Pergamon

Appl. Math. Lett. Vol. 10, No. 6, pp. 129–133, 1997

0893-9659/97 $17.00 + 0.00

PII: S0893-9659(97)00116-X

A New Method for a Class of Nonlinear Variational Inequalities with Fuzzy Mappings

NAN-JING HUANG
Department of Mathematics, Sichuan University
Chengdu, Sichuan 610064, P.R. China

(Received December 1995; revised and accepted May 1996)

Abstract—In this paper, we construct a new iterative algorithm of solution for a new class of nonlinear variational inequalities with fuzzy mappings and give some convergence analysis of iterative sequences generated by algorithm.

Keywords—Variational inequality, Fuzzy mapping, Algorithm, Convergence.

1. INTRODUCTION

Variational inequalities not only have stimulated new results dealing with nonlinear partial differential equations, but also have been used in a large variety of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium, and engineering sciences, etc. (See [1–6] and the references cited therein.)

Recently, the variational inequalities for fuzzy mappings was introduced and studied by Chang and Zhu [7] and later developed by Noor [8], Chang [9], Chang and Huang [10,11] and Lee *et al.* [12]. On the other hand, in the recent papers [13–16], He proposed some new projection and contraction methods for a class of linear or nonlinear variational inequalities with monotone mappings.

The purpose of this paper is to construct a new iterative algorithm of solution for a new class of nonlinear variational inequalities with fuzzy mappings. We also give some convergence analysis of iterative sequences generated by algorithm. Our results develop some projection and contraction methods for solving monotone variational inequalities.

2. PRELIMINARIES

Let $\mathcal{F}(R^n)$ be the family of all fuzzy sets over R^n. A mapping F from R^n into $\mathcal{F}(R^n)$ is called fuzzy mapping over R^n. If F is a fuzzy mapping over R^n, the $F(x)$ (denoted by F_x in the sequel) is a fuzzy set over R^n, and $F_x(y)$ is the membership function of the point y in F_x. Let $B \in \mathcal{F}(R^n)$, $p \in [0,1]$. Then $(B)_p = \{x \in R^n : B(x) \geq p\}$ is called a p-cut set of B.

A set-valued mapping $A : R^n \to 2^{R^n}$ (where 2^{R^n} denotes the family of all nonempty subsets of R^n) is called to be upper semicontinuous [17], if for any $\{u_k\} \subset R^n$, $y_k \in A(u_k)$,

$$u_k \to u_*, \qquad y_k \to y_* \Longrightarrow y_* \in A(u_*).$$

The author is very thankful to the referees for their valuable suggestions.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Let $F : R^n \to \mathcal{F}(R^n)$ and $p : R^n \to [0,1]$. We can define a set-valued mapping $\widetilde{F}$ as follows:

$$\widetilde{F} : R^n \to 2^{R^n}, \qquad x \mapsto (F_x)_{p(x)}.$$

In the sequel, we shall call $\widetilde{F}$ the set-valued mapping induced by fuzzy mapping F.

A fuzzy mapping $F : R^n \to \mathcal{F}(R^n)$ is called to be upper semicontinuous monotone, if $\widetilde{F}$ is upper semicontinuous monotone.

Let $K \subset R^n$ be a closed convex set, $F : R^n \to \mathcal{F}(R^n)$ and $p : R^n \to [0,1]$. We consider the following problem: find $u \in K$, $y \in R^n$ such that $F_u(y) \geq p(u)$ and

$$(v-u)^\top y \geq 0, \qquad \forall\, v \in K, \tag{2.1}$$

which is called a nonlinear variational inequality for fuzzy mapping.

If $F : R^n \to 2^{R^n}$ and $p(x) \equiv 1$, $\forall\, x \in R^n$, then the problem (2.1) is equivalent to finding $u \in K$, $y \in F(u)$ such that

$$(v-u)^\top y \geq 0, \qquad \forall\, v \in K, \tag{2.2}$$

which is called a set-valued nonlinear variational inequality. Obviously, the problems (2.1) and (2.2) include many variational inequalities as special cases.

Let S denote the solution set of (2.1) and $P_K(\cdot)$ denote the projection to K. Throughout this paper, we assume that $S \neq \emptyset$. In following the $\|\cdot\|$, G and $\|u\|_G$ will be denoted the Euclidean norm, positive definite matrix and $(u^\top G u)^{1/2}$ respectively.

3. ITERATIVE ALGORITHM

Let $\beta > 0$ be a constant, $F : R^n \to \mathcal{F}(R^n)$ and $p : R^n \to [0,1]$. By [11], we know that $u \in K$, $y \in R^n$ are the solutions of problem (2.1) if and only if $u \in K$, $y \in R^n$ such that $F_u(y) \geq p(u)$ and

$$e(u,y,\beta) \stackrel{\text{def}}{=} u - P_K[u - \beta y] = 0. \tag{3.1}$$

Let $\gamma \in (0,2)$ be a constant and F be a upper semicontinuous monotone fuzzy mapping. Since F is monotone, we know that $\widetilde{F}$ is also monotone. By [18], $(I + \beta\widetilde{F})^{-1}$ is a single-valued nonexpansive mapping.

Now, we give the algorithm of solution for the problem (2.1) as follows.

ALGORITHM 3.1. Given $u^0 \in R^n$, $y^0 \in \widetilde{F}(u^0)$, for $k = 0,1,2,\ldots$, if $(u^k, y^k) \notin S$ and $u^k + \beta y^k - \gamma\rho_k G^{-1} e(u^k, y^k, \beta)$ is in the image of $I + \beta\widetilde{F}$, then

$$u^{k+1} = \left(I + \beta\widetilde{F}\right)^{-1}\left(u^k + \beta y^k - \gamma\rho_k G^{-1} e\left(u^k, y^k, \beta\right)\right), \tag{3.2}$$

and $y^{k+1} \in \widetilde{F}(u^{k+1})$ such that

$$u^{k+1} + \beta y^{k+1} = u^k + \beta y^k - \gamma\rho_k G^{-1} e\left(u^k, y^k, \beta\right), \tag{3.3}$$

where

$$\rho_k = \frac{\left\|e\left(u^k, y^k, \beta\right)\right\|^2}{e\left(u^k, y^k, \beta\right)^\top G^{-1} e\left(u^k, y^k, \beta\right)}. \tag{3.4}$$

REMARK 3.1.

(i) If F is a set-valued mapping, then Algorithm 3.1 reduces to a new iterative algorithm of solution for the set-valued nonlinear variational inequality (2.2).

(ii) If F is a single-valued mapping, then Algorithm 3.1 reduces to the PC methods for monotone variational inequalities of [13].

(iii) If F is a single-valued mapping, $G = I$ and $\gamma = 1$, then Algorithm 3.1 reduces to the Douglas-Rachford operator splitting methods (see [13,19,20]).

(iv) In addition, if F is a single-valued mapping, $G = I, \gamma = 1$, and $K = R^n$, then Algorithm 3.1 reduces the the Levenberg-Marquardt method for unconstrained optimization [13,16].

4. CONVERGENCE

We first give the following lemma.

LEMMA 4.1. *Let $(u^*, y^*) \in S$, then*

$$(u - u^* + \beta(y - y^*))^\top e(u, y, \beta) \geq \|e(u, y, \beta)\|^2 + \beta(u - u^*)^\top (y - y^*), \quad \forall\, u \in R^n, \qquad \forall\, y \in \widetilde{F}(u). \tag{4.1}$$

PROOF. Since (u^*, y^*) is a solution of the problem (2.1) and $P_K(\cdot) \in K$, it follows from (2.1) that

$$\beta y^{*\top}(P_K(u - \beta y) - u^*) \geq 0, \qquad \forall\, u \in R^n, \quad \forall\, y \in \widetilde{F}(u). \tag{4.2}$$

A well know basic property of the projection mapping is

$$(v - P_K v)^\top (P_K v - u) \geq 0, \qquad \forall\, v \in R^n, \quad \forall\, u \in K. \tag{4.3}$$

Letting $v := u - \beta y$ and $u := u^*$ in (4.3), we have

$$(e(u, y, \beta) - \beta y)^\top (P_K(u - \beta y) - u^*) \geq 0. \tag{4.4}$$

Adding (4.2) and (4.4), we obtain

$$(e(u, y, \beta) - \beta(y - y^*))^\top (u - u^* - e(u, y, \beta)) \geq 0. \tag{4.5}$$

From the above inequality follows the assertion of this lemma immediately. This completes the proof.

THEOREM 4.1. *The sequences $\{u^k\}$ and $\{y^k\}$ generated by Algorithm 3.1 satisfy the following inequality:*

$$\begin{aligned} \left\|u^{k+1} - u^* + \beta\left(y^{k+1} - y^*\right)\right\|_G^2 \leq{} & \left\|u^k - u^* + \beta\left(y^k - y^*\right)\right\|_G^2 \\ & - \gamma(2 - \gamma)\rho_k \left\|e\left(u^k, y^k, \beta\right)\right\|^2 \\ & - 2\gamma\rho_k\beta\left(u^k - u^*\right)^\top \left(y^k - y^*\right), \qquad \forall\, (u^*, y^*) \in S. \end{aligned} \tag{4.6}$$

PROOF. From (3.2) and (3.3), we have

$$u^{k+1} - u^k + \beta\left(y^{k+1} - y^k\right) = -\gamma\rho_k G^{-1} e\left(u^k, y^k, \beta\right). \tag{4.7}$$

It follows from (3.4), (4.1), and (4.7) that

$$\begin{aligned} \left\|u^{k+1} - u^* + \beta\left(y^{k+1} - y^*\right)\right\|_G^2 &= \left\|u^k - u^* + \beta\left(y^k - y^*\right) - \gamma\rho_k G^{-1} e(u^k, y^k, \beta)\right\|_G^2 \\ &= \left\|u^k - u^* + \beta\left(y^k - y^*\right)\right\|_G^2 \\ &\quad - 2\gamma\rho_k\left(u^k - u^* + \beta\left(y^k - y^*\right)\right)^\top e\left(u^k, y^k, \beta\right) \\ &\quad + \gamma^2\rho_k^2 e\left(u^k, y^k, \beta\right)^\top G^{-1} e\left(u^k, y^k, \beta\right) \\ &\leq \left\|u^k - u^* + \beta\left(y^k - y^*\right)\right\|_G^2 \\ &\quad - \gamma(2 - \gamma)\rho_k \|e(u^k, y^k, \beta)\|^2 \\ &\quad - 2\gamma\rho_k\beta\left(u^k - u^*\right)^\top \left(y^k - y^*\right), \qquad \forall\, (u^*, y^*) \in S. \end{aligned}$$

This completes the proof.

THEOREM 4.2. *The sequences $\{u^k\}$ and $\{y^k\}$ generated by Algorithm 3.1 converge to $\bar{u}$ and $\bar{y}$, respectively, and $(\bar{u}, \bar{y})$ is a solution of the problem (2.1).*

PROOF. Let $(\hat{u}, \hat{y})$ be a solution of the problem (2.1). Since $\widetilde{F}$ is monotone, it follows from (4.6) that the sequences $\{u^k\}$ and $\{y^k\}$ are bounded. Also from (4.6), we have

$$\sum_{k=0}^{\infty} \gamma(2-\gamma)\rho_k \left\|e\left(u^k, y^k, \beta\right)\right\|^2 \leq \left\|u^0 - \hat{u} + \beta\left(y^0 - \hat{y}\right)\right\|_G^2 .$$

This implies that

$$\lim_{k\to\infty} e\left(u^k, y^k, \beta\right) = 0.$$

Let $\bar{u}$ be a cluster point of $\{u^k\}$, the subsequence $\{u^{k_j}\}$ converge to $\bar{u}$, and $\bar{y}$ be a cluster point of $\{y^k\}$, the subsequence $\{y^{k_j}\}$ converge to $\bar{y}$. Since $P_K(\cdot)$ is continuous, (3.1) shows that $e(u, y, \beta)$ is continuous, and therefore,

$$e(\bar{u}, \bar{y}, \beta) = \lim_{j\to\infty} e\left(u^{k_j}, y^{k_j}, \beta\right) = 0. \tag{4.8}$$

Since $y^k \in \widetilde{F}(u^k)$, $u^{k_j} \to \bar{u}$, $y^{k_j} \to \bar{y}$, and $\widetilde{F}$ is upper semicontinuous, we have $\bar{y} \in \widetilde{F}(\bar{u})$ and so (4.8) implies that $(\bar{u}, \bar{y})$ is a solution of the problem (2.1).

Now, we prove that $u^k \to \bar{u}$ and $y^k \to \bar{y}$. In fact, from $(\bar{u}, \bar{y}) \in S$ and (4.6), we get

$$\left\|u^{k+1} - \bar{u} + \beta\left(y^{k+1} - \bar{y}\right)\right\|_G^2 \leq \left\|u^k - \bar{u} + \beta\left(y^k - \bar{y}\right)\right\|_G^2 ,$$

and this yields that

$$\lim_{k\to\infty} u^k = \bar{u}, \qquad \lim_{k\to\infty} y^k = \bar{y}.$$

This completes the proof.

REFERENCES

1. C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities, Application to Free Boundary Problems*, Wiley, New York, (1984).
2. A. Bensoussan and J.L. Lions, *Impulse Control and Quasivariational Inequalities*, Gauthiers-Villers, Bordas, Paris, (1984).
3. Shih-sen Chang, *Variational Inequality and Complementarity Problem Theory with Applications*, Shanghai Scientific and Tech. Literature Publishing House, Shanghai, (1991).
4. T. Harker and J.P. Pang, Finite-dimensional variational inequality and nonlinear complementarity problems: A survey of theory algorithms and applications, *Math. Programming* **48**, 161–220, (1990).
5. U. Mosco, Implicit variational problems and quasi-variational inequalities, Lecture Notes in Mathematics, Volume 543, pp. 83–156, Springer-Verlag, Berlin, (1976).
6. M.A. Noor, K.I. Noor and T.M. Rassias, Some aspects of variational inequalities, *J. Comput. Appl. Math.* **47**, 285–312, (1993).
7. Shih-sen Chang and Yuan-guo Zhu, On variational inequalities for fuzzy mappings, *Fuzzy Sets and Systems* **32**, 359–367, (1989).
8. M.A. Noor, Variational inequalities for fuzzy mappings (I), *Fuzzy Sets and Systems* **55**, 309–312, (1993).
9. Shih-sen Chang, Coincidence theorems and variational inequalities for fuzzy mappings, *Fuzzy Sets and Systems* **61**, 359–368, (1994).
10. Shih-sen Chang and Nan-jing Huang, Generalized complementarity problems for fuzzy mappings, *Fuzzy Sets and Systems* **55**, 227–234, (1993).
11. Shih-sen Chang and Nan-jing Huang, Generalized quasi-complementarity problems for a pair of fuzzy mappings, *Fuzzy Sets and Systems* (to appear).
12. G.M. Lee, D.S. Kim, B.S. Lee and S.J. Cho, Generalized vector variational inequality and fuzzy extension, *Appl. Math. Lett.* **6** (6), 47–51, (1993).
13. B.S. He, A class of new methods for monotone variational inequalities, Reports of the Institute of Mathematics, Nanjing University, (1995).
14. B.S. He, A projection and contraction methods for a class of linear complementarity problem and its application in convex quadratic programming, *Appl. Math. Optim.* **25**, 247–262, (1992).
15. B.S. He, A new method for a class of linear variational inequalities, *Math. Programming* **66**, 137–144, (1994).

16. B.S. He, Solving a class of linear projection equations, *Numerische Mathematik* **68**, 71–80, (1994).
17. S. Kakutani, A generalization of Brouwer's fixed point theorem, *Duke Math. J.* **8**, 457–459, (1941).
18. D. Pascali and S. Sburlan, *Nonlinear Mappings of Monotone Type*, Sijthoff and Noordhoff International Publishers, Bucaresti, (1978).
19. J. Douglas and H.H. Rachford, On the numerical solution of the heat conduction problem in 2 and 3 space variables, *Trans. Amer. Math. Soc.* **82**, 421–439, (1956).
20. P.L. Lions and B. Mercier, Splitting algorithms for the sum of two nonlinear operators, *SIAM J. Numerical Anal.* **16**, 964–979, (1979).