Note

List coloring of Cartesian products of graphs

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Abstract

A well-established generalization of graph coloring is the concept of list coloring. In this setting, each vertex \(v\) of a graph \(G\) is assigned a list \(L(v)\) of \(k\) colors and the goal is to find a proper coloring \(c\) of \(G\) with \(c(v) \in L(v)\). The smallest integer \(k\) for which such a coloring \(c\) exists for every choice of lists is called the list chromatic number of \(G\) and denoted by \(\chi_l(G)\).

We study list colorings of Cartesian products of graphs. We show that unlike in the case of ordinary colorings, the list chromatic number of the product of two graphs \(G\) and \(H\) is not bounded by the maximum of \(\chi_l(G)\) and \(\chi_l(H)\). On the other hand, we prove that \(\chi_l(G \times H) \leq \min\{\chi_l(G) + \text{col}(H), \text{col}(G) + \chi_l(H)\} - 1\) and construct examples of graphs \(G\) and \(H\) for which our bound is tight.

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1. Introduction

List coloring is a widely studied generalization of the classical notion of graph coloring. List colorings were introduced independently by Erdős et al. [3] and Vizing [7]. Each vertex \(v\) of a graph \(G\) is equipped with a list \(L(v)\) of \(k\) colors and the goal is to find a proper coloring \(c\) of \(G\) such that \(c(v) \in L(v)\) for every vertex \(v\) of \(G\). The smallest integer \(k\) for which a coloring \(c\) exists for every list assignment \(L\) is called the list chromatic number of \(G\) and denoted by \(\chi_l(G)\). Clearly, the list chromatic number of a graph \(G\) is at least its chromatic number but it can be strictly larger.

In particular, there exist bipartite graphs with arbitrary large list chromatic number. The reader is referred to surveys [1,5,6,8] for additional results on list coloring.

In this short paper, we investigate the list chromatic number of Cartesian products of graphs. The Cartesian product \(G \times H\) of graphs \(G\) and \(H\) is a graph with vertex set \(V(G) \times V(H)\). Two vertices \((u, v)\) and \((u', v')\) are adjacent in \(G \times H\) if either \(u = u'\) and \(v v' \in E(H)\) or \(v = v'\) and \(uu' \in E(G)\) and \(v = v'\). Note that \(G \times H\) contains \(|G|\) copies of \(H\) (each corresponds to a single vertex of \(G\)) and it similarly contains \(|H|\) copies of \(G\). It is well-known that the chromatic
number of \( G \times H \) is equal to the maximum of the chromatic numbers of the graphs \( G \) and \( H \). Let us remark that for the simplicity, we write \( L(u, v) \) for the list of a vertex \((u, v)\) instead of more correct \( L((u, v))\).

One naturally asks whether an analogous statement holds for list coloring of graphs. Since the product of \( K_{1,2} \) and \( K_{1,4} \) contains the complete bipartite graph \( K_{2,4} \) and \( \chi_l(K_{2,4}) = 3 \), the statement can hold with the maximum of the list chromatic numbers. Hence, one can at least ask whether the list chromatic number of \( G \times H \) can be bounded by \( \max\{\chi_l(G), \chi_l(H)\} + C \) for a constant \( C \) (or even for \( C = 1 \)). We show that even such a statement is false by constructing graphs \( G \) with \( \chi_l(G \times G) = 2\chi_l(G) - 1 \).

Another graph parameter closely related to the chromatic number and the list chromatic number is the coloring number. The coloring number \( \text{col}(G) \) of a graph \( G \) is the smallest integer \( d \) for which there exists an ordering \( v_1, \ldots, v_n \) of the vertices of \( G \) such that each vertex \( v_i \) has at most \( d - 1 \) neighbors among the vertices \( v_1, \ldots, v_{i-1} \). A graph \( G \) with \( \text{col}(G) = d \) is also called \((d - 1)\)-degenerate. Clearly, \( \chi_l(G) \leq \text{col}(G) \) and \( \chi_l(C) \leq \text{col}(G) \). Our main result is the following upper bound on the list chromatic number of the Cartesian product of two graphs \( G \) and \( H \):

\[
\chi_l(G \times H) \leq \min\{\chi_l(G) + \text{col}(H), \text{col}(G) + \chi_l(H)\} - 1.
\]

The bound can be generalized to products of more graphs (see Corollary 2). In Section 3, we show that this bound cannot be improved. In particular, for every pair of positive integers \( k \) and \( \ell \), there exist a graph \( G \) with \( \chi_l(G) = \text{col}(G) = k \) and a graph \( H \) with \( \chi_l(H) = \text{col}(H) = \ell \) such that \( \chi_l(G \times H) = k + \ell - 1 \).

2. Upper bound

We start with establishing our upper bound on the list chromatic number of the Cartesian product of graphs.

**Theorem 1.** Let \( G \) and \( H \) be two graphs. The list chromatic number \( \chi_l(G \times H) \) of the Cartesian product \( G \times H \) can be bounded as follows:

\[
\chi_l(G \times H) \leq \min\{\chi_l(G) + \text{col}(H), \text{col}(G) + \chi_l(H)\} - 1.
\]

**Proof.** By symmetry, it is enough to prove that \( \chi_l(G \times H) \leq \chi_l(G) + \text{col}(H) - 1 \). Let \( v_1, \ldots, v_n \) be the vertices of \( H \) ordered in such a way that each vertex \( v_i \) has at most \( \text{col}(H) - 1 \) neighbors among the vertices \( v_1, \ldots, v_{i-1} \). Let \( V_i \) be the vertices of \( G \times H \) that are contained in the copy of \( G \) corresponding to a vertex \( v_i \).

Fix a list assignment \( L \) for \( G \times H \) such that \( |L(v)| = \chi_l(G) + \text{col}(H) - 1 \) for every vertex \( v \in V(G \times H) \). We construct a proper coloring \( c \) of \( G \times H \) with \( c(v) \in L(v) \) for every \( v \in V(G \times H) \). First, color the subgraph of \( G \times H \) induced by \( V_1 \). Since this subgraph is isomorphic to \( G \) and each vertex has a list of size at least \( \chi_l(G) \), such a coloring exists.

Assume that we have already constructed a proper coloring \( c \) of the subgraph of \( G \times H \) induced by \( V_1 \cup \cdots \cup V_{i-1} \). We now extend the coloring \( c \) to the vertices of \( V_i \). First, remove from the list \( L(v) \) of each vertex \( v \) of \( V_i \) the colors of its neighbors among the vertices contained in \( V_1 \cup \cdots \cup V_{i-1} \). Since the vertex \( v \) has at most \( \text{col}(H) - 1 \) such neighbors (one neighbor for each neighbor of \( v_i \) that precedes \( v_i \) in the ordering of the vertices of \( H \)), the new list \( L'(v) \) has size at least \( \chi_l(G) \). Since the list chromatic number of \( G \) is \( \chi_l(G) \), the copy of \( G \) induced in \( G \times H \) by the vertices of \( V_i \) can be colored from the new lists. In this way, the coloring is eventually extended to the entire graph \( G \times H \). \( \square \)

An immediate corollary of Theorem 1 is the following upper bound on the list chromatic number of the Cartesian product of several graphs:

**Corollary 2.** If \( G_1, \ldots, G_k \) are graphs, then the following holds:

\[
\chi_l(G_1 \times \cdots \times G_k) \leq \chi_l(G_1) + \text{col}(G_2) + \cdots + \text{col}(G_k) - (k - 1).
\]

3. Lower bound

In this section, we show that there exists a graph \( G \) with \( \text{col}(G) = \chi_l(G) \) such that \( \chi_l(G \times G) = \text{col}(G) + \chi_l(G) - 1 \). Let us start with the following lemma:
Theorem 5. For every pair of positive integers $k$ and $t$, \[ \chi_t(G + H) = 2^{\chi_t(G)} - 1. \]

Proof. Let $H$ be the Cartesian product of $G$ and $K_{k,t}$. Fix a list assignment $L$ that assigns each vertex of $G$ a set of $\chi_t(G) - 1$ colors such that the vertices of $G$ cannot be properly colored from their lists. Let $L_0$ be the union of all the lists $L(v)$ and let $X$ be the smaller part of $K_{k,t}$ and $Y$ the larger one. Finally, let $X_H$ be the vertices of $H$ that are contained in the copies of $G$ corresponding to the vertices of $X$. Note that $|X_H| = kn$.

We now construct a list assignment $L_H$ from which $H$ cannot be colored. The lists $L_H(v, x), v \in V(G)$ and $x \in X$, i.e., the lists of the vertices of $X_H$, are disjoint sets of $k + \chi_t(G) - 1$ that are distinct from the colors of $L_0$. Next, associate with each of the (at most $t = (k + \chi_t(G) - 1)^{kn}$) colorings $c$ of the vertices of $X_H$ a vertex $y_c$ of $Y$. The list $L_H(v, y_c)$ is the union of the lists $L(v)$ and the set of $k$ colors assigned to the $k$ neighbors of the vertex $(v, y_c)$ in $X_H$. Observe that the size of the list $L_H(w)$ is $k + \chi_t(G) - 1$ for every vertex $w \in V(H)$. We show that $H$ cannot be colored from the lists $L_H$.

Assume that there exists a coloring $c_H$ of $H$ such that $c_H(v) \in L_H(v)$ for every $v \in V(H)$. Let $c$ be the restriction of $c_H$ to the vertices of $X_H$. Observe now that $c_H(v, y_c) \in L(v)$ for every vertex $v$; indeed, $c_H(v, y_c)$ cannot be any of the $k$ colors assigned to the neighbors of $(v, y_c)$ in $X_H$. Since these $k$ colors are precisely the $k$ colors of $L_H(v, y_c)\setminus L_0(v)$, it follows that $c_H(v, y_c) \in L(v)$. Hence, the coloring $c_H$ restricted to the copy of $G$ corresponding to the vertex $y_c$ in $H$ is a proper coloring of $G$ from the lists $L$. This contradicts the choice of the list assignment $L$.

Since $H$ cannot be colored from the lists $L_H$, $\chi_t(H) > \chi_t(G) + k - 1$. Since the graph $K_{k,t}$ is $k$-degenerate, its coloring number is $k + 1$ and $\chi_t(H) \leq \chi_t(G) + k$ by Theorem 1. Hence, $\chi_t(H) = \chi_t(G) + k$. \[ \square \]

We now apply Lemma 3 to obtain our lower bound:

Theorem 4. Let $G$ be the complete bipartite graph with parts of sizes $k$ and $(2k)^{k(k+1)}$. Both the list chromatic number and the coloring number of $G$ are equal to $k + 1$ and the list chromatic number of the Cartesian product of two copies of $G$ is $2k + 1 = \chi_t(G) + \text{col}(G) - 1$.

Proof. Let $G'$ be a subgraph of $G$ that is isomorphic to the complete bipartite graph with parts of sizes $k$ and $k^k$. It is well-known that $\chi_t(G') = k + 1$ (one way to observe this is to apply Lemma 3 to the product of $K_1$ and $G' \cong K_{k,k}$).

Now apply Lemma 3 to the product of the graph $G'$ and the graph $G \cong K_{k,t}$ with $t = (2k)^{k(G')}$. It follows that $\chi_t(G' \times G) = \chi_t(G') + k = 2k + 1$. Hence, $\chi_t(G \times G) \geq 2k + 1$.

Since $G$ is $k$-degenerate, it holds that $\text{col}(G) \leq k + 1$ and consequently $\chi_t(G) \leq k + 1$. In particular, $\chi_t(G) = \chi_t(G) = k + 1$. By Theorem 1, $\chi_t(G \times G) \leq \text{col}(G) + \chi_t(G) - 1 = 2k + 1$. Since $G' \times G$ is a subgraph of $G \times G$, we conclude that $\chi_t(G \times G) = 2k + 1$. \[ \square \]

An analogous argument yields the following:

Theorem 5. For every pair of positive integers $k$ and $\ell$, there exist graphs $G$ and $H$ such that $\chi_t(G) = \text{col}(G) = k$, $\chi_t(H) = \text{col}(H) = \ell$ and $\chi_t(G \times H) = k + \ell - 1$.

Proof. Let $k' = k - 1$ and $\ell' = \ell - 1$. Proceed as in the proof of Theorem 4 with the pair of graphs $G = K_{k',s}$ and $H = K_{\ell',t}$ with $s = (k' + \ell')^{k'(\ell' + k')}$ and $t = (k' + \ell')^{k'(\ell' + k')}$.

\[ \square \]

4. Open problems

We have initiated study of the list chromatic number of the Cartesian product of two graphs. Our original motivation was the question whether the list chromatic number $\chi_t(G \times H)$ of two graphs $G$ and $H$ could be bounded by $\max\{\chi_t(G), \chi_t(H)\}$ as in the case of usual colorings. We have shown that this does not hold for list colorings, in particular, $\chi_t(G \times G) = 2\chi_t(G) - 1$ for the graph $G$ constructed in Theorem 4. However, $\chi_t(G \times H)$ can be bounded by a function of $\chi_t(G)$ and $\chi_t(H)$: by the result of Alon [2,1], the coloring number of $G$ does not exceed $2^{O(\chi_t(G))}$. Similarly, $\text{col}(H) \leq 2^{O(\chi_t(H))}$. Hence, $\text{col}(G \times H) \leq \min\{\chi_t(G) + 2^{O(\chi_t(H))}, \chi_t(H) + 2^{O(\chi_t(G))}\}$. However, we suspect that a much better upper bound can be established:
Conjecture 6. There exists a constant $A$ such that the following holds for every pair of graphs $G$ and $H$:

$$\chi_l(G \times H) \leq A(\chi_l(G) + \chi_l(H)).$$

Also note that if the $(am, bm)$-conjecture of Erdős et al. [3] is true, then $\chi_l(G \times H) \leq \chi_l(G)\chi_l(H)$.

Another problem is to bound the list chromatic number of $G \times H$ in terms of the maximum degrees of $G$ and $H$. If $G$ and $H$ are complete graphs of orders $a$ and $b$, then $G \times H$ is isomorphic to the line graph of the complete bipartite graph with parts of sizes $a$ and $b$. Kahn [4] showed that the list chromatic number of the line-graph of a graph with maximum degree $\Delta$ does not exceed $\Delta + o(\Delta)$. In particular, $\chi_l(K_a \times K_b) = \max\{a, b\} + o(a + b)$. This leads us to the following problem:

**Conjecture 7.** Let $G$ and $H$ be two graphs with maximum degree at most $\Delta$. The list chromatic number of $G \times H$ does not exceed $\Delta + o(\Delta)$.

References