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The discriminant of an algebraic torus

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ABSTRACT

For a torus T defined over a global field K, we revisit an analytic class number formula obtained by Shyr in the 1970s as a generalization of Dirichlet's class number formula. We prove a local–global presentation of the quasi-discriminant of T, which enters into this formula, in terms of cocharacters of T. This presentation can serve as a more natural definition of this invariant.

0. Introduction

The well-known class number formula expresses the class number of a number field K in terms of other arithmetic invariants of K:

$$h = \frac{w|\Delta|^{1/2}\rho}{2^r(2\pi)^t R}$$

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where w is the number of roots of unity in K, Δ is the discriminant of K, ρ is the residue of the Dedekind Zeta-function of K at $s=1$, R is the regulator of K and r (resp. t) is the number of real (resp. pairs of complex) embeddings of K.

In the early 1960s, T. Ono defined in [Ono] analogues of these invariants for algebraic tori defined over both kinds of global fields, namely, the number field case (denoted by case (N)) and the case of algebraic function field in one variable over a finite field of constants $\mathbb{F}_q$ (denoted by case (F)). One of these invariants is the *quasi-discriminant*. As in the case of the discriminant of a global field, the quasi-discriminant of T is the volume of the fundamental domain of the maximal compact subgroup of $T(A_K)/T(K)$ – where A_K is the adele ring – with respect to the Tamagawa measure. J.M. Shyr gave in [Shyr1] a similar definition in the case of algebraic $\mathbb{Q}$-tori. In this new construction, other arithmetic invariants are taken from Ono's definition. This led him to a relation which can be viewed as an "analogue of the class number formula" for algebraic $\mathbb{Q}$-tori. This relation can be generalized to tori defined over any global field K, as follows:

$$c_T^{\mathrm{Shyr}} := |\Delta_K|^{-d/2} C_\infty \prod_{\mathfrak{p}} L_{\mathfrak{p}}(1,\chi_T)\cdot\omega_{\mathfrak{p}}\big(T_{\mathfrak{p}}(\mathcal{O}_{\mathfrak{p}})\big) = \frac{\rho_T\tau_T w_T}{h_T R_T} \tag{0.1}$$

where C_∞ is an archimedean factor and Δ_K is the discriminant of K, $d=\dim T$. For any prime $\mathfrak{p}$ of K, $K_{\mathfrak{p}}$ is the complete localization of K at $\mathfrak{p}$, $T_{\mathfrak{p}} = T\otimes K_{\mathfrak{p}}$, $T_{\mathfrak{p}}(\mathcal{O}_{\mathfrak{p}})$ is the maximal compact subgroup of $T_{\mathfrak{p}}(K_{\mathfrak{p}})$, $L_{\mathfrak{p}}(s,\chi_T)$ is the local Artin L-function and $\omega_{\mathfrak{p}}$ is some Haar measure on $T_{\mathfrak{p}}(K_{\mathfrak{p}})$. ρ_T is the residue of the global Artin L-function at $s=1$, w_T is the cardinality of the torsion part of the group of units of T, h_T is the class number of T, R_T is the regulator of T (equal to 1 in case (F)), and τ_T is the Tamagawa number of T. This work is described in the fourth section and on Appendix A. We call c_T^{Shyr} the *Shyr invariant*.

Locally, let K be a henselian local field with a ring of integers $\mathcal{O}_{\mathfrak{p}}$ and a finite residue field k. Let T be an algebraic K-torus splitting over a finite Galois extension L/K with Galois group Γ. We investigate the invariant $L_{\mathfrak{p}}(1,\chi_T)\cdot\omega_{\mathfrak{p}}(T(\mathcal{O}_{\mathfrak{p}}))$. Ono defined the group $T(\mathcal{O}_{\mathfrak{p}})$ using the dual Γ-module, namely its group of characters. In order to measure it, we would like to describe it as a group of $\mathcal{O}_{\mathfrak{p}}$-points of some integral model. In the first section we exhibit the properties and relation between two integral models of T, namely, the standard integral model X defined by V.E. Voskresenskiĭ (which is of finite type), and the well-known Néron–Raynaud integral model $\mathcal{T}$ (which is locally of finite type). After applying a smoothening process to X, the identity components of both models coincide. Let Φ_T be the k-scheme of the group of components of the reduction of $\mathcal{T}$ modulo $\mathfrak{p}$. In the second section we prove that:

$$L_{\mathfrak{p}}(1,\chi_T)\cdot\omega_{\mathfrak{p}}\big(T(\mathcal{O}_{\mathfrak{p}})\big) = \big|\Phi_T(k)_{\mathrm{tor}}\big|.$$

In the third section, we use a construction of Kottwitz in [Ko], to prove an isomorphism of k-schemes $\Phi_T(k)\cong (X_\bullet(T)_I)^{\langle F\rangle}$ where $X_\bullet(T)$ is the cocharacter group of T, I is the inertia subgroup of Γ and F is the Frobenius automorphism generating Γ/I. This isomorphism gives us another computation of the local component in Shyr's invariant:

$$L_{\mathfrak{p}}(1,\chi_T)\cdot\omega_{\mathfrak{p}}\big(T_{\mathfrak{p}}(\mathcal{O}_{\mathfrak{p}})\big) = \big|\ker\big(1-F|X_\bullet(T)_I\big)_{\mathrm{tor}}\big|.$$

Globally, together with the infinite place information, we prove in the fourth section our main theorem, which can be viewed as a local–global result, in spirit of the Artin–Hasse conductor-discriminant formula:

Main theorem.

$$c_T^{\mathrm{Shyr}} = |\Delta_K|^{-d/2} C_\infty \prod_{\mathfrak{p}} \big|\ker\big(1-F_{\mathfrak{p}}|X_\bullet(T_{\mathfrak{p}})_{I_{\mathfrak{p}}}\big)_{\mathrm{tor}}\big| = \frac{\rho_T\tau_T w_T}{h_T R_T}.$$

From this formula one can see that the Shyr invariant can be decomposed into the product of the "arithmetic–geometric part" (related to the discriminant of the ground field) and the "algebraic" part (reflecting the Galois action on the cocharacters).

1. Integral models of algebraic tori

Let K be any field. An *algebraic torus* T is an algebraic K-group such that $T \otimes_K L \cong \mathbb{G}_{m,L}^d$ for some finite Galois extension L/K where $\mathbb{G}_m$ is the multiplicative group and d is the dimension of T. The smallest among such extensions is called the *splitting field* of T. We write $T \in \mathcal{C}(L/K)$. We denote by $X^\bullet(T) = \mathrm{Hom}(T \otimes_K L, \mathbb{G}_{m,L})$ the group of characters of T. For any intermediate field $K \subseteq F \subseteq L$, $X^\bullet(T)_F$ is the sublattice of characters defined over F.

Now let K be a local field which is the complete localization of a global field with respect to a prime $\mathfrak{p}$. We denote by $\mathcal{O}_\mathfrak{p}$ the ring of integers of K and by $U_\mathfrak{p} = \mathcal{O}_\mathfrak{p}^\times$ its subgroup of units. Let $k = \mathcal{O}_\mathfrak{p}/\mathfrak{p}$ be the residue field with cardinality q. Let $T \in \mathcal{C}(L/K)$ and denote by $\Gamma = Gal(L/K)$ the Galois group and by I its inertia subgroup.

An $\mathcal{O}_\mathfrak{p}$*-integral model* of T is an $\mathcal{O}_\mathfrak{p}$-scheme M whose generic fiber $M \otimes_{\mathcal{O}_\mathfrak{p}} K$ is isomorphic to T. The *reduction modulo* $\mathfrak{p}$ of M is its special fiber $M_\mathfrak{p} = M \otimes_{\mathcal{O}_\mathfrak{p}} k$. Its identity component M^0 is an open subscheme of M which is the identity component of the special fiber, i.e. such that $M^0 \otimes k = (M \otimes k)^0$ (see [BLR, p. 154]).

T. Ono defined $T(\mathcal{O}_\mathfrak{p})$ as the maximal compact subgroup of $T(K)$ with respect to the $\mathfrak{p}$-adic topology. This was done using the group of characters (see [Ono, 2.1.3]):

$$T(\mathcal{O}_\mathfrak{p}) = \bigl\{x \in T(K):\ \chi(x) \in U_\mathfrak{p}\ \forall \chi \in X^\bullet(T)_K\bigr\}.$$

However, in order to measure this group with respect to some local measure, this description may not be enough. We would like to write $T(\mathcal{O}_\mathfrak{p})$ as the group of $\mathcal{O}_\mathfrak{p}$-points of some group scheme. Toward this end, we consider two integral models of T.

1.1. The standard integral model

The following construction is due to V.E. Voskresenskiĭ and can be found in [Pop, §1]. Notations are as above and suppose that $(L:K) = n$ and $\dim T = d$. Then $X^\bullet(T)$ is spanned as a $\mathbb{Z}$-lattice by a basis $\{\chi_i\}_{i=1}^d$. The Hopf algebra $B = (L[X^\bullet(T)])^\Gamma$ is the coordinate ring of T. The isomorphism $T \otimes_K L \cong \mathbb{G}_m^d$ is equivalent to the isomorphism of L-algebras: $B \otimes_K L = LB \cong L[X^\bullet(T)]$. Let $\{\omega_i\}_{i=1}^n$ be an integral basis of L over K. Then

$$L\bigl[X^\bullet(T)\bigr] = B\omega_1 \oplus \cdots \oplus B\omega_n.$$

Thus there are linear combinations:

$$\chi_i = x_i^{(1)}\omega_1 + \cdots + x_i^{(n)}\omega_n, \quad x_i^{(j)} \in B,$$
$$\chi_i^{-1} = y_i^{(1)}\omega_1 + \cdots + y_i^{(n)}\omega_n, \quad y_i^{(l)} \in B.$$

Definition 1. $A(X^\bullet(T)) = \mathcal{O}_\mathfrak{p}[x_i^{(j)}, y_m^{(l)}]$ is a Hopf $\mathcal{O}_\mathfrak{p}$-algebra. The group $\mathcal{O}_\mathfrak{p}$-scheme $X = \mathrm{Spec}\, A(X^\bullet(T))$ is called the *standard integral model* of T.

Remark 1.1. (See [Vos, §10.3].) Being obtained from a linear representation, X is of finite type over $\mathcal{O}_\mathfrak{p}$, and is reduced and faithfully flat. Further, $X(\mathcal{O}_\mathfrak{p}) = T(\mathcal{O}_\mathfrak{p})$ is the maximal compact subgroup of $T(K)$ with respect to the $\mathfrak{p}$-adic topology.

1.2. The Néron–Raynaud integral model and relations to the standard model

Let K^{sh} be the strict henselization of K and $\mathcal{O}_{\mathfrak{p}}^{\mathrm{sh}}$ be its ring of integers. We refer to the Néron–Raynaud (NR for short) model $\mathcal{T}$ of T which is locally of finite type over $\mathcal{O}_{\mathfrak{p}}^{\mathrm{sh}}$, as defined in [BLR, Chapter 10] and satisfying $\mathcal{T}(\mathcal{O}_{\mathfrak{p}}^{\mathrm{sh}}) = T(K^{\mathrm{sh}})$.

Remark 1.2. A local ring is strictly henselian if its residue field is separably closed. In our cases (N) and (F) the residue field of any complete localization is perfect, thus K^{sh} is the maximal unramified extension of K and it is a Galois over K. Let $\Gamma_{\mathrm{sh}} = \mathit{Gal}(K^{\mathrm{sh}}/K)$. The Γ_{sh}-invariant subgroup of $\mathcal{T}(\mathcal{O}_{\mathfrak{p}}^{\mathrm{sh}}) = T(K^{\mathrm{sh}})$ is then $\mathcal{T}(\mathcal{O}_{\mathfrak{p}}) = T(K)$. Indeed, as $\mathcal{T}$ is separated, the canonical map $\mathcal{T}(\mathcal{O}_{\mathfrak{p}}) \to T(K)$ is injective. It is also surjective by the universal property of the NR-model.

The following construction can be found in [VKM]. Let $T_L = T \otimes L \cong \mathbb{G}_{m,L}^d$. Let $\mathcal{O}_{\mathfrak{P}}$ be the ring of integers of L and let $\mathcal{T}_L$ be the NR-model of T_L defined over it. The $\mathcal{O}_{\mathfrak{p}}$-scheme $\mathcal{S} = R_{\mathcal{O}_{\mathfrak{P}}/\mathcal{O}_{\mathfrak{p}}}(\mathcal{T}_L)$ obtained by the Weil restriction of scalars, is the NR-model of $R = R_{L/K}(T_L)$. Its identity component is $\mathcal{S}^0 = R_{\mathcal{O}_{\mathfrak{P}}/\mathcal{O}_{\mathfrak{p}}}(\mathbb{G}_{m,\mathcal{O}_{\mathfrak{P}}}^d)$. Let $\mathcal{N}$ be the schematic closure of T in $\mathcal{S}$ induced by the canonical embedding $T \to R$. The standard $\mathcal{O}_{\mathfrak{p}}$-model X of T is isomorphic to $\mathcal{N} \cap \mathcal{S}^0$ (see [VKM, Proposition 6], the proof there is for p-adic fields but the arguments are valid also in case (F)).

Lemma 1.3. $X^0 = \mathcal{N}^0$.

Proof. $\mathcal{N}^0 = (\mathcal{N} \cap \mathcal{S}^0)^0 = X^0$. $\square$

The schemes $\mathcal{N}$ and X are not necessarily smooth, i.e., their special fibers may not be reduced. To achieve the desired smooth NR-model $\mathcal{T}$, one should apply the *smoothening process* (see [BLR, Chapter 3]). It is sufficient to control the defect of smoothness over $X = \mathcal{N} \cap \mathcal{S}^0$ [BLR, Proposition 10.1/4]. Thus the equality of the identity components of the two models is preserved. We denote the obtained *smooth* standard model by X_{sm}. As the ring representing X is Notherian, this process consists of blowing up finitely many maximal ideals and X_{sm} remains of finite type. Moreover, by definition, the generic fibers of X_{sm} and X are isomorphic.

Corollary 1.4. $X_{\mathrm{sm}}^0 = \mathcal{T}^0$.

2. Reductions and local volume computations

2.1. Rational points of the group of components

Denote by $i : \operatorname{Spec} k \to \operatorname{Spec} \mathcal{O}_{\mathfrak{p}}$ the canonical closed immersion of the special point. We call the k-scheme $\Phi_T = i^*(\mathcal{T}/\mathcal{T}^0)$ the *group of components* of $\mathcal{T}$. There is an exact sequence of k-schemes:

$$1 \to \mathcal{T}_{\mathfrak{p}}^0 \to \mathcal{T}_{\mathfrak{p}} \to \Phi_T \to 1,$$

where $\mathcal{T}_{\mathfrak{p}} = i^*(\mathcal{T})$ and $\mathcal{T}_{\mathfrak{p}}^0 = i^*(\mathcal{T}^0)$. Let l be the residue field of L and let $\mathfrak{g} = \mathit{Gal}(l/k)$. Since k is finite and $\mathcal{T}_{\mathfrak{p}}^0$ is affine and connected, by Lang's Theorem (see [Ser, Chapter VI, Proposition 5]), $H^1(\mathfrak{g}, \mathcal{T}_{\mathfrak{p}}^0(l))$ is trivial implying the exactness of:

$$1 \to \mathcal{T}_{\mathfrak{p}}^0(k) \to \mathcal{T}_{\mathfrak{p}}(k) \xrightarrow{\varphi} \Phi_T(k) \to 1. \tag{2.1}$$

Consider the map composition:

$$T(K) = \mathcal{T}(\mathcal{O}_{\mathfrak{p}}) \xrightarrow{r} \mathcal{T}_{\mathfrak{p}}(k) \xrightarrow{\varphi} \Phi_T(k)$$

where r is the reduction modulo $\mathfrak{p}$ map and φ is the map in (2.1). As $\mathcal{T}$ is smooth and $\mathcal{O}_{\mathfrak{p}}$ is complete (and therefore henselian), r is surjective (see [BLR, Proposition 2.3/5]). Since φ is also surjective and the kernel of $r \circ \varphi$ is well known to be $\mathcal{T}^0(\mathcal{O}_{\mathfrak{p}})$ (see, e.g., [Gon, p. 1153]), we obtain:

Lemma 2.1. $T(K)/\mathcal{T}^0(\mathcal{O}_{\mathfrak{p}}) \cong \mathcal{T}_{\mathfrak{p}}(k)/\mathcal{T}^0_{\mathfrak{p}}(k) = \Phi_T(k)$.

The same construction for the smooth standard model X_{sm} with its group of components $\phi_T = i^*(X_{\mathrm{sm}}/X^0_{\mathrm{sm}})$ leads to the corresponding isomorphism of abelian groups:

$$X_{\mathrm{sm}}(\mathcal{O}_{\mathfrak{p}})/X^0_{\mathrm{sm}}(\mathcal{O}_{\mathfrak{p}}) \cong \phi_T(k). \tag{2.2}$$

Lemma 2.2. $X_{\mathrm{sm}}(\mathcal{O}_{\mathfrak{p}})/\mathcal{T}^0(\mathcal{O}_{\mathfrak{p}}) \cong \phi_T(k) = \Phi_T(k)_{\mathrm{tor}}$.

Proof. By Corollary 1.4, $X^0_{\mathrm{sm}}(\mathcal{O}_{\mathfrak{p}}) = \mathcal{T}^0(\mathcal{O}_{\mathfrak{p}})$. Recall that $\mathcal{T}$ is the smooth schematic closure $\mathcal{N}_{\mathrm{sm}}$ of T in $\mathcal{S} = R_{\mathcal{O}_{\mathfrak{P}}/\mathcal{O}_{\mathfrak{p}}}(\mathcal{T}_L)$, whereas $X_{\mathrm{sm}} = \mathcal{N}_{\mathrm{sm}} \cap \mathcal{S}^0$ (see (1.2)). Thus as an abelian group:

$$\mathcal{T}(\mathcal{O}_{\mathfrak{p}})/X_{\mathrm{sm}}(\mathcal{O}_{\mathfrak{p}}) \subseteq \mathcal{S}(\mathcal{O}_{\mathfrak{p}})/\mathcal{S}^0(\mathcal{O}_{\mathfrak{p}}) \cong \Phi_R(k),$$

where Φ_R is the group of component of $R_{L/K}(\mathbb{G}_m^d)$ and it is free (see [Xar, Lemma 2.6]). We get a decomposition of abelian groups:

$$\Phi_T(k) \cong T(K)/\mathcal{T}^0(\mathcal{O}_{\mathfrak{p}}) \cong \mathcal{T}(\mathcal{O}_{\mathfrak{p}})/X_{\mathrm{sm}}(\mathcal{O}_{\mathfrak{p}}) \times X_{\mathrm{sm}}(\mathcal{O}_{\mathfrak{p}})/\mathcal{T}^0(\mathcal{O}_{\mathfrak{p}})$$

on which the first factor is free and the second is finite (see Remark 1.1) and therefore is the torsion part of $\Phi_T(k)$. □

2.2. Local volume computations

As K is locally compact, it admits a left invariant Haar measure. We normalize such an additive measure dx on K by requiring: $dx(\mathfrak{p}) = q^{-1}$, which is equivalent to $dx(\mathcal{O}_{\mathfrak{p}}) = 1$. This induces a multiplicative Haar measure $\omega_{\mathfrak{p}}$ on the group of points $T(K)$ (see [Weil, §2.2]).

Definition 2. Let M be one of the aforementioned $\mathcal{O}_{\mathfrak{p}}$-models of a K-torus T, namely the (smooth) standard model or the NR one. We call the reduction of M "*good*" if $M^0_{\mathfrak{p}}$ is a k-torus. As the identity components of these two models coincide, the definition of good reduction does not depend on the choice of a model.

Proposition 2.3. *(See [NX, Proposition 1.1].) A K-torus T has good reduction if and only if it splits over an unramified extension. This means that I acts trivially on $X^\bullet(T)$.*

Remark 2.4. (See [NX, Proposition 1.2].) Let Y and N be the kernel and image of the map

$$tr : X^\bullet(T) \to X^\bullet(T)^I, \quad \chi \mapsto \sum_{\sigma \in I} \chi^\sigma.$$

Then the exact sequence:

$$0 \to Y \to X^\bullet(T) \to N \to 0$$

induces an exact sequence of K-tori:

$$1 \to T_I \to T \to T_a \to 1 \tag{2.3}$$

on which T_I is the maximal subtorus of T having good reduction whereas T_a is *I-anisotropic*, i.e. $X^\bullet(T_a)^I = \{0\}$, standing extremely on the other edge with regard to the good reduction one. The sequence (2.3) shows that any torus is an extension of such two pieces.

Remark 2.5. (See [NX, Theorem 1.3].) Let $T_{(\mathfrak{p})}$ be the toric part of $\mathcal{T}^0_{\mathfrak{p}}$. There is an isomorphism of Γ/I-modules:

$$X^\bullet(T_{(\mathfrak{p})}) \cong X^\bullet(T_I) \cong X^\bullet(T)/\ker\big(X^\bullet(T) \overset{tr}{\to} X^\bullet(T)^I\big).$$

Remark 2.6. A K-torus T admits a finite type NR-model $\mathcal{T}$ if and only if $T \otimes_K K^{\mathrm{sh}}$ does not contain any subgroup of type $\mathbb{G}_m$ (see [BLR, 10.2.1]), i.e., it is I-anisotropic. In this case $\mathcal{T}$ coincides with the smooth standard model X_{sm} (see [Pop, Proposition 10.8]).

We briefly introduce now the local Artin functions which serve as a system of convergence factors in the infinite product of local measures on the global Shyr invariant (see formula (0.1)). The following definitions can be found in [Vos, §13] and in [Neu, Chapter VII, §10.1]. The Galois group of the maximal unramified subextension in L/K, namely Γ/I, is isomorphic to $\mathfrak{g} = Gal(l/k)$ and is generated by the Frobenius automorphism F. $\mathfrak{g}$ acts naturally on $X^\bullet(T)^I$, inducing an integral representation:

$$h : \mathfrak{g} \to Aut\big(X^\bullet(T)^I\big) \cong \mathbf{GL}_{d_I}(\mathbb{Z}), \quad d_I = \operatorname{rank}\big(X^\bullet(T)^I\big).$$

Denote the character of this representation by χ_T.

Definition 2.7. The *local Artin L-function* for T is defined by

$$L_{\mathfrak{p}}(s, \chi_T, L/K) = L_{\mathfrak{p}}(s, \chi_T) = \det\left(1_d - \frac{h(F)}{q^s}\right)^{-1},$$

where $s \in \mathbb{C}$ with $Re(s) > 1$.

Theorem 2.8. *(See [Vos, Theorem 14.3/3].) If T splits over an unramified extension then*: $|T(k)| \cdot q^{-d} = L_{\mathfrak{p}}(1, \chi_T)^{-1}$, $d = \dim T$.

Definition 2.9. Let G_1, G_2 be algebraic groups defined over a field K. An *isogeny* $\lambda : G_1 \to G_2$ is a surjective homomorphism of algebraic groups with finite kernel. We denote it by $\lambda : G_1 \rightleftharpoons G_2$.

Theorem 2.10. *(See [Xar, 2.19].) Let T be an algebraic torus defined over a local field K splitting over a Galois extension L with inertia subgroup I. Φ_T is torsion-free if and only if $H^1(I, X^\bullet(T)) = 0$.*

As one can observe from Theorem 2.10, the abelian group $H^1(I, X^\bullet(T))$ which is isomorphic by Tate-duality to $H^1(I, T)$, measures the lack of connectedness of the torsion part in $\Phi_T(k)$. As this property is not invariant under isogeny, according to Lemma 2.2, this prevents at the same time the Shyr invariant to be respected by an isogeny of tori.

Proposition 2.11. *(See [Ono, §1.3.1].) Let K be any field and L be a finite Galois extension with $\Gamma = Gal(L/K)$. Let $T^*, T \in \mathcal{C}(L/K)$. Then*:

$$T^* \rightleftharpoons T \quad \Leftrightarrow \quad X^\bullet\big(T^*\big) \otimes \mathbb{Q} \cong X^\bullet(T) \otimes \mathbb{Q} \quad \text{as } \Gamma\text{-modules}.$$

Lemma 2.12. *Let T^*, T be K-isogenous tori. Then the toric parts of their reductions are k-isogenous.*

Proof. Consider the two exact sequences of Γ-modules for T and T^*, induced by the canonical decomposition (2.3). Since $\mathbb{Q}$ is flat as a $\mathbb{Z}$-module, exactness is preserved after tensoring with it:

$$0 \to X^\bullet(T_a) \otimes \mathbb{Q} \to X^\bullet(T) \otimes \mathbb{Q} \to X^\bullet(T_I) \otimes \mathbb{Q} \to 0,$$
$$0 \to X^\bullet\big(T_a^*\big) \otimes \mathbb{Q} \to X^\bullet\big(T^*\big) \otimes \mathbb{Q} \to X^\bullet\big(T_I^*\big) \otimes \mathbb{Q} \to 0.$$

Recall from Remark 2.5 that:

$$X^\bullet(T_I) \cong X^\bullet(T)/\ker\big(X^\bullet(T) \xrightarrow{tr} X^\bullet(T)^I\big).$$

As T and T^* are K-isogenous, according to Theorem 2.11, $X^\bullet(T) \otimes \mathbb{Q} \cong X^\bullet(T^*) \otimes \mathbb{Q}$ and therefore:

$$\ker\big(X^\bullet(T) \otimes \mathbb{Q} \xrightarrow{tr} X^\bullet(T)^I \otimes \mathbb{Q}\big) = \ker\big(X^\bullet\big(T^*\big) \otimes \mathbb{Q} \xrightarrow{tr} X^\bullet\big(T^*\big)^I \otimes \mathbb{Q}\big)$$

implying that $X^\bullet(T_I) \otimes \mathbb{Q} \cong X^\bullet(T_I^*) \otimes \mathbb{Q}$. By Remark 2.5 this is equivalent to an isomorphism of their reduction toric part as $\mathfrak{g} = Gal(l/k)$-modules:

$$X^\bullet\big(T^*_{(\mathfrak{p})}\big) \otimes \mathbb{Q} \cong X^\bullet(T_{(\mathfrak{p})}) \otimes \mathbb{Q},$$

which again by Proposition 2.11 implies that $T^*_{(\mathfrak{p})}$ and $T_{(\mathfrak{p})}$ are k-isogenous. □

We consider another good reduction torus associated to T, namely, the factor torus T^I corresponding to the Γ-module $X^\bullet(T)^I$.

Lemma 2.13. *T_I and T^I are K-isogenous.*

Proof. Consider again the canonical decomposition of Γ-modules (2.4):

$$0 \to X^\bullet(T_a) \to X^\bullet(T) \xrightarrow{tr} X^\bullet(T_I) \to 0.$$

Taking the I-invariants gives the long exact sequence:

$$0 \to X^\bullet(T_a)^I = 0 \to X^\bullet(T)^I \to X^\bullet(T_I)^I = X^\bullet(T_I) \to H^1(I, T_a).$$

The finiteness of $H^1(I, T_a)$ implies the one of $\operatorname{coker}(X^\bullet(T)^I \to X^\bullet(T_I))$, which means that the $\mathbb{Z}$-lattice $X^\bullet(T)^I$ is a sublattice of finite index in the $\mathbb{Z}$-lattice $X^\bullet(T_I)$. Back to K-tori, this indicates that the corresponding epimorphism $T_I \to T^I$ has a finite kernel. □

Proposition 2.14. $\omega_\mathfrak{p}(\mathcal{T}^0(\mathcal{O}_\mathfrak{p})) = |\mathcal{T}^0_\mathfrak{p}(k)| \cdot q^{-d} = L_\mathfrak{p}(1, \chi_T)^{-1}$.

Proof. $\mathcal{T}^0$ is the reduction preimage of the k-group $\mathcal{T}^0_{\mathfrak{p}}$, thus it is smooth and therefore the reduction of points $\mathcal{T}^0(\mathcal{O}_{\mathfrak{p}}) \to \mathcal{T}^0_{\mathfrak{p}}(k)$ is surjective. Consider the exact sequence:

$$1 \to \mathcal{T}^1(\mathcal{O}_{\mathfrak{p}}) \to \mathcal{T}^0(\mathcal{O}_{\mathfrak{p}}) \to \mathcal{T}^0_{\mathfrak{p}}(k) \to 1. \tag{2.4}$$

The reduction image of $\mathcal{T}^1(\mathcal{O}_{\mathfrak{p}})$ is the d-tuple $(1,..,1)$ in $\mathcal{T}^0_{\mathfrak{p}}(k)$ where $d = \dim \mathcal{T}^0 = \dim T$. It is isomorphic to another preimage of this map, namely $(1+\mathfrak{p})^d$, which is homeomorphic to the additive group $\mathfrak{p}^d$ implying that $\omega_{\mathfrak{p}}(\mathcal{T}^1(\mathcal{O}_{\mathfrak{p}})) = \bigwedge_{i=1}^d dx_i(\mathfrak{p}^d) = q^{-d}$ and consequently by (2.4):

$$\omega_{\mathfrak{p}}\big(\mathcal{T}^0(\mathcal{O}_{\mathfrak{p}})\big) = \big|\mathcal{T}^0_{\mathfrak{p}}(k)\big| \cdot q^{-d}.$$

As for the right hand equality of the proposition, $\mathcal{T}^0_{\mathfrak{p}}$ is an affine smooth and connected k-group (see [NX, §1]). It has a canonical decomposition over k:

$$\mathcal{T}^0_{\mathfrak{p}} = T_{(\mathfrak{p})} \times U$$

where $T_{(\mathfrak{p})}$ is a k-torus and U is a unipotent k-group. U is isomorphic to an affine space $\mathbb{A}_k^{\dim U}$ thus $|U(k)| = q^{\dim U}$ and therefore: $|\mathcal{T}^0_{\mathfrak{p}}(k)| \cdot q^{-d} = |T_{(\mathfrak{p})}(k)| \cdot q^{-d_I}$ where $d_I = \dim T_{(\mathfrak{p})}$. Let T_I be the maximal subtorus of T with good reduction. From Remark 2.5, $X_\bullet(T_I) \cong X_\bullet(T_{(\mathfrak{p})})$ as Γ/I-modules. Thus we may deduce that $T_{(\mathfrak{p})}$ is the reduction of T_I, splitting over an unramified extension. Hence by Theorem 2.8 $|T_{(\mathfrak{p})}(k)| \cdot q^{-d_I} = L_{\mathfrak{p}}(1, \chi_{T_I})^{-1}$. According to Lemma 2.13, T_I and T^I are K-isogenous. These tori have good reduction, thus due to Lemma 2.12 their reductions (being k-tori) are k-isogenous, implying that their L-functions coincide (see [Vos, p. 106]). In particular $L_{\mathfrak{p}}(1, \chi_{T_I})$ is equal to $L_{\mathfrak{p}}(1, \chi_{T^I})$ which is by definition equal to $L_{\mathfrak{p}}(1, \chi_T)$. □

As noted in Remark 1.1, $X(\mathcal{O}_{\mathfrak{p}})$ is the maximal compact subgroup of $T(K)$. Further, it is equal to $X_{\mathrm{sm}}(\mathcal{O}_{\mathfrak{p}})$ (see [BLR, §3.1, Definition 1]). From Lemma 2.2 and Proposition 2.14 the local component in the Shyr invariant can be computed by:

Corollary 2.15. $L_{\mathfrak{p}}(1, \chi_T) \cdot \omega_{\mathfrak{p}}(X_{\mathrm{sm}}(\mathcal{O}_{\mathfrak{p}})) = (X_{\mathrm{sm}}(\mathcal{O}_{\mathfrak{p}}) : \mathcal{T}^0(\mathcal{O}_{\mathfrak{p}})) = |\Phi_T(k)_{\mathrm{tor}}|$.

3. Relation with the cocharacter group

The following construction can be found in [Ko, §7.2]. It was originally defined over a p-adic field but as we shall see, it can be applied also in case (F). Let K^{sh} be the strict henselization of the local field K in a separable closure K_s. As K^{sh} is the maximal unramified extension of K, the group $Gal(K_s/K^{\mathrm{sh}})$ is the inertia subgroup I of the absolute one $Gal(K_s/K)$. R. Kottwitz extends the canonical epimorphism $(K^{\mathrm{sh}})^\times \to \mathbb{Z}$ with kernel equal to the group of units of K^{sh}, to an epimorphism

$$\mathcal{K}_T : T\big(K^{\mathrm{sh}}\big) \to X_\bullet(T)_I$$

where the latter group is the I-coinvariants of the cocharacter group. Let $\mathcal{T}$ be the NR-model of T defined over the ring of integers $\mathcal{O}^{\mathrm{sh}}_{\mathfrak{p}}$ of K^{sh}.

Lemma 3.1. $\ker(\mathcal{K}_T) = \mathcal{T}^0(\mathcal{O}^{\mathrm{sh}}_{\mathfrak{p}})$.

Proof. As noted in the first line of the proof of [HR, Proposition 3, p. 189], $\ker(\mathcal{K}_T)$ is the unique Iwahori subgroup of $T(K^{\mathrm{sh}})$. Thus, by [HR, Definition 1, p. 188], and the statement of the cited proposition, $\ker(\mathcal{K}_T)$ coincides with $\mathcal{T}^0(\mathcal{O}^{\mathrm{sh}}_{\mathfrak{p}})$ (see [RP, Remarks 2.2(iii)]). Note that the proof applies to any strictly henselian discretely-valued field and therefore covers both cases (N) and (F). □

Since the residue field of K^{sh} is k_s, the group of components of $\mathcal{T}$ splits over it, i.e. $\Phi_T(k_s) = \Phi_T$. Hence together with Lemma 2.1, the Kottwitz construction gives rise to an isomorphism

$$T\big(K^{\mathrm{sh}}\big)/\mathcal{T}^0\big(\mathcal{O}_{\mathfrak{p}}^{\mathrm{sh}}\big) \cong \Phi_T \cong X_\bullet(T)_I. \tag{3.1}$$

Now let T be defined over K and let $\mathcal{T}$ be its NR-model. The absolute group $\mathfrak{g}_{\mathrm{sep}} = Gal(k_s/k)$ being generated by the Frobenius automorphism F, is identified with $\Gamma_{\mathrm{sh}} = Gal(K^{\mathrm{sh}}/K)$. The scheme $\mathcal{T}^0$ has a geometrically connected fiber. Moreover, it is affine over $\mathcal{O}_{\mathfrak{p}}$ (see [KM, Proposition 3] and [BLR, p. 290]). Thus by Lang's Theorem, the cohomology group

$$H^1\big(\langle F\rangle, \mathcal{T}^0\big(\mathcal{O}_{\mathfrak{p}}^{\mathrm{sh}}\big)\big) = H^1\big(\langle F\rangle, \mathcal{T}^0(k_s)\big)$$

where k is considered as an $\mathcal{O}_{\mathfrak{p}}$-algebra, is trivial. Hence taking the Γ_{sh}-invariants of (3.1) gives rise to an epimorphism (see [Ko, 7.6.2]):

$$T(K) \to \big(X_\bullet(T)_I\big)^{\langle F\rangle}$$

with kernel equals to $\mathcal{T}^0(\mathcal{O}_{\mathfrak{p}}^{\mathrm{sh}}) \cap T(K) = \mathcal{T}^0(\mathcal{O}_{\mathfrak{p}})$. Again by Lemma 2.1 and Corollary 2.15 we get:

Corollary 3.2.

$$T(K)/\mathcal{T}^0(\mathcal{O}_{\mathfrak{p}}) \cong \Phi_T(k) \cong \ker\big(1 - F|X_\bullet(T)_I\big)$$

and:

$$X_{\mathrm{sm}}(\mathcal{O}_{\mathfrak{p}})/\mathcal{T}^0(\mathcal{O}_{\mathfrak{p}}) \cong \Phi_T(k)_{\mathrm{tor}} \cong \ker\big(1 - F|X_\bullet(T)_I\big)_{\mathrm{tor}}.$$

Example 3.3. Let L be a cyclic extension of K with Galois group $\Gamma = \langle\sigma\rangle$. Let $R = R_{L/K}(\mathbb{G}_m)$ be the corresponding Weil torus, i.e., such that for any K-algebra B, $R(B) = (B \otimes L)^\times$. The *norm torus* $T' = R^{(1)}_{L/K}(\mathbb{G}_m)$ is the kernel of the norm map $N_{L/K} : R \to \mathbb{G}_{m,K}$, mapping any element of $R(B)$ to the product of its images under all Galois automorphisms. Suppose L/K is totally ramified, i.e., $I = \Gamma$. Then T' is an I-anisotropic torus. Its NR-model $\mathcal{T}'$ which is of finite type, coincides with the smooth standard one X'_{sm} (see Remark 2.6). Note that the character group $X^\bullet(R) = \mathbb{Z}[\Gamma]$ (see [Vos, §3.12]) coincides as a Γ-module with the group of cocharacters:

$$X_\bullet(R) = \mathrm{Hom}(\mathbb{G}_m, R \otimes L) = \mathrm{Hom}\big(\mathbb{Z}, \mathbb{Z}[\Gamma]\big) = \mathbb{Z}[\Gamma] = X^\bullet(R).$$

Since Γ is cyclic, T is isomorphic as a Γ-module to the projective group $P = R/\mathbb{G}_m$ (see [LL, p. 22]). The exact sequences of K-tori:

$$1 \to T' \to R \to \mathbb{G}_m \to 1,$$
$$1 \to \mathbb{G}_m \to R \to P \to 1$$

induce the exact sequences of dual modules:

$$0 \to \mathbb{Z} \to X^\bullet(R) \to X^\bullet\big(T'\big) \to 0,$$
$$0 \to \mathbb{Z} \to X_\bullet(R) \to X_\bullet(P) \to 0.$$

We get an isomorphism of Γ-modules: $X_\bullet(T') \cong X_\bullet(P) \cong X^\bullet(T')$. Explicitly we have

$$X'_{\mathrm{sm}}(\mathcal{O}_\mathfrak{p})/\mathcal{T}'^0(\mathcal{O}_\mathfrak{p}) \cong X_\bullet(T')_I = \left(\mathbb{Z}[\sigma]\Big/\sum_i \sigma^i\right)\Big/(1-\sigma) = \mu_n$$

thus by Corollary 2.15 we get $L_\mathfrak{p}(1, \chi_{T'}) \cdot \omega_\mathfrak{p}(X'_{\mathrm{sm}}(\mathcal{O}_\mathfrak{p})) = n$. More generally, for any extension L/K, we have $X'_{\mathrm{sm}}(\mathcal{O}_\mathfrak{p})/\mathcal{T}'^0(\mathcal{O}_\mathfrak{p}) \cong \phi_{T'} = \mu_e$ where e is the ramification index (see [Pop, Theorem 3]).

4. Ono and Shyr invariants

In the following section we briefly describe the analogues of the arithmetic invariants of number fields for algebraic tori as defined by Ono, and the analogue of the classical class number formula for algebraic $\mathbb{Q}$-tori, as formulated by Ono and Shyr. This construction is generalized to K-tori where K is any global field. Finally, our local results will be inserted in these invariant formulas.

4.1. Arithmetical invariants of algebraic tori

Notation 4.1. Let K be a global field, i.e., either a number field or an algebraic function field in one variable over a finite field of constants $\mathbb{F}_q$. We denote:

Δ_K – the discriminant of K. In case (F) equals to q^{2g-2} where g is the genus of K.
S – a finite set of valuations of K which contains the set S_∞ of the archimedean ones.
K_v – the completion of K with respect to a valuation $v \in S$.
$\mathcal{O}_v$ – the ring of integers of K_v and $U_v = \mathcal{O}_v^\times$ – its subgroup of units.
$T_v = T \otimes K_v$ and $T_v(\mathcal{O}_v) = \{x \in T_v(K_v)\colon\ \chi(x) \in U_v\ \forall \chi \in X^\bullet(T)_{K_v}\}$.
If $v = \mathfrak{p}$ is a prime, $T_\mathfrak{p}(\mathcal{O}_\mathfrak{p})$ is the maximal compact subgroup of $T_\mathfrak{p}(K_\mathfrak{p})$.
$\mathfrak{P}$ is a prime of L lying over $\mathfrak{p}$, $\Gamma_\mathfrak{p} = Gal(L_\mathfrak{P}/K_\mathfrak{p})$ and $I_\mathfrak{p}$ is the inertia subgroup.

In Definition 2.7 we defined the local Artin L-function. Globally, consider the action of Γ on $X^\bullet(T)$ and the corresponding representation $\Gamma \to Aut(X^\bullet(T)) \cong \mathbf{GL}_n(Z)$. Its character χ_T is decomposed into a sum of irreducible characters of Γ with integral coefficients:

$$\chi_T = \sum_{i=1}^m a_i \chi_i, \quad a_i \in \mathbb{Z},$$

where χ_1 is the *principal character*.

The *global Artin L-function* is defined by the Euler product:

$$L(s, \chi_T) = L(s, \chi_T, L/K) = \prod_\mathfrak{p} L_\mathfrak{p}(s, \chi_{T_\mathfrak{p}}, L_\mathfrak{P}/K_\mathfrak{p})$$

again with $Re(s) > 1$, having a pole at $s = 1$ of order a_1.

Definition 3. The *quasi-residue* of T is the limit:

$$\rho_T = \lim_{s \to 1} (s-1)^{a_1} L(s, \chi_T).$$

Following Ono in [Ono], for any finite set of places S which contains S_∞ we define $T_A(S) = \prod_{v \in S} T_v \prod_{v \notin S} T_v(\mathcal{O}_v)$. Then the *adelic group* T_A is the inductive limit of $T_A(S)$ with respect to S. The *group of S-units* is $T_K(S) = T_K \cap T_A(S)$. The *group of units* of T is then $T_K(S_\infty)$ where S_∞ is the set of archimedean places, which are the elements of Γ composed with the absolute norm $|\cdot|_\infty$.

Theorem 4.2 *(Shyr's generalization of Dirichlet Unit Theorem). (See [Shyr2].) The group $T_K(S)$ is a direct product of the finite group $T_K \cap T_A^c$ where: $T_A^c = \prod_v T_v(\mathcal{O}_v)$ and a group isomorphic to $\mathbb{Z}^{r(S)-r}$ where: $r(S) = \sum_{v \in S} r_v$.*

Let T be a $\mathbb{Q}$-torus and let $r_{\mathbb{Q}} = \mathrm{rank}(X^{\bullet}(T)_{\mathbb{Q}})$, $r_\infty = \mathrm{rank}(X^{\bullet}(T)_{\mathbb{R}})$. Let $\{\xi_i\}_{i=1}^{r_\infty}$ be a $\mathbb{Z}$-basis of $X^{\bullet}(T)_{\mathbb{R}}$ such that $\{\xi_i\}_{i=1}^{r_{\mathbb{Q}}}$ is a $\mathbb{Z}$-basis of $X^{\bullet}(T)_{\mathbb{Q}}$. Then the group of units $T_K(S_\infty)$ is decomposed into $W \times E$ where W is finite and $E \cong \mathbb{Z}^{r_\infty - r_{\mathbb{Q}}}$. We denote $w_T = |W|$.

Definition 4. Let $\{e_j\}_{j=r_{\mathbb{Q}}+1}^{r_\infty}$ be a $\mathbb{Z}$-basis of E. The number $R_T = |\det(\ln|\xi_i(e_j)|_{r_{\mathbb{Q}}+1 \leqslant j \leqslant r_\infty})|$ is called the *regulator* of T over $\mathbb{Q}$. Geometrically, this number represents (as for number fields) the volume of the fundamental domain for the free part of group of units.

We set: $T_A^1 = \{x \in T_A\colon\ \chi(x) \in I_K^1\ \forall \chi \in X^{\bullet}(T)_K\}$ where $I_K^1 = \{a \in I_K\colon\ \|a\| = 1\}$ and I_K denotes the idele group. It is the maximal subgroup of $T(A_K)$ such that $T_A^1/T(K)$ is compact.

Definition 5. The *class number* of T is the finite index:

$$h_T = \begin{cases} [T_A : T_K \cdot T_A^{S_\infty}] & \text{case (N)}, \\ [T_A^1 : T_K \cdot T_A^{S_\infty}] & \text{case (F)}. \end{cases}$$

The global measure is obtained by the infinite product of these local measures, multiplied by a set of convergence local factors, namely, the Artin local L-functions $L_{\mathfrak{p}}(1, \chi_T)$:

$$\tau = \rho_T^{-1} |\Delta_K|^{-\frac{d}{2}} \prod_{v|\infty} \omega_v \prod_{\mathfrak{p}} L_{\mathfrak{p}}(1, \chi_T) \omega_{\mathfrak{p}}.$$

This is called the *Tamagawa measure*. This measure applied to $\prod_{\mathfrak{p}} T_{\mathfrak{p}}(\mathcal{O}_{\mathfrak{p}})$ is convergent – almost all primes are unramified on which $L_{\mathfrak{p}}(1, \chi_T) \cdot \omega_{\mathfrak{p}}(T_{\mathfrak{p}}(\mathcal{O}_{\mathfrak{p}})) = 1$.

4.2. The quasi-discriminant definition

The geometrical meaning of a global field discriminant is the volume of a fundamental domain of its ring of integers. An analogue for the case of an algebraic torus T defined over a global field, called the "*quasi-discriminant*" of T, is the volume – with respect to some normalized Haar measure – of the fundamental domain of the maximal compact subgroup of $T(A_K)/T(K)$. In the next section we refer to two similar analogues, given by T. Ono and J.M. Shyr, to this invariant in the case of algebraic tori.

4.2.1. Ono's invariant

The following construction can be found in [Ono]. Let $\{\chi_i\}_{i=1}^{r_K}$ be a $\mathbb{Z}$-basis of $X^{\bullet}(T)_K$. Consider the epimorphism $\psi : T_A \to \mathbb{R}_+^{r_K}$ defined by

$$\alpha \mapsto \left(\ln\left|\chi_i(\alpha)\right|\right)_{1 \leqslant i \leqslant r_K}.$$

It yields the isomorphism $T_A/T_A^1 \cong \mathbb{R}_+^{r_K}$. For a Haar measure d defined on T_A, consider the decomposition

$$d(T_A/T_K) = d\left(T_A/T_A^1\right) \cdot d\left(T_A^1/T_K\right).$$

Let t_K be the pullback of the Haar measure $\frac{dx_1 \cdots dx_r}{x_1 \cdots x_r}$ on $\mathbb{R}_+^{r_K}$ from T_A/T_A^1. Define the *normalized* Haar measure Ω_T on T_A, i.e., such that $\int_{T_A^1/T_K} d(\Omega_T/t_K) = 1$.

Definition 6. Comparing the measure Ω_T/t_K with the Tamagawa measure gives the constant

$$c_T^{\text{Ono}} = \frac{\omega_T}{\Omega_T/d_K}.$$

Now assume $K=\mathbb{Q}$. Let $\{\xi_i\}_{i=1}^{r_\infty}$ be a $\mathbb{Z}$-basis of $X^\bullet(T)_\mathbb{R}$ such that $\{\xi_i\}_{i=1}^{r_\mathbb{Q}}$ is a $\mathbb{Z}$-basis of $X^\bullet(T)_\mathbb{Q}$. Define $\Phi_0: T_\mathbb{R} \to \mathbb{R}^{r_\infty}$ by

$$x \mapsto \left(\ln\left|\xi_i(x)\right|\right)_{1\leqslant i\leqslant r_\infty}.$$

By the Unit Theorem, $\operatorname{rank}(U) = r_\infty - r$. Let $\{e_j\}_{j=r_\mathbb{Q}+1}^{r_\infty}$ be a $\mathbb{Z}$-basis of E and consider the parallelotope:

$$P_0 = \left\{ \sum_{j=r_\mathbb{Q}+1}^{r_\infty} \lambda_j \Phi_0(e_j)\colon\ 0\leqslant \lambda_j \leqslant 1 \right\}.$$

This is the fundamental domain of E on $\mathbb{R}^{r_\infty - r_\mathbb{Q}}$ and its Euclidean volume is the regulator R_T. This domain may be extended to dimension r_∞ by the cube $(1\leqslant \lambda_j \leqslant e)_{1\leqslant j\leqslant r_\mathbb{Q}}$. The extended embedding $\Phi: T_A^{S_\infty} \to \mathbb{R}^{r_\infty}$ defined by $\Phi(x) = \Phi_0(x_\infty)$ (ignoring the non-archimedean places components), gives rise to a parallelotope P by a continuation of a unit cube on $\mathbb{R}^{r_\infty}$. Thus the Euclidean volume of P on $\mathbb{R}^{r_\infty}$ remains R_T.

Lemma 4.3. *(See [Ono, 3.8.5].)* $\int_{\Phi^{-1}(P)} d(\Omega_T/t_\mathbb{Q}) = w_T/h_T$.

Denote by I the unit cube in $\mathbb{R}^{r_\infty}$. Define by M_∞ the ∞ component of $\Phi^{-1}(I)$, i.e.,

$$M_\infty = \left\{x\in T_\mathbb{R}\colon\ 1\leqslant \left|\xi_i(x)\right|_\infty \leqslant e,\ 1\leqslant i\leqslant r_\infty\right\}.$$

The regulator, being the Euclidean volume of P on $\mathbb{R}^{r_\infty}$, is obtained by

$$\frac{\int_{\Phi^{-1}(P)} d(\Omega_T/t_\mathbb{Q})}{\int_{\Phi^{-1}(I)} d(\Omega_T/t_\mathbb{Q})}$$

hence we get

$$\begin{aligned} c_T^{Ono} &= \frac{\omega_T}{\Omega_T/t_\mathbb{Q}}\left(\Phi^{-1}(P)\right) = \frac{R_T\omega_T(\Phi^{-1}(I))}{\Omega_T/t_\mathbb{Q}(\Phi^{-1}(P))} = \frac{R_T\int_{M_\infty}\omega_\infty \prod_\mathfrak{p} L_\mathfrak{p}(1,\chi_T)\omega_\mathfrak{p}}{w_T/h_T} \\ &= \frac{R_T h_T}{w_T}\int\limits_{M_\infty}\omega_\infty \prod_\mathfrak{p}\int\limits_{T_\mathfrak{p}(\mathcal{O}_\mathfrak{p})} L_\mathfrak{p}(1,\chi_T)\omega_\mathfrak{p}. \end{aligned}$$

According to Theorem 4.2, the group of units of T defined over any number field K has a decomposition $T_\mathbb{Z} \cong W\times E$ on which the group E is isomorphic to $\mathbb{Z}^{r_\infty - r_K}$ and $|W| = w_T$. Hence Ono's result can be generalized to an algebraic torus defined over any number field K, by including the discriminant which is different from 1 in the general case. Recall that

$$\omega_T = |\Delta_k|^{-d/2}\omega_\infty \prod_\mathfrak{p} L_\mathfrak{p}(1,\chi_T)\omega_\mathfrak{p}.$$

Hence

$$c_T^{\mathrm{Ono}} = \frac{\omega_T}{\Omega_T/t_K}\bigl(\Phi^{-1}(P)\bigr) = \frac{R_T\omega_T(\Phi^{-1}(I))}{\Omega_T/t_K(\Phi^{-1}(P))} = \frac{R_T|\Delta_k|^{-d/2}\int_{M_\infty}\omega_\infty\prod_{\mathfrak p}L_{\mathfrak p}(1,\chi_T)\omega_{\mathfrak p}}{w_T/h_T}$$
$$= \frac{R_T h_T}{w_T}|\Delta_k|^{-d/2}\int\limits_{M_\infty}\omega_\infty\prod_{\mathfrak p}\int\limits_{T_{\mathfrak p}(\mathcal{O}_{\mathfrak p})}L_{\mathfrak p}(1,\chi_T)\omega_{\mathfrak p}. \tag{4.1}$$

Remark 4.4. This generalization is not required in case (F) on which the group of units of T has no free part, and thus has no regulator. Indeed, Ono's result in that case is not restricted to $\mathbb{F}_q(x)$ being the analogue of $\mathbb{Q}$, and Ono's formula does include the discriminant there. We will write this formula explicitly in Appendix A.

4.2.2. Shyr's invariant

J.M. Shyr in [Shyr1] gave a similar definition to the one of Ono in the case of algebraic $\mathbb{Q}$-tori. Consider the decomposition of a local measure d into $dT_v(K_v) = d(T_v(K_v)/T_v(\mathcal{O}_v))\cdot dT_v(\mathcal{O}_v)$. The measure ν_v is defined by the pullback of the measure $\frac{dx_1\cdots dx_{r_v}}{x_1\cdots x_{r_v}}$ on $\mathbb{R}_+^{r_v}$ (and the canonical discrete measure on $\mathbb{Z}^{r_{\mathfrak p}}$, respectively) for $T_v(K_v)/T_v(\mathcal{O}_v)$, matched together with the normalized Haar measure on $T_v(\mathcal{O}_v)$. Then the measure ν_T is defined by the infinite product $\prod_v \nu_v$ and $\nu_T(T_A/T_K) = \nu_T(T_A/T_A^c)$. In this new construction, the other arithmetic invariants are taken from Ono's definition. Explicitly global Shyr invariant is computed by:

$$c_T^{\mathrm{Shyr}} = \int\limits_{M_\infty}\omega_\infty\prod_{\mathfrak p}\int\limits_{T_{\mathfrak p}(\mathcal{O}_{\mathfrak p})}L_{\mathfrak p}(1,\chi_T)\omega_{\mathfrak p}.$$

Shyr obtained a relation between the Haar measures: $\tau_T = \rho_T^{-1}c_T^{\mathrm{Shyr}}\nu_T$. This led him to a formula reflecting the relation between the other arithmetic invariants of T which can be viewed as a torus analogue of the class number formula, namely:

$$c_T^{\mathrm{Shyr}} = \frac{\rho_T\tau_T w_T}{h_T R_T}. \tag{4.2}$$

As we have done in (4.1), due to the generalization of the Unit Theorem, this result can be generalized to an algebraic torus T defined over any number field K on which the field discriminant may not be 1. In that general case we would get

$$c_T^{\mathrm{Shyr}} = \omega_T\bigl(\Phi^{-1}(I)\bigr) = |\Delta_K|^{-d/2}\int\limits_{M_\infty}\omega_\infty\prod_{\mathfrak p}\int\limits_{T_{\mathfrak p}(\mathcal{O}_{\mathfrak p})}L_{\mathfrak p}(1,\chi_T)\omega_{\mathfrak p}. \tag{4.3}$$

The number $D_T = 1/(c_T^{\mathrm{Shyr}})^2$ is called the *quasi-discriminant* of T over K.

4.3. Main theorem

With the above notations, using our previous local results, namely, Corollaries 2.15 and 3.2, we now get the following computation of the Shyr invariant as appears in (4.3) with the relation in (4.2).

Theorem 4.5. *For any prime $\mathfrak p$ of K, let $F_{\mathfrak p}$ be the local Frobenius automorphism, let $X_\bullet(T_{\mathfrak p})$ be the cocharacter group of $T_{\mathfrak p} = T\otimes K_{\mathfrak p}$ and let $X_\bullet(T_{\mathfrak p})_{I_{\mathfrak p}}$ be its coinvariants factor group. Then*:

$$c_T^{\mathrm{Shyr}} = |\Delta_K|^{-d/2} C_\infty \prod_{\mathfrak{p}} \bigl|\ker\bigl(1 - F_{\mathfrak{p}} | X_\bullet(T_{\mathfrak{p}})_{I_{\mathfrak{p}}}\bigr)_{\mathrm{tor}}\bigr| = \frac{\rho_T \tau_T w_T}{h_T R_T},$$

where

$$C_\infty = \begin{cases} \int_{M_\infty} \omega_\infty & \textit{case } (\mathrm{N}), \\ (\ln q)^{-r_K} & \textit{case } (\mathrm{F}), \end{cases}$$

and $\Delta_K = q^{2g-2}$, $R_T = 1$ *in case* (F).

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Appendix A. Tori defined over algebraic function fields

Let X be a smooth, projective and irreducible algebraic curve of genus g defined over $\mathbb{F}_q$ and let Y be a finite Galois cover of X. Let $K = \mathbb{F}_q(X)$ and $L = \mathbb{F}_q(Y)$ be the corresponding fields of rational functions. Then L/K is a finite Galois extension with $\Gamma = Gal(L/K)$. Let $T \in \mathcal{C}(L/K)$ be an algebraic torus of dimension d.

Just as for number fields, let $\omega_{\mathfrak{p}}$ be a normalized invariant form on $K_{\mathfrak{p}}$, i.e., such that $\omega_p(\mathcal{O}_{\mathfrak{p}}) = 1$. Since $\deg \mathfrak{p} = (\mathcal{O}_{\mathfrak{p}}/\mathfrak{p} : \mathbb{F}_q)$, we have $|\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}| = q^{\deg \mathfrak{p}}$. Hence the normalization condition implies $\omega_{\mathfrak{p}}(\mathfrak{p}) = q^{-\deg \mathfrak{p}}$. The infinite product $\prod_{\mathfrak{p}} \omega_{\mathfrak{p}}$ which is multiplied by the set $L_{\mathfrak{p}}(1, \chi_T, L/K)$ as a system of correcting factors, induces the Tamagawa measure on T (see [Weil, §2.2]):

$$\omega_T = q^{-(g-1)d} \prod_{\mathfrak{p}} L_{\mathfrak{p}}(1, \chi_T)\omega_{\mathfrak{p}}, \quad d = \dim T.$$

Define the normalized measure Ω_T on T_A by the condition $\int_{T_A^1/T_K} d\Omega_T = 1$.

Then $\omega_T = (\ln q)^{r_K} c_T^{\mathrm{Ono}} \Omega_T$, reflecting the decomposition

$$d(T_A/T_K) = d\bigl(T_A/T_A^1\bigr) \cdot d\bigl(T_A^1/T_K\bigr)$$

on which T_A/T_A^1 is given the measure $(\ln q)^{r_K}$ (see [Ono, §3.2]). Hence in this case we get the relation:

$$\tau_T = \frac{c_T^{\mathrm{Ono}}}{\rho_T} = \frac{h_T \prod_{\mathfrak{p}} \int_{T_{\mathfrak{p}}(\mathcal{O}_{\mathfrak{p}})} L_{\mathfrak{p}}(1, \chi_T)\omega_{\mathfrak{p}}}{w_T \rho_T (\ln q)^{r_K} q^{d(g-1)}}. \tag{A.1}$$

As in Shyr's approach, consider the decomposition

$$T_A/T_K \cong \bigl(T_A/T_A^1\bigr) \times \bigl(T_A^1/T_A^{S_\infty} T_K\bigr) \times \bigl(T_A^{S_\infty} T_K/T_K\bigr).$$

By the same construction as in Section 4.2.2, $\mu_T|_{T_A/T_A^1} = t_K$ and therefore:

$$\rho_T^{-1} \omega_T(T_A/T_K) = t_K\bigl(T_A/T_A^1\bigr) \cdot \tau_T = \frac{\mu_T(T_A/T_K)}{\mu_T(T_A/T_A^1)} \cdot \tau_T.$$

Now, as we gave the measure $(\ln q)^{r_K}$ to each point in T_A/T_A^1, we get

$$\rho_T^{-1}\omega_T(T_A/T_K) = \tau_T \cdot \frac{\mu_T(T_A/T_K)}{(\ln q)^{r_K}}.$$

Thus over T_A/T_K we have $\rho_T^{-1}\omega_T \cdot (\ln q)^{r_K} = \tau_T \mu_T$.

Since here there are no archimedean places, we have $T_A^{S_\infty} = \prod_{\mathfrak{p}} T_{\mathfrak{p}}(\mathcal{O}_{\mathfrak{p}}) = T_A^{1,S_\infty}$.

Now $h_T = (T_A^1 : T_A^{S_\infty} T_K)$ and $T_A^{S_\infty} T_K/T_K \cong T_A^{S_\infty}/W$ where W is finite.

Thus $c_T^{\mathrm{Shyr}} = \frac{\omega_T}{\nu_T}(T_A^{S_\infty})$. But: $T_A^{S_\infty} \subset T_A^1$ and so: $\nu_T(T_A^{S_\infty}) = 1$. Hence

$$c_T^{\mathrm{Shyr}} = \frac{\omega_T}{\nu_T}\big(T_A^{S_\infty}\big) = \frac{(\ln q)^{-r_K}}{q^{d(g-1)}} \prod_{\mathfrak{p}} L_{\mathfrak{p}}(1, \chi_T)\omega_{\mathfrak{p}}\big(T_{\mathfrak{p}}(\mathcal{O}_{\mathfrak{p}})\big). \tag{A.2}$$

From the equation above $\tau_T = \rho_T^{-1} c_T^{\mathrm{Shyr}} \nu_T$, we get the following relation, which can be viewed as a *class number formula analogue* for algebraic tori defined over function fields:

$$c_T^{\mathrm{Shyr}} = \frac{\omega_T}{\nu_T}\big(T_A^{S_\infty}\big) = \rho_T \tau_T \mu_T\big(T_A^{S_\infty}\big) = \frac{\rho_T \tau_T w_T}{h_T}. \tag{A.3}$$

Note that this result is none other than formula (A.1) obtained by T. Ono (see [Ono, 3.8.10′]).

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