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On a new integral-type operator from the Bloch space to Bloch-type spaces on the unit ball

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A R T I C L E I N F O

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A B S T R A C T

We introduce the following integral-type operator on the space $H(\mathbb{B})$ of all holomorphic functions on the unit ball $\mathbb{B} \subset \mathbb{C}^n$

$$P_\varphi^g(f)(z) = \int_0^1 f\big(\varphi(tz)\big)g(tz)\frac{dt}{t}, \quad z \in \mathbb{B},$$

where $g \in H(\mathbb{B})$, $g(0) = 0$ and φ is a holomorphic self-map of $\mathbb{B}$. The boundedness and compactness of the operator from the Bloch space $\mathcal{B}$ or the little Bloch space $\mathcal{B}_0$ to the Bloch-type space $\mathcal{B}_\mu$ or the little Bloch-type space $\mathcal{B}_{\mu,0}$, are characterized. In the main results we calculate the essential norm of the operators $P_\varphi^g : \mathcal{B}$ (or $\mathcal{B}_0$) $\to \mathcal{B}_\mu$ (or $\mathcal{B}_{\mu,0}$) in an elegant way.

1. Introduction and preliminaries

Let $\mathbb{B}$ be the open unit ball in $\mathbb{C}^n$, $\mathbb{D}$ the unit disk in $\mathbb{C}$, $H(\mathbb{B})$ the class of all holomorphic functions on the unit ball and $H^\infty = H^\infty(\mathbb{B})$ the space of all bounded holomorphic functions on $\mathbb{B}$ with the norm

$$\|f\|_\infty = \sup_{z\in\mathbb{B}}\big|f(z)\big|.$$

Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in $\mathbb{C}^n$, $\langle z, w\rangle = \sum_{k=1}^n z_k\bar{w}_k$ and $|z| = \sqrt{\langle z, z\rangle}$.

For $f \in H(\mathbb{B})$ with the Taylor expansion $f(z) = \sum_{|\beta|\geqslant 0} a_\beta z^\beta$, let

$$\Re f(z) = \sum_{|\beta|\geqslant 0} |\beta| a_\beta z^\beta$$

be the radial derivative of f, where $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ is a multi-index, $|\beta| = \beta_1 + \cdots + \beta_n$ and $z^\beta = z_1^{\beta_1}\cdots z_n^{\beta_n}$ (see [29]).

A positive continuous function μ on $[0, 1)$ is called normal [30] if there is $\delta \in [0, 1)$ and a and b, $0 < a < b$ such that

$$\frac{\mu(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1) \quad\text{and}\quad \lim_{r\to 1}\frac{\mu(r)}{(1-r)^a} = 0,$$
$$\frac{\mu(r)}{(1-r)^b} \text{ is increasing on } [\delta, 1) \quad\text{and}\quad \lim_{r\to 1}\frac{\mu(r)}{(1-r)^b} = \infty.$$

If we say that a function $\mu : \mathbb{B} \to [0, \infty)$ is normal we will also assume that it is radial, that is, $\mu(z) = \mu(|z|)$, $z \in \mathbb{B}$.

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The weighted space $H_\mu^\infty = H_\mu^\infty(\mathbb{B})$ consists of all $f \in H(\mathbb{B})$ such that

$$\sup_{z\in\mathbb{B}} \mu(z)\left|f(z)\right| < \infty,$$

where μ is normal. For $\mu(z) = (1-|z|^2)^\beta$, $\beta > 0$ we obtain the (classical) weighted space $H_\beta^\infty = H_\beta^\infty(\mathbb{B})$.

The little weighted space $H_{\mu,0}^\infty = H_{\mu,0}^\infty(\mathbb{B})$ is a subspace of H_μ^∞ consisting of all $f \in H(\mathbb{B})$ such that

$$\lim_{|z|\to 1} \mu(z)\left|f(z)\right| = 0.$$

The Bloch-type space, denoted by $\mathcal{B}_\mu = \mathcal{B}_\mu(\mathbb{B})$, consists of all $f \in H(\mathbb{B})$ such that

$$B_\mu(f) = \sup_{z\in\mathbb{B}} \mu(z)\left|\Re f(z)\right| < \infty,$$

where μ is normal. With the norm

$$\|f\|_{\mathcal{B}_\mu} = \left|f(0)\right| + B_\mu(f)$$

the Bloch-type space becomes a Banach space.

The α-Bloch space $\mathcal{B}^\alpha$ is obtained for $\mu(z) = (1-|z|^2)^\alpha$, $\alpha \in (0,\infty)$ (see, e.g., [26,35,38] and references therein).

The little Bloch-type space $\mathcal{B}_{\mu,0}$ is a subspace of $\mathcal{B}_\mu$ consisting of those f such that

$$\lim_{|z|\to 1} \mu(z)\left|\Re f(z)\right| = 0.$$

Bearing in mind the following asymptotic relation from [36] (see also [8] for the case of the α-Bloch space)

$$b_\mu(f) := \sup_{z\in\mathbb{B}} \mu(z)\left|\nabla f(z)\right| \asymp \sup_{z\in\mathbb{B}} \mu(z)\left|\Re f(z)\right| \tag{1}$$

we see that $\mathcal{B}_\mu$ can be defined as the class of all $f \in H(\mathbb{B})$ such that $b_\mu(f)$ is finite. Also the little Bloch-type space is equivalent with the subspace of $\mathcal{B}_\mu$ consisting of all $f \in H(\mathbb{B})$ such that

$$\lim_{|z|\to 1} \mu(z)\left|\nabla f(z)\right| = 0.$$

From this observation and for some technical benefits, for the norm of the α-Bloch space we choose the second definition, that is, $f \in \mathcal{B}^\alpha$ if and only if

$$\|f\|_{\mathcal{B}^\alpha} = \left|f(0)\right| + \sup_{z\in\mathbb{B}} \left(1-|z|^2\right)^\alpha \left|\nabla f(z)\right| < \infty.$$

If $\mu(z) = (1-|z|^2)$, then the quantity $b_\mu(f)$ in (1), will be denoted by $b(f)$.

Let φ be a holomorphic self-map of $\mathbb{B}$. For $f \in H(\mathbb{B})$ the composition operator is defined by $C_\varphi f(z) = f(\varphi(z))$ (see, e.g., the monograph [9] or recent papers [10,18,27,37]).

Let $g \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. Products of integral and composition operators on $H(\mathbb{D})$ were introduced by S. Li and S. Stević in a private communication (see, e.g., [23,24] and [25], as well as related paper [20]) as follows

$$C_\varphi J_g f(z) = \int_0^{\varphi(z)} f(\zeta)g(\zeta)\,d\zeta \quad \text{and} \quad J_g C_\varphi f(z) = \int_0^z f\big(\varphi(\zeta)\big)g(\zeta)\,d\zeta. \tag{2}$$

Operators in (2) are extensions of the following integral operator

$$T_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)\,d\zeta$$

which was introduced in [28]. Some other results on the operator T_g can be found, e.g., in [1–3,31]. For some results on n-dimensional extensions of the operator, see [4–7,11–17,19,21,22,32–34,36] and references therein.

One of the interesting questions is to extend operators in (2) in the unit ball settings and to study their function theoretic properties between spaces of holomorphic functions on the unit ball in terms of inducing functions.

Assume that $g \in H(\mathbb{B})$, $g(0) = 0$ and φ is a holomorphic self-map of $\mathbb{B}$, then we introduce the following operator on the unit ball

$$P_\varphi^g(f)(z) = \int_0^1 f\big(\varphi(tz)\big)g(tz)\frac{dt}{t}, \quad f \in H(\mathbb{B}),\ z \in \mathbb{B}. \tag{3}$$

If $n=1$, then $g\in H(\mathbb{D})$ and $g(0)=0$, so that $g(z)=zg_0(z)$, for some $g_0\in H(\mathbb{D})$. By the change of variable $\zeta=tz$, it follows that

$$P_\varphi^g f(z)=\int_0^1 f\big(\varphi(tz)\big)tzg_0(tz)\frac{dt}{t}=\int_0^z f\big(\varphi(\zeta)\big)g_0(\zeta)\,d\zeta.$$

Thus operator (3) is a natural extension of the second operator in (2).

Here we study the boundedness and compactness of operator P_φ^g from the Bloch space $\mathcal{B}$ or the little Bloch space $\mathcal{B}_0$ to the Bloch-type space $\mathcal{B}_\mu$ or the little Bloch-type space $\mathcal{B}_{\mu,0}$. In Sections 4 and 5 we calculate the essential norm of the operators $P_\varphi^g:\mathcal{B}$ (or $\mathcal{B}_0$) $\to\mathcal{B}_\mu$ (or $\mathcal{B}_{\mu,0}$).

Throughout the paper C will denote a positive constant not necessarily the same at each occurrence. The notation $A\asymp B$ means that there is a positive constant C such that $A/C\leqslant B\leqslant CA$.

The following lemmas are used in the proofs of the main results.

Lemma 1. *Suppose $g\in H(\mathbb{B})$, $g(0)=0$, μ is normal and φ is a holomorphic self-map of $\mathbb{B}$. Then the operator $P_\varphi^g:\mathcal{B}$ (or $\mathcal{B}_0$) $\to\mathcal{B}_\mu$ is compact if and only if $P_\varphi^g:\mathcal{B}$ (or $\mathcal{B}_0$) $\to\mathcal{B}_\mu$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{B}$ (or $\mathcal{B}_0$) converging to zero uniformly on compacts of $\mathbb{B}$, we have $\|P_\varphi^g f_k\|_{\mathcal{B}_\mu}\to 0$ as $k\to\infty$.*

The proof of Lemma 1 follows by standard arguments (see, for example, the proofs of Proposition 3.11 in [9] and Lemma 3 in [34]). Hence, we omit its proof.

Lemma 2. *Suppose $f,g\in H(\mathbb{B})$ and $g(0)=0$. Then*

$$\Re P_\varphi^g(f)(z)=f\big(\varphi(z)\big)g(z).$$

Proof. Assume that the holomorphic function $f(\varphi(z))g(z)$ has the expansion $\sum_\beta a_\beta z^\beta$. Since $g(0)=0$, note that $a_0=0$. Then

$$\Re\big[P_\varphi^g(f)\big](z)=\Re\int_0^1\sum_{\beta\neq 0}a_\beta(tz)^\beta\frac{dt}{t}=\Re\bigg(\sum_{\beta\neq 0}\frac{a_\beta}{|\beta|}z^\beta\bigg)=\sum_{\beta\neq 0}a_\beta z^\beta,$$

which is what we wanted to prove. □

Lemma 3. *Let $f\in\mathcal{B}(\mathbb{B})$. Then the following inequality holds*

$$\big|f(z)\big|\leqslant\|f\|_{\mathcal{B}}\max\bigg\{1,\frac{1}{2}\ln\frac{1+|z|}{1-|z|}\bigg\}. \tag{4}$$

Proof. The proof of the lemma follows from the following inequality

$$\big|f(z)-f(0)\big|=\bigg|\int_0^1\big\langle\nabla f(tz),\bar z\big\rangle dt\bigg|\leqslant b(f)\int_0^1\frac{|z|dt}{1-|z|^2t^2}=b(f)\frac{1}{2}\ln\frac{1+|z|}{1-|z|},$$

where $b(f)=\sup_{z\in\mathbb{B}}(1-|z|^2)|\nabla f(z)|$. □

2. The norm of the operator $P_\varphi^g:\mathcal{B}$ (or $\mathcal{B}_0$) $\to\mathcal{B}_\mu$

In this section we calculate the norms $\|P_\varphi^g\|_{\mathcal{B}\to\mathcal{B}_\mu}$ and $\|P_\varphi^g\|_{\mathcal{B}_0\to\mathcal{B}_\mu}$.

Theorem 1. *Assume $g\in H(\mathbb{B})$, $g(0)=0$, μ is normal, φ is a holomorphic self-map of $\mathbb{B}$ and $P_\varphi^g:\mathcal{B}$ (or $\mathcal{B}_0$) $\to\mathcal{B}_\mu$ is bounded. Then*

$$\big\|P_\varphi^g\big\|_{\mathcal{B}\to\mathcal{B}_\mu}=\big\|P_\varphi^g\big\|_{\mathcal{B}_0\to\mathcal{B}_\mu}=\sup_{z\in\mathbb{B}}\mu(z)\big|g(z)\big|\max\bigg\{1,\frac{1}{2}\ln\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\bigg\}. \tag{5}$$

Proof. If $f\in\mathcal{B}$, then by Lemma 2 and (4) we obtain

$$\big\|P_\varphi^g f\big\|_{\mathcal{B}_\mu}=\sup_{z\in\mathbb{B}}\mu(z)\big|g(z)f\big(\varphi(z)\big)\big|\leqslant\|f\|_{\mathcal{B}}\sup_{z\in\mathbb{B}}\mu(z)\big|g(z)\big|\max\bigg\{1,\frac{1}{2}\ln\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\bigg\}, \tag{6}$$

from which it follows that

$$\left\| P_\varphi^g \right\|_{\mathcal{B}\to\mathcal{B}_\mu} \leqslant \sup_{z\in\mathbb{B}} \mu(z)\left|g(z)\right| \max\left\{1, \frac{1}{2}\ln\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right\}. \tag{7}$$

The same inequality holds for $P_\varphi^g : \mathcal{B}_0 \to \mathcal{B}_\mu$.

Now we prove the reverse inequality. By taking the function given by $f_0(z) \equiv 1 \in \mathcal{B}_0$ and using the boundedness of $P_\varphi^g : \mathcal{B} \to \mathcal{B}_\mu$, we obtain

$$\left\| P_\varphi^g \right\|_{\mathcal{B}\to\mathcal{B}_\mu} = \|f_0\|_{\mathcal{B}} \left\| P_\varphi^g \right\|_{\mathcal{B}\to\mathcal{B}_\mu} \geqslant \left\| P_\varphi^g f_0 \right\|_{\mathcal{B}_\mu} = \sup_{z\in\mathbb{B}} \mu(z)\left|g(z)\right|\left|f_0\big(\varphi(z)\big)\right| = \sup_{z\in\mathbb{B}} \mu(z)\left|g(z)\right|. \tag{8}$$

The same inequality holds for $P_\varphi^g : \mathcal{B}_0 \to \mathcal{B}_\mu$.

For $w \in \mathbb{B}$, set

$$f_w(z) = \frac{1}{2}\ln\frac{1+\langle z, w\rangle}{1-\langle z, w\rangle}, \tag{9}$$

with $\ln 1 = 0$. Since $f_w(0) = 0$ and

$$\big(1-|z|^2\big)\left|\nabla f_w(z)\right| = \frac{(1-|z|^2)|w|}{|1-\langle z, w\rangle^2|} \leqslant \frac{1-|z|^2}{1-|w|^2|z|^2} \leqslant \min\left\{1, \frac{1-|z|^2}{1-|w|^2}\right\},$$

it follows that $\sup_{w\in\mathbb{B}} \|f_w\|_{\mathcal{B}} \leqslant 1$, and $f_w \in \mathcal{B}_0$ for each fixed $w \in \mathbb{B}$.

From this and the boundedness of $P_\varphi^g : \mathcal{B}$ (or $\mathcal{B}_0$) $\to \mathcal{B}_\mu$ we have that when $\varphi(w) \neq 0$ and for every $t \in (0, 1)$ the following inequality holds

$$\begin{aligned}\left\| P_\varphi^g \right\|_{\mathcal{B}_0\to\mathcal{B}_\mu} &\geqslant \left\| P_\varphi^g f_{t\varphi(w)/|\varphi(w)|} \right\|_{\mathcal{B}_\mu} \\ &= \sup_{z\in\mathbb{B}} \mu(z)\left|g(z)\right| \frac{1}{2}\left|\ln\frac{1+t\langle\varphi(z), \varphi(w)/|\varphi(w)|\rangle}{1-t\langle\varphi(z), \varphi(w)/|\varphi(w)|\rangle}\right| \\ &\geqslant \frac{1}{2}\mu(w)\left|g(w)\right|\ln\frac{1+t|\varphi(w)|}{1-t|\varphi(w)|}.\end{aligned} \tag{10}$$

Note that (10) obviously holds if $\varphi(w) = 0$.

Letting $t \to 1$ in (10), we obtain that for each $w \in \mathbb{B}$,

$$\left\| P_\varphi^g \right\|_{\mathcal{B}_0\to\mathcal{B}_\mu} \geqslant \frac{1}{2}\mu(w)\left|g(w)\right|\ln\frac{1+|\varphi(w)|}{1-|\varphi(w)|}.$$

From this and since w is an arbitrary element of $\mathbb{B}$, it follows that

$$\left\| P_\varphi^g \right\|_{\mathcal{B}_0\to\mathcal{B}_\mu} \geqslant \frac{1}{2}\sup_{z\in\mathbb{B}} \mu(z)\left|u(z)\right|\ln\frac{1+|\varphi(z)|}{1-|\varphi(z)|}. \tag{11}$$

Note also that

$$\left\| P_\varphi^g \right\|_{\mathcal{B}\to\mathcal{B}_\mu} \geqslant \left\| P_\varphi^g \right\|_{\mathcal{B}_0\to\mathcal{B}_\mu}. \tag{12}$$

From (8), (11) and (12) we obtain that

$$\left\| P_\varphi^g \right\|_{\mathcal{B}\to\mathcal{B}_\mu} \geqslant \left\| P_\varphi^g \right\|_{\mathcal{B}_0\to\mathcal{B}_\mu} \geqslant \sup_{z\in\mathbb{B}} \mu(z)\left|g(z)\right| \max\left\{1, \frac{1}{2}\ln\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right\}. \tag{13}$$

From (7) and (13), equalities in (5) follow. □

Corollary 1. *Assume $g \in H(\mathbb{B})$, $g(0) = 0$, μ is normal and φ is a holomorphic self-map of $\mathbb{B}$. Then $P_\varphi^g : \mathcal{B}$ (or $\mathcal{B}_0$) $\to \mathcal{B}_\mu$ is bounded if and only if*

$$\sup_{z\in\mathbb{B}} \mu(z)\left|g(z)\right| \max\left\{1, \frac{1}{2}\ln\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right\} < \infty. \tag{14}$$

Proof. If $P_\varphi^g : \mathcal{B}$ (or $\mathcal{B}_0$) $\to \mathcal{B}_\mu$ is bounded, then (14) follows from Theorem 1. If (14) holds, then the boundedness of $P_\varphi^g : \mathcal{B}$ (or $\mathcal{B}_0$) $\to \mathcal{B}_\mu$ follows from (6). □

3. The boundedness of the operator $P_\varphi^g : \mathcal{B}_0 \to \mathcal{B}_{\mu,0}$

Here we characterize the boundedness of the operator $P_\varphi^g : \mathcal{B}_0 \to \mathcal{B}_{\mu,0}$.

Theorem 2. *Assume $g \in H(\mathbb{B})$, $g(0) = 0$, μ is normal and φ is a holomorphic self-map of $\mathbb{B}$. Then $P_\varphi^g : \mathcal{B}_0 \to \mathcal{B}_{\mu,0}$ is bounded if and only if $P_\varphi^g : \mathcal{B}_0 \to \mathcal{B}_\mu$ is bounded and $g \in H_{\mu,0}^\infty$.*

Proof. Assume that $P_\varphi^g : \mathcal{B}_0 \to \mathcal{B}_{\mu,0}$ is bounded. Then clearly $P_\varphi^g : \mathcal{B}_0 \to \mathcal{B}_\mu$ is bounded. Taking the test function $f_0(z) = 1 \in \mathcal{B}_0$ we obtain $g \in H_{\mu,0}^\infty$.

Conversely, assume $P_\varphi^g : \mathcal{B}_0 \to \mathcal{B}_\mu$ is bounded and $g \in H_{\mu,0}^\infty$. Then, for every polynomial p, we have

$$\mu(z)\big|\Re P_\varphi^g p(z)\big| = \mu(z)\big|g(z)p\big(\varphi(z)\big)\big| \leqslant \mu(z)\big|g(z)\big|\|p\|_\infty \to 0, \quad \text{as } |z| \to 1.$$

Since the set of all polynomials is dense in $\mathcal{B}_0$, for each $f \in \mathcal{B}_0$ there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \|f - p_k\|_{\mathcal{B}} = 0. \tag{15}$$

From (15) and since the operator $P_\varphi^g : \mathcal{B}_0 \to \mathcal{B}_\mu$ is bounded, it follows that

$$\big\|P_\varphi^g f - P_\varphi^g p_k\big\|_{\mathcal{B}_\mu} \leqslant \big\|P_\varphi^g\big\|_{\mathcal{B}_0\to\mathcal{B}_\mu} \|f - p_k\|_{\mathcal{B}_0} \to 0,$$

as $k \to \infty$. Hence $P_\varphi^g(\mathcal{B}_0) \subset \mathcal{B}_{\mu,0}$. Since $\mathcal{B}_{\mu,0}$ is a closed subset of $\mathcal{B}_\mu$ the boundedness of $P_\varphi^g : \mathcal{B}_0 \to \mathcal{B}_{\mu,0}$ follows. □

4. Essential norm of $P_\varphi^g : \mathcal{B}$ (or $\mathcal{B}_0$) $\to \mathcal{B}_\mu$

Let X and Y be Banach spaces, and $L : X \to Y$ be a bounded linear operator. The essential norm of the operator $L : X \to Y$, denoted by $\|L\|_{e,X\to Y}$, is defined as follows

$$\|L\|_{e,X\to Y} = \inf\big\{\|L + K\|_{X\to Y}\colon\ K \text{ is compact from } X \text{ to } Y\big\},$$

where $\|\cdot\|_{X\to Y}$ denote the operator norm.

From this definition and since the set of all compact operators is a closed subset of the set of bounded operators it follows that operator L is compact if and only if $\|L\|_{e,X\to Y} = 0$.

Here we prove the main result in the paper, namely, we calculate the essential norm of the operator $P_\varphi^g : \mathcal{B}$ (or $\mathcal{B}_0$) $\to \mathcal{B}_\mu$.

Theorem 3. *Assume $g \in H(\mathbb{B})$, $g(0) = 0$, μ is normal, φ is a holomorphic self-map of $\mathbb{B}$ and $P_\varphi^g : \mathcal{B}$ (or $\mathcal{B}_0$) $\to \mathcal{B}_\mu$ is bounded. If $\|\varphi\|_\infty = 1$, then*

$$\big\|P_\varphi^g\big\|_{e,\mathcal{B}\to\mathcal{B}_\mu} = \big\|P_\varphi^g\big\|_{e,\mathcal{B}_0\to\mathcal{B}_\mu} = \frac{1}{2}\limsup_{|\varphi(z)|\to 1} \mu(z)\big|g(z)\big| \ln\frac{1 + |\varphi(z)|}{1 - |\varphi(z)|}, \tag{16}$$

while if $\|\varphi\|_\infty < 1$, then

$$\big\|P_\varphi^g\big\|_{e,\mathcal{B}\to\mathcal{B}_\mu} = \big\|P_\varphi^g\big\|_{e,\mathcal{B}_0\to\mathcal{B}_\mu} = 0. \tag{17}$$

Proof. First assume that $\|\varphi\|_\infty = 1$. Set the following family of test functions

$$f_w^\varepsilon(z) = \left(\ln\frac{(1 + |w|)^2}{1 - \langle z, w\rangle}\right)^{\varepsilon+1}\left(\ln\frac{1 + |w|}{1 - |w|}\right)^{-\varepsilon}, \quad w \in \mathbb{B} \setminus \{0\}.$$

It is easy to see that

$$\big|f_w^\varepsilon(0)\big| \leqslant \big(\ln\big(1 + |w|\big)^2\big)^{\varepsilon+1}\left(\ln\frac{1 + |w|}{1 - |w|}\right)^{-\varepsilon} \leqslant 2^{\varepsilon+1}\ln 2$$

and

$$\lim_{|w|\to 1}\big|f_w^\varepsilon(0)\big| = 0. \tag{18}$$

Further we have

$$\big(1-|z|^2\big)\big|\nabla f_w^\varepsilon(z)\big| = (\varepsilon+1)\frac{(1-|z|^2)|w|}{|1-\langle z,w\rangle|}\left|\ln\frac{(1+|w|)^2}{1-\langle z,w\rangle}\right|^\varepsilon\left(\ln\frac{1+|w|}{1-|w|}\right)^{-\varepsilon}$$

$$\leqslant (\varepsilon+1)\frac{(1-|z|^2)|w|}{1-|z||w|}\left(\ln\frac{(1+|w|)^2}{1-|z||w|}+2\pi\right)^\varepsilon\left(\ln\frac{1+|w|}{1-|w|}\right)^{-\varepsilon} \tag{19}$$

$$\leqslant (\varepsilon+1)\big(1+|z|\big)|w|\left(\ln\frac{(1+|w|)^2}{1-|w|}+2\pi\right)^\varepsilon\left(\ln\frac{1+|w|}{1-|w|}\right)^{-\varepsilon}. \tag{20}$$

From (20) it follows that

$$\limsup_{|w|\to 1} b\big(f_w^\varepsilon\big) \leqslant 2(\varepsilon+1) \tag{21}$$

and from (19) that, for each fixed $w\in\mathbb{B}\setminus\{0\}$, $f_w^\varepsilon\in\mathcal{B}_0$.

Hence (18) and (21) imply

$$\lim_{|w|\to 1}\big\|f_w^\varepsilon\big\|_{\mathcal{B}} \leqslant 2(\varepsilon+1). \tag{22}$$

Now, assume that $(\varphi(z_k))_{k\in\mathbb{N}}$ is a sequence in $\mathbb{B}$ such that $|\varphi(z_k)|\to 1$ as $k\to\infty$. Note that from (22) it follows that the sequence $F_k(z)=f^\varepsilon_{\varphi(z_k)}(z)$, $k\in\mathbb{N}$ is such that

$$\lim_{k\to\infty}\|F_k\|_{\mathcal{B}} \leqslant 2(\varepsilon+1), \tag{23}$$

and that F_k converges to zero uniformly on compacts of $\mathbb{B}$ as $k\to\infty$. By Theorem 3.16 in [38] it follows that $F_k\to 0$ weakly in $\mathcal{B}_0$ as $k\to\infty$. Hence for every compact operator $K:\mathcal{B}_0\to\mathcal{B}_\mu$ we have that

$$\lim_{k\to\infty}\|KF_k\|_{\mathcal{B}_\mu} = 0. \tag{24}$$

Assume that $K:\mathcal{B}_0\to\mathcal{B}_\mu$ is an arbitrary compact operator. Then from the boundedness of $P_\varphi^g:\mathcal{B}_0\to\mathcal{B}_\mu$ we have that for each $k\in\mathbb{N}$

$$\|F_k\|_{\mathcal{B}}\big\|P_\varphi^g+K\big\|_{\mathcal{B}_0\to\mathcal{B}_\mu} \geqslant \big\|\big(P_\varphi^g+K\big)(F_k)\big\|_{\mathcal{B}_\mu} \geqslant \big\|P_\varphi^g F_k\big\|_{\mathcal{B}_\mu} - \|KF_k\|_{\mathcal{B}_\mu}. \tag{25}$$

Letting $k\to\infty$ in (25) and using (24) we obtain

$$\begin{aligned}\lim_{k\to\infty}\|F_k\|_{\mathcal{B}}\big\|P_\varphi^g+K\big\|_{\mathcal{B}_0\to\mathcal{B}_\mu} &\geqslant \limsup_{k\to\infty}\big(\big\|P_\varphi^g F_k\big\|_{\mathcal{B}_\mu} - \|KF_k\|_{\mathcal{B}_\mu}\big)\\ &= \limsup_{k\to\infty}\big\|P_\varphi^g F_k\big\|_{\mathcal{B}_\mu}\\ &= \limsup_{k\to\infty}\sup_{z\in\mathbb{B}}\mu(z)\big|g(z)\big|\big|F_k\big(\varphi(z)\big)\big|\\ &\geqslant \limsup_{k\to\infty}\mu(z_k)\big|g(z_k)F_k\big(\varphi(z_k)\big)\big|\\ &= \limsup_{k\to\infty}\mu(z_k)\big|g(z_k)\big|\ln\frac{1+|\varphi(z_k)|}{1-|\varphi(z_k)|}.\end{aligned}$$

From this and (23) we have

$$2(\varepsilon+1)\big\|P_\varphi^g+K\big\|_{\mathcal{B}_0\to\mathcal{B}_\mu} \geqslant \limsup_{k\to\infty}\mu(z_k)\big|g(z_k)\big|\ln\frac{1+|\varphi(z_k)|}{1-|\varphi(z_k)|}. \tag{26}$$

Taking the infimum in (26) over the set of all compact operators $K:\mathcal{B}_0\to\mathcal{B}_\mu$ and since ε is an arbitrary positive number, we obtain

$$\big\|P_\varphi^g\big\|_{e,\mathcal{B}_0\to\mathcal{B}_\mu} \geqslant \limsup_{k\to\infty}\frac{1}{2}\mu(z_k)g(z_k)\ln\frac{1+|\varphi(z_k)|}{1-|\varphi(z_k)|},$$

which implies the inequality

$$\|P_\varphi^g\|_{e,\mathcal{B}_0\to\mathcal{B}_\mu} \geqslant \limsup_{|\varphi(z)|\to 1}\frac{1}{2}\mu(z)g(z)\ln\frac{1+|\varphi(z)|}{1-|\varphi(z)|}. \tag{27}$$

Now we prove the reverse inequality. Assume that $(r_l)_{l\in\mathbb{N}}$ is a sequence which increasingly converges to 1. Consider the operators defined by

$$P^g_{r_l\varphi}(f)(z) = \int_0^1 g(tz) f\big(r_l\varphi(tz)\big)\frac{dt}{t}, \quad l \in \mathbb{N}. \tag{28}$$

Assume that $(h_k)_{k\in\mathbb{N}}$ is a bounded sequence in $\mathcal{B}$ (or $\mathcal{B}_0$) converging to zero uniformly on compacts of $\mathbb{B}$. Since $g \in H^\infty_\mu$, we have

$$\mu(z)\big|\Re P^g_{r_l\varphi}(h_k)(z)\big| = \mu(z)\big|g(z)h_k\big(r_l\varphi(z)\big)\big| \leqslant \|g\|_{H^\infty_\mu} \sup_{|w|\leqslant r_l}\big|h_k(w)\big| \to 0,$$

as $k\to\infty$. Hence by Lemma 1, for each $l\in\mathbb{N}$, $P^g_{r_l\varphi}:\mathcal{B}$ (or $\mathcal{B}_0$) $\to \mathcal{B}_\mu$ is compact.

Let $\rho\in(0,1)$ be fixed for a moment. Employing Lemma 2, the fact that $g\in H^\infty_\mu$, and the following formula (see, [38, Theorem 3.14])

$$\sup_{f\in\mathcal{B},\,\|f\|_\mathcal{B}\leqslant 1}\big|f(z)-f(w)\big| = \frac{1}{2}\ln\frac{1+|\varphi_w(z)|}{1-|\varphi_w(z)|}, \quad z,w\in\mathbb{B}$$

(where φ_w is the involutive automorphism of $\mathbb{B}$ that interchanges 0 and w), we have

$$\begin{aligned}
\big\|P^g_\varphi - P^g_{r_l\varphi}\big\|_{\mathcal{B}\to\mathcal{B}_\mu} &= \sup_{\|f\|_\mathcal{B}\leqslant 1}\sup_{z\in\mathbb{B}}\mu(z)\big|g(z)\big|\big|f\big(\varphi(z)\big)-f\big(r_l\varphi(z)\big)\big| \\
&\leqslant \sup_{\|f\|_\mathcal{B}\leqslant 1}\sup_{|\varphi(z)|\leqslant\rho}\mu(z)\big|g(z)\big|\big|f\big(\varphi(z)\big)-f\big(r_l\varphi(z)\big)\big| \\
&\quad + \sup_{\|f\|_\mathcal{B}\leqslant 1}\sup_{|\varphi(z)|>\rho}\mu(z)\big|g(z)\big|\big|f\big(\varphi(z)\big)-f\big(r_l\varphi(z)\big)\big| \\
&\leqslant \|g\|_{H^\infty_\mu}\sup_{\|f\|_\mathcal{B}\leqslant 1}\sup_{|\varphi(z)|\leqslant\rho}\big|f\big(\varphi(z)\big)-f\big(r_l\varphi(z)\big)\big| &(29)\\
&\quad + \sup_{|\varphi(z)|>\rho}\mu(z)\big|g(z)\big|\frac{1}{2}\ln\frac{1+|\varphi_{\varphi(z)}(r_l\varphi(z))|}{1-|\varphi_{\varphi(z)}(r_l\varphi(z))|}. &(30)
\end{aligned}$$

Since

$$\big|\varphi_{\varphi(z)}\big(r_l\varphi(z)\big)\big| = \left|\frac{\varphi(z)-P_{\varphi(z)}(r_l\varphi(z))-s_qQ_{\varphi(z)}(r_l\varphi(z))}{1-\langle r_l\varphi(z),\varphi(z)\rangle}\right| = \frac{|\varphi(z)|(1-r_l)}{1-r_l|\varphi(z)|^2} \leqslant \big|\varphi(z)\big|,$$

and since the function

$$h(x) = \ln\frac{1+x}{1-x} \tag{31}$$

is increasing on the interval $[0,1)$, we obtain

$$\sup_{|\varphi(z)|>\rho}\mu(z)\big|g(z)\big|\ln\frac{1+|\varphi_{\varphi(z)}(r_l\varphi(z))|}{1-|\varphi_{\varphi(z)}(r_l\varphi(z))|} \leqslant \sup_{|\varphi(z)|>\rho}\mu(z)\big|g(z)\big|\ln\frac{1+|\varphi(z)|}{1-|\varphi(z)|}. \tag{32}$$

Now we estimate the quantity in (29). Let

$$I_l := \sup_{\|f\|_\mathcal{B}\leqslant 1}\sup_{|\varphi(z)|\leqslant\rho}\big|f\big(\varphi(z)\big)-f\big(r_l\varphi(z)\big)\big|.$$

By using the mean value theorem and the definition of the Bloch space, we obtain

$$\begin{aligned}
I_l &\leqslant \sup_{\|f\|_\mathcal{B}\leqslant 1}\sup_{|\varphi(z)|\leqslant\rho}(1-r_l)\big|\varphi(z)\big|\sup_{|w|\leqslant\rho}\big|\nabla f(w)\big| \\
&\leqslant \rho\frac{1-r_l}{1-\rho^2}\sup_{\|f\|_\mathcal{B}\leqslant 1}\|f\|_\mathcal{B} \\
&= C_\rho(1-r_l)\to 0 \quad \text{as } l\to\infty. &(33)
\end{aligned}$$

Letting $l\to\infty$ in (29) and (30), using (32) and (33), and then letting $\rho\to 1$ we obtain the inequality

$$\big\|P^g_\varphi\big\|_{e,\mathcal{B}\to\mathcal{B}_\mu} \leqslant \limsup_{|\varphi(z)|\to 1}\frac{1}{2}\mu(z)g(z)\ln\frac{1+|\varphi(z)|}{1-|\varphi(z)|}. \tag{34}$$

From (27), (34) and since

$$\big\|P^g_\varphi\big\|_{e,\mathcal{B}\to\mathcal{B}_\mu} \geqslant \big\|P^g_\varphi\big\|_{e,\mathcal{B}_0\to\mathcal{B}_\mu},$$

both equalities in (16) follow.

Now assume $\|\varphi\|_\infty < 1$, then similar to operators in (28) it is proved that the operator $P_\varphi^g : \mathcal{B}$ (or $\mathcal{B}_0) \to \mathcal{B}_\mu$ is compact, which is equivalent with (17), finishing the proof of the theorem. □

The following result regarding the compactness of the operator $P_\varphi^g : \mathcal{B}$ (or $\mathcal{B}_0) \to \mathcal{B}_\mu$ is a direct consequence of Theorem 3.

Corollary 2. *Assume $g \in H(\mathbb{B})$, $g(0) = 0$, μ is normal, φ is a holomorphic self-map of $\mathbb{B}$ such that $\|\varphi\|_\infty = 1$, and the operator $P_\varphi^g : \mathcal{B}$ (or $\mathcal{B}_0) \to \mathcal{B}_\mu$ is bounded. Then the operator $P_\varphi^g : \mathcal{B}$ (or $\mathcal{B}_0) \to \mathcal{B}_\mu$ is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} \mu(z)\big|g(z)\big| \ln \frac{1+|\varphi(z)|}{1-|\varphi(z)|} = 0. \tag{35}$$

5. Essential norm of the operator $P_\varphi^g : \mathcal{B}$ (or $\mathcal{B}_0) \to \mathcal{B}_{\mu,0}$

Here we calculate the essential norm of the operator $P_\varphi^g : \mathcal{B}$ (or $\mathcal{B}_0) \to \mathcal{B}_{\mu,0}$.

Theorem 4. *Assume $g \in H(\mathbb{B})$, $g(0) = 0$, μ is normal, φ is a holomorphic self-map of $\mathbb{B}$ and the operator $P_\varphi^g : \mathcal{B}$ (or $\mathcal{B}_0) \to \mathcal{B}_{\mu,0}$ is bounded. Then*

$$\big\|P_\varphi^g\big\|_{e,\mathcal{B}\to\mathcal{B}_{\mu,0}} = \big\|P_\varphi^g\big\|_{e,\mathcal{B}_0\to\mathcal{B}_{\mu,0}} = \frac{1}{2} \limsup_{|z|\to 1} \mu(z)\big|g(z)\big| \ln \frac{1+|\varphi(z)|}{1-|\varphi(z)|}. \tag{36}$$

Proof. Since $P_\varphi^g : \mathcal{B}$ (or $\mathcal{B}_0) \to \mathcal{B}_{\mu,0}$ is bounded, then for the test function $f_0(z) \equiv 1 \in \mathcal{B}_0$, we obtain that $g \in H_{\mu,0}^\infty$.

First assume $\|\varphi\|_\infty < 1$. Then, similar to operators in (28) it can be proved that $P_\varphi^g : \mathcal{B}\,(\text{or}\,\mathcal{B}_0) \to \mathcal{B}_{\mu,0}$ is compact. Hence

$$\big\|P_\varphi^g\big\|_{e,\mathcal{B}\ (\text{or}\ \mathcal{B}_0)\to\mathcal{B}_{\mu,0}} = 0.$$

On the other hand, since $\|\varphi\|_\infty < 1$, and $g \in H_{\mu,0}^\infty$, it follows that

$$\limsup_{|z|\to 1} \mu(z)\big|g(z)\big| \ln \frac{1+|\varphi(z)|}{1-|\varphi(z)|} \leqslant \left(\ln \frac{1+\|\varphi\|_\infty}{1-\|\varphi\|_\infty}\right) \lim_{|z|\to 1} \mu(z)\big|g(z)\big| = 0,$$

from which (36) follows in this case.

Now assume $\|\varphi\|_\infty = 1$. It is clear that

$$\limsup_{|z|\to 1} \mu(z)\big|g(z)\big| \ln \frac{1+|\varphi(z)|}{1-|\varphi(z)|} \geqslant \limsup_{|\varphi(z)|\to 1} \mu(z)\big|g(z)\big| \ln \frac{1+|\varphi(z)|}{1-|\varphi(z)|}. \tag{37}$$

Assume that $(z_k)_{k\in\mathbb{N}}$ is such a sequence that

$$\limsup_{|z|\to 1} \mu(z)\big|g(z)\big| \ln \frac{1+|\varphi(z)|}{1-|\varphi(z)|} = \lim_{k\to\infty} \mu(z_k)\big|g(z_k)\big| \ln \frac{1+|\varphi(z_k)|}{1-|\varphi(z_k)|}.$$

If $\sup_{k\in\mathbb{N}} |\varphi(z_k)| < 1$, then in view of the fact $g \in H_{\mu,0}^\infty$, the last limit is zero and consequently the second limit in (37) is also zero. Otherwise, there is a subsequence $(\varphi(z_{k_l}))_{l\in\mathbb{N}}$ such that $|\varphi(z_{k_l})| \to 1$ as $l \to \infty$, so that both limits in (37) are equal, that is,

$$\limsup_{|z|\to 1} \mu(z)\big|g(z)\big| \ln \frac{1+|\varphi(z)|}{1-|\varphi(z)|} = \limsup_{|\varphi(z)|\to 1} \mu(z)\big|g(z)\big| \ln \frac{1+|\varphi(z)|}{1-|\varphi(z)|}.$$

From this and by Theorem 3 the result follows in this case, finishing the proof of the theorem. □

Corollary 3. *Assume $g \in H(\mathbb{B})$, $g(0) = 0$, μ is normal, φ is a holomorphic self-map of $\mathbb{B}$ and the operator $P_\varphi^g : \mathcal{B}$ (or $\mathcal{B}_0) \to \mathcal{B}_{\mu,0}$ is bounded. Then the operator $P_\varphi^g : \mathcal{B}$ (or $\mathcal{B}_0) \to \mathcal{B}_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to 1} \mu(z)\big|g(z)\big| \ln \frac{1+|\varphi(z)|}{1-|\varphi(z)|} = 0. \tag{38}$$

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