \cdot q_2 \cdot \ldots \cdot q_d) \). Then \( r_j = \alpha p_j + (1 - \alpha) q_j, j = 1, 2, \ldots, d \) for \( \alpha \) in \( GF(m) \). In Kishen’s construction, the block containing \( \overline{X}_1 \) and \( \overline{X}_2 \) is given by an equation of the form

\[
\beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_d X_d = C,
\]
where each $\beta_j$, $j = 1, 2, \ldots, d$, is fixed and $GF(m^h)$, and at least one $\beta_j$ is non-zero. Then

$$\beta_1 r_1 + \beta_2 r_2 + \cdots + \beta_d r_d$$

$$= \alpha(\beta_1 p_1 + \beta_2 p_2 + \cdots + \beta_d p_d) + (1 - \alpha)(\beta_1 q_1 + \cdots + \beta_d q_d)$$

$$= \alpha C + (1 - \alpha) C = C,$$

so that $Z = (r_1, r_2, \ldots, r_d)$ lies in any block containing $X_1$ and $X_2$.

Suppose in another parallel class, $X_1, X_2, \text{ and } Z$ have distinct labels $C_1, C_2, \text{ and } C_3$ in $GF(m)$. Then there exists a set of parameters $\gamma_1, \gamma_2, \ldots, \gamma_d$ from $GF(m^h)$ such that

$$\gamma_1 p_1 + \gamma_2 p_2 + \cdots + \gamma_d p_d = C_1$$

$$\gamma_1 q_1 + \gamma_2 q_2 + \cdots + \gamma_d q_d = C_2$$

$$\gamma_1 r_1 + \gamma_2 r_2 + \cdots + \gamma_d r_d = C_3.$$

Then for $\alpha$ in $GF(m)$,

$$\alpha C_1 + (1 - \alpha) C_2$$

$$= \gamma_1(\alpha p_1 + (1 - \alpha) q_1) + \cdots + \gamma_d(\alpha p_d + (1 - \alpha) q_d) = C_3.$$

Thus the labels $C_1, C_2, \text{ and } C_3$ are collinear in $AG(h, m)$. In summary, if $X^*, j = 1, 2, \ldots, m$, are $m$ collinear points in $AG(dh, m)$ then the matching points $X_j, j = 1, 2, \ldots, m$, are either in the same blocks in the hypercubes (and carry the same label) or their labels form a line in $AG(h, m)$.

The next theorem extends Theorem 5.1 from two dimensions to $d$ dimensions.

**Theorem 5.11.** For $n = m^h$, a complete set of YFHC($m; n^d$) hypercubes derived from a complete set of YFHC($n; n^d$) hypercubes is equivalent to $AG(dh, m)$ if and only if the substitutions correspond to hyperplanes of $AG(h, m)$ and the YFHC($n; n^d$) hypercubes are $m$-partitionable.

**Proof.** The proof depends on showing that the intersection of all blocks in the YFHC($m; n^d$) hypercubes containing an arbitrary two points is a fixed size. As in Theorem 5.1 the condition on the substitutions guarantees that when the $m^h$ symbols of the YFHC($n; n^d$) hypercubes are mapped onto the $m$ symbols of the YFHC($m; n^d$) hypercubes, the intersection of all substitutions (i.e., $AG(h, m)$ hyperplanes) containing a given two YFHC($n; n^d$) symbols will contain exactly $m$ symbols. Combined with the partitionability condition this guarantees that the intersection of all YFHC($m; n^d$) blocks containing two arbitrary points is exactly $m$ in size.
The arguments required to establish this follow directly those of the proof of Theorem 5.1.

If we set \(d = 2\) and \(h = 1\) in this last result we again recapture Bose's result.

With \(h = 2, d\) dimensional analogues of Corollaries 5.3 and 5.4 are obtained.

**Corollary 5.12.** Every complete set of \(\text{YFHC}(m;m^{2^d})\) hypercubes derived from a complete set of \(m\)-partitionable \(\text{YFHC}(m^2;m^{2^d})\) hypercubes is equivalent to \(\text{AG}(2d,m)\).

**Corollary 5.13.** If \(m\) is not a prime power, then a complete set of \(\text{MOLS}\) of order \(m\) and complete set of \(m\)-partitionable \(\text{YFHC}(m^2;m^{2^d})\) hypercubes cannot both exist.

6. **QYFHCs, Nets, and Affine Designs**

So far our generalizations of Bose's equivalence have been obtained by proceeding from complete sets of MOLS to complete sets of mutually orthogonal hypercubes of various types. In this section we generalize by relaxing the completeness condition. The results in this section are motivated by the following, see, for example, [10, p. 3].

**Theorem 6.1.** The following are equivalent:

1. there exist \(t\) MOLS of order \(m\),
2. there exists an \((m, t + 2; 1)\)-net,
3. there exists an affine \(S_{t+2}(1, m; m^2)\) design.

We generalize as follows

**Theorem 6.2.** The following are equivalent:

1. there exist \(t\) mutually orthogonal QYFHCs of dimension \(d\), frequency \(n^{d-1}/m\), based on \(m\) symbols,
2. there exists an \((m, t + d; n^d/m^2)\)-net,
3. there exists an affine \(S_{t+d}(1, n^d/m; n^d)\) design.

**Proof.** Let \(F_1, \ldots, F_t\) be \(t\) mutually orthogonal QYFHCs of frequency \(n^{d-1}/m\) on \(m\) symbols and let \(n = \mu m\). Define \(t\) parallel classes each containing \(m\) blocks in the usual manner by letting a block \(B_i, \alpha, i = 1, \ldots, t, \alpha = 1, \ldots, m, \) be the set of points where \(F_i(x_1, \ldots, x_d) = \alpha\). We note that each block has \(n^d/m\) points and each parallel class contains \(m\) blocks which
partition $\mathcal{P}$, the set of points $(x_1, \ldots, x_d)$ with $1 \leq x_i \leq n$. Furthermore, any two nonparallel blocks intersect in $n^d/m^2$ points since the hypercubes are mutually orthogonal. Hence we obtain an incidence structure which is an $(m, t; n^d/m^2)$-net. Moreover, since each $F_i$ is a QYFHC, we can extend the net by adding $d$ more parallel classes, one for each dimension, to obtain an $(m, t + d; n^d/m^2)$-net. Hence (1) implies (2).

Let $\Pi$ be an $(m, t + d; n^d/m^2)$-net and let $\mathcal{D}_1, \ldots, \mathcal{D}_d$, $\mathcal{B}_1, \ldots, \mathcal{B}_t$ denote the $t + d$ parallel classes. The point set $\mathcal{P}$ can then be partitioned into $m^d$ sets $D_1 \cap \cdots \cap D_d$ with $D_i \in \mathcal{D}_i$ of size $n^d/m^d$. Hence we partition $\mathcal{P}$ as the set of cells $(x_1, \ldots, x_d)$, where $1 \leq x_i \leq n$ and $n = \mu m$.

For $1 \leq j \leq t$ let $\mathcal{B}_j = \{B_1^{(j)}, \ldots, B_m^{(j)}\}$ be the $j$th parallel class. We define an $n \times \cdots \times n$ array $F_j$ of dimension $d$ based on $1, \ldots, m$ by defining $F_j(x_1, \ldots, x_d) = k$ if and only if $(x_1, \ldots, x_d) \in B_k^{(j)}$. Since $\Pi$ is an $(m, t + d; n^d/m^2)$-net, $|B_k^{(j)} \cap D_i| = \cdots = |B_k^{(j)} \cap D_d| = n^d/m^2$ for $D_i \in \mathcal{D}_i$, $i = 1, \ldots, d$. Moreover, $|B_k^{(j)} \cap B_k^{(j')}| = n^d/m^d$ for $j \neq j'$. Hence each $F_i$ is a QYFHC of frequency $n^{d-1}/m$ and any $F_i$ is orthogonal to $F_j$ whenever $i \neq j$. Since (2) is equivalent to (3), the proof is complete.

As a special case when $d = 2$ we obtain the following result of Huang and Laurent [10, Theorem 3.1].

**Corollary 6.3.** The following are equivalent

1. there exist $t$ MOQFSs of frequency $n/m$ on $m$ symbols,
2. there exists an $(m, t + 2; n^2/m^2)$-net,
3. there exists an affine $S_{t+2}(1, n^2/m; n^2)$ design.

We note that in Theorem 6.1 when $t = -d + (n^d - 1)/(m - 1)$, the maximal number of mutually orthogonal QYFHCs, the resulting nets and affine 1-designs become the affine resolvable BIB designs of Sections 3 and 4. In particular, an affine $S_{(m^d-1)(m-1)}(1, m^{d-1}; m^{d-1})$ design is an $AD(m, (m^{d-2} - 1)/(m - 1))$ design.

**Acknowledgment**

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**References**


10. T. Huang and M. Laurent, $(s, r; μ)$-nets and alternating forms graphs, Discrete Math., to appear.


