ELSEVIER

Topology and its Applications 75 (1997) 51–79

TOPOLOGY AND ITS APPLICATIONS

Long chains of topological group topologies— A continuation ☆

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Received 10 July 1995; revised 21 May 1996

Abstract

We continue the work initiated in our earlier article (J. Pure Appl. Algebra 70 (1991) 53–72); as there, for G a group let $\mathcal{B}(G)$ (respectively $\mathcal{N}(G)$) be the set of Hausdorff group topologies on G which are (respectively are not) totally bounded. In this abstract let $\mathcal{A}$ be the class of (discrete) maximally almost periodic groups G such that $|G| = |G/G'|$. We show (Theorem 3.3(A)) for $G \in \mathcal{A}$ with $|G| = \gamma \geqslant \omega$ that the condition that $\mathcal{B}(G)$ contains a chain $\mathcal{C}$ with $|\mathcal{C}| = \beta$ is equivalent to a natural and purely set-theoretic condition, namely that the partially ordered set $\langle \mathcal{P}(2^\gamma), \subseteq \rangle$ contains a chain of length β. (Thus the algebraic structure of G is irrelevant.) Similar results hold for chains in $\mathcal{B}(G)$ of fixed local weight, and for chains in $\mathcal{N}(G)$.

Theorem 6.4. If $\mathcal{T}_1 \in \mathcal{B}(G)$ and the Weil completion $\overline{\langle G, \mathcal{T}_1 \rangle}$ is connected, then for every Hausdorff group topology $\mathcal{T}_0 \subseteq \mathcal{T}_1$ with $w\langle G, \mathcal{T}_0 \rangle < \alpha_1 = w\langle G, \mathcal{T}_1 \rangle$ there are 2^{α_1}-many group topologies between $\mathcal{T}_0$ and $\mathcal{T}_1$.

From Theorem 7.4. Let F be a compact, connected Lie group with trivial center. Then the product topology $\mathcal{T}_0$ on F^ω is the only pseudocompact group topology on F^ω, but there are chains $\mathcal{C} \subseteq \mathcal{B}(F^\omega)$ and $\mathcal{C}' \subseteq \mathcal{B}(F^\omega)$ with $|\mathcal{C}| = (2^{\mathfrak{c}})^+$ and $|\mathcal{C}'| = 2^{(\mathfrak{c}^+)}$ such that $\mathcal{T}_0 \subseteq \bigcap \mathcal{C}$ and $\mathcal{T}_0 \subseteq \bigcap \mathcal{C}'$.

Keywords: Topological group; Group topology; Pseudocompact topological group; Totally bounded group topology; Pre-compact group topology; Weight; Local weight; Minimally almost periodic group; Maximally almost periodic group; Bohr compactification; Lie group; Van der Waerden group

AMS classification: Primary 22A05; 54H11, Secondary 22E46

☆ Although this paper was completed in all its essentials (including detailed proofs) in the summer of 1994, it was submitted for publication only in July 1995: Several results of this paper are taken from the "Habilitationsschrift" [49] of the second-listed author, which was submitted to the Department of Mathematics of Universität Hannover in November 1995.

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PII S0166-8641(96)00078-8

1. Introduction and summary

Given an infinite Abelian group G and a cardinal γ such that $\log|G| \leqslant \gamma \leqslant 2^{|G|}$, the first-listed author with Berhanu and Reid [6] determined the width, height and depth of the poset $\mathcal{B}_\gamma(G)$ of totally bounded group topologies on G with (local) weight γ. These results were extended by the second-listed author [48] to groups in the class $\mathcal{A}$ defined as follows: $G \in \mathcal{A}$ if G in its discrete topology is maximally almost periodic and the commutator subgroup G' of G satisfies $|G/G'| = |G| \geqslant \omega$. Motivated by these results, we initiated in [15] the study of long chains of Hausdorff group topologies (with emphasis on totally bounded group topologies) on a given group. We showed there *inter alia* that an Abelian group G with $|G| = \alpha$ admits such a chain of length β if and only if the obviously necessary set-theoretic condition is satisfied, namely that the power set $\mathcal{P}(\alpha)$ admits a chain of length β. In Section 3 of the present paper we extend this result to groups $G \in \mathcal{A}$, and we give in addition the correct extension of this characterization to chains of topologies of fixed local weight. Independently Dikranjan [26,27] and co-authors [5] studied these and similar issues in a larger, more general class of groups (the so-called weakly Abelian groups—see [5, (7.6)] for a definition) and improved our Theorem 3.3 (cf. [26, Theorem 1]; [27, Theorem 2.2]; [5, Theorem 7.7]); according to [27, p. 140], the proofs follow the proof of Theorem 2.13 of [48]. We announced Theorem 3.3 in the abstract [12].

To our knowledge, chains (and anti-chains) of pseudocompact and of countably compact group topologies on Abelian groups were investigated for the first time in [13]. For compact, connected groups, this study is continued here in Section 7. Recently Dikranjan [28] has presented many new and interesting results concerning chains of pseudocompact group topologies for pre-compact varieties, for relatively free groups, and for Abelian groups; these results appear to be not closely related to ours in Section 7.

The results of Section 3 give rise naturally to these questions: If G is an infinite discrete maximally almost periodic group with $|G| = |G/G'|$, must G admit a totally bounded group topology of weight $\log|G|$? Does every totally bounded group topology on an infinite Abelian group G contain a (smaller) group topology of weight $\log|G|$? We respond negatively to these questions in Theorems 3.15 and 3.12, respectively.

While our principal focus throughout is on Hausdorff group topologies, we retreat briefly in Section 4 to show that every group G with $|G/G'| \geqslant \omega$ admits a minimally almost periodic (in general, non-Hausdorff) group topology of countable local weight. This gives in Section 5 for such groups G an order-isomorphism from the set $\mathcal{B}(G)$ of totally bounded Hausdorff group topologies on G into the set $\mathcal{N}(G)$ of those which are not totally bounded, as well as a partial "fixed local weight" analogue in $\mathcal{N}(G)$ of the set-theoretic characterization of Section 3 for $\mathcal{B}(G)$.

The results of Section 6, all taken from the unpublished thesis of Remus [49], are included here in the interest of readability and completeness. The principal contribution is the exact computation, for suitably restricted group topologies $\mathcal{T}_i$ $(i = 0, 1)$ on certain groups G, of the cardinality of the "topological interval"

$$[\mathcal{T}_0, \mathcal{T}_1] = \{\mathcal{S}\colon\ \mathcal{T}_0 \subseteq \mathcal{S} \subseteq \mathcal{T}_1,\ \mathcal{S} \text{ is a group topology for } G\}.$$

(Of course, the "interval" $[\mathcal{T}_0, \mathcal{T}_1]$ is partially ordered, not necessarily linearly ordered.) In Sections 6 and 7, given a totally bounded group $G = \langle G, \mathcal{T}\rangle$ of maximal weight $2^{|G|}$ whose completion $\overline{\langle G, \mathcal{T}\rangle}$ is connected, we show that there are $2^{2^{|G|}}$-many totally bounded (necessarily connected) group topologies $\mathcal{S}$ on G contained in $\mathcal{T}$; and we classify exactly those cardinals β for which the set $\{\mathcal{S} \in \mathcal{B}(G)\colon\ \mathcal{S} \subseteq \mathcal{T}\}$ contains a chain of length β. And given a compact, connected group $G = \langle G, \mathcal{T}\rangle$ with $\mathrm{cf}(w(G)) > \omega$, we show that there are $2^{2^{|G|}}$-many pseudocompact (necessarily connected) group topologies $\mathcal{S}$ on G containing $\mathcal{T}$; and we classify exactly those cardinals β for which the set

$$\{\mathcal{S} \in \mathcal{B}(G)\colon\ \mathcal{S} \text{ is pseudocompact and } \mathcal{S} \supseteq \mathcal{T}\}$$

contains a chain of length β. Finally in Section 7 we give examples of compact metric groups $\langle G, \mathcal{T}\rangle$ such that $\mathcal{T}$ is the only pseudocompact group topology on G yet there exist long chains of totally bounded group topologies on G.

Some of our results were announced in [12,11], and [14, 3.10I].

2. Preliminaries

Notation 2.1. The symbols α, β and γ denote (generally, infinite) cardinal numbers, and ω is the least infinite cardinal.

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the set of integers, rationals, reals and complex numbers, in each case with the usual topology and the usual algebraic operations, and $\mathbb{T} = \{\zeta \in \mathbb{C}\colon\ |\zeta| = 1\}$. The set of primes is denoted $\mathbb{P}$.

The cyclic group of order n is denoted $\mathbb{Z}(n)$.

Except when dealing explicitly with the additive group $\mathbb{C}$ or one of its subgroups, we denote the identity element of each group G we consider (whether or not G is Abelian) by 1 or 1_G.

The center of a group G is denoted $Z(G)$.

The commutator subgroup of a group G is denoted G'. We note that when $\langle G, \mathcal{T}\rangle$ is a topological group the group G' may fail to be $\mathcal{T}$-closed, so the quotient topology $\mathcal{T}_q$ on G/G' may lack the Hausdorff separation property.

The notation $N \lhd G$ indicates that N is a normal subgroup of the group G.

With every topological group G we associate the Bohr compactification $\mathrm{b}G$ of G and the Bohr homomorphism b from G into $\mathrm{b}G$.

The following statement of the properties we need is (in its essentials) taken from Heyer [37, V§4].

Theorem 2.2. *For every topological group $G = \langle G, \mathcal{T}\rangle$ there are a compact group* $\mathrm{b}G$ *and a continuous homomorphism* b *from G onto a dense subgroup of* $\mathrm{b}G$ *such that: for every continuous homomorphism h from G into a compact group K there is a continuous homomorphism $\bar{h}$ from* $\mathrm{b}G$ *into K such that $h = \bar{h} \circ \mathrm{b}$.*

(For an early construction of bG when G is a locally compact Abelian Hausdorff group, see Anzai and Kakutani [3]; for such groups G, they refer to bG as the *universal Bohr compactification of* G. To our knowledge, bG was defined and examined in the present unrestricted setting by Weil [56], then independently by Alfsen and Holm [2]. In the terminology of [56] and [2], the group here denoted bG is called the *groupe compact attaché à* G and the *maximal compact representation of* G, respectively.)

Terminology and notation 2.3. We say that a group $G = \langle G, \mathcal{T} \rangle$ is *pre-compact* if the topology $\mathcal{T}$ is the initial topology imposed on G by the Bohr homomorphism $\mathrm{b}: G \to \mathrm{b}G$. A pre-compact Hausdorff group is said to be *totally bounded*. Thus the totally bounded groups are exactly those Hausdorff groups G whose Weil completion [55] $\overline{G}$ is a compact group. Since a topological group is a dense topological subgroup of at most one compact group, we have: A topological group G is totally bounded if and only if $\mathrm{b}G = \overline{G}$.

Notation 2.4. The lattice of pre-compact group topologies on a group G is denoted $PK(G)$; the set of totally bounded group topologies on a group G is denoted $\mathcal{B}(G)$; the set of Hausdorff group topologies which are not totally bounded on a group G is denoted $\mathcal{N}(G)$.

The weight of a space X, and the local weight of X at $p \in X$, are denoted $w(X)$ and $\chi(p, X)$ respectively; for a topological group G we write $\chi(G)$ in place of $\chi(1, G)$.

The following theorem is well known in the Hausdorff context, but we could not find the general case stated explicitly in the literature. We include a proof for the reader's convenience.

Theorem 2.5. *Let* $\langle G, \mathcal{T} \rangle$ *be a topological group.*

(a) *If* D *is dense in* G *then every* $p \in D$ *satisfies* $\chi(p, D) = \chi(G)$.

(b) *If* $\mathcal{T} \in PK(G)$ *and* $\chi\langle G, \mathcal{T} \rangle = \alpha \geqslant \omega$, *then* $w\langle G, \mathcal{T} \rangle = \alpha$.

Proof. (a) The inequality $\leqslant$ being obvious, we show $\geqslant$. It is enough to take the case $p = 1$ and to show that if $\mathcal{A}$ is a set of neighborhoods of 1 in G such that $\{U \cap D\colon U \in \mathcal{A}\}$ is a local base at 1 in D, then $\mathcal{A}$ is a local base at 1 in G. Given a neighborhood W of 1 in G let V be a neighborhood of 1 in G such that $V^2 \subseteq W$; there is $U \in \mathcal{A}$ such that $U \cap D \subseteq V \cap D$, so

$$1 \in \overline{U} = \overline{U \cap D} \subseteq \overline{V \cap D} = \overline{V} \subseteq V^2 \subseteq W$$

as required.

(b) Let $\mathcal{A}$ be a local $\mathcal{T}$-base at 1, and for $U \in \mathcal{A}$ let $1 \in V_U \in \mathcal{T}$ and finite $F_U \subseteq G$ satisfy $V_U^{-1} \cdot V_U \subseteq U$ and $F_U \cdot V_U = G$. Then $\{xU\colon U \in \mathcal{A},\ x \in F_U\}$ is a base for $\mathcal{T}$. □

It follows from Theorem 2.5 that for $\alpha \geqslant \omega$ and G a group with $\mathcal{T} \in \mathcal{B}(G)$, the four conditions "$\chi\langle G, \mathcal{T} \rangle = \alpha$", "$\chi\overline{\langle G, \mathcal{T} \rangle} = \alpha$", "$w\langle G, \mathcal{T} \rangle = \alpha$", and "$w\overline{\langle G, \mathcal{T} \rangle} = \alpha$" are equivalent. In what follows we use these conditions interchangeably.

For $\alpha \geqslant \omega$ and G a group we write

$$\mathcal{B}_\alpha(G) = \{\mathcal{T} \in \mathcal{B}(G)\colon\ \chi\langle G, \mathcal{T}\rangle = \alpha\} \quad \text{and}$$
$$\mathcal{N}_\alpha(G) = \{\mathcal{T} \in \mathcal{N}(G)\colon\ \chi\langle G, \mathcal{T}\rangle = \alpha\}.$$

The following two statements are very useful. Clearly (a) is a consequence of (b), but historically (a) preceded (b). A detailed proof of (a) is available in [36, 28.58(b)]. To our knowledge, (b) is due to Pełczyński [43, 8.10]; for a direct proof using modern techniques and for references to relevant papers of Hagler and Gerlits and Efimov and Šapirovskiĭ, see Shakhmatov [52].

Theorem 2.6. *Let G be an infinite compact group. Then*
(a) $|G| = 2^{w(G)}$, *and*
(b) *G contains a copy of the space* $\{0,1\}^{w(G)}$.

Theorem 2.7. *Let $\langle G, \mathcal{T}_1\rangle$ be a totally bounded topological group and let $w\langle G, \mathcal{T}_1\rangle = \alpha_1 > \alpha_0 \geqslant |G| \geqslant \omega$. Then there is a Hausdorff group topology $\mathcal{T}_0$ on G such that $\mathcal{T}_0 \subseteq \mathcal{T}_1$ and $w\langle G, \mathcal{T}_0\rangle = \alpha_0$.*

Proof. It is enough to find a continuous isomorphism π from $\langle G, \mathcal{T}_1\rangle$ onto a topological group H with $w(H) = \alpha_0$.

We note first that for arbitary $\alpha \geqslant \omega$ the generalized Cantor set $F(\alpha) := \{0,1\}^\alpha$ contains a compact set S such that $|S| = \alpha$ and $w(S) = \alpha$. Indeed for $\xi < \alpha$ define $p(\xi) \in F(\xi)$ by

$$p(\xi)_\eta = \begin{cases} 0 & \text{if } \eta \neq \xi, \\ 1 & \text{if } \eta = \xi \end{cases}$$

and set $X = \{p(\xi)\colon\ \xi < \alpha\}$. Then the set $S := \overline{X}^{F(\alpha)} = X \cup \{O\}$ (with $O_\xi = 0$ for all $\xi < \alpha$) is as required.

It is a consequence of the Peter–Weyl theorem as generalized to arbitrary compact groups by van Kampen (cf. [35, 22.14]) that the compact group $\overline{\langle G, \mathcal{T}_1\rangle}$ embeds as a topological subgroup of a group $K = \prod_{i\in I} K_i$ with each K_i a compact metrizable group (of matrices) and with $|I| = w(G, \mathcal{T}_1) = w\overline{(G, \mathcal{T}_1)} = \alpha_1$. In what follows we identify $\overline{\langle G, \mathcal{T}_1\rangle}$ with its image in K—that is, we write $\overline{\langle G, \mathcal{T}_1\rangle} \subseteq K$. Using 2.6(b) and the preceding paragraph we write

$$K \supseteq \overline{\langle G, \mathcal{T}_1\rangle} \supseteq F(\alpha_1) \supseteq F(\alpha_0) \supseteq S$$

with S compact and $|S| = w(S) = \alpha_0$; and using $|G \cup S| = \alpha_0$ we find $J \subseteq I$ so that $|J| \leqslant \alpha_0$ and the projection $\pi_J : K \to K_J := \prod_{i\in J} K_i$ is one-to-one on $G \cup S$.

We write $\pi = \pi_J|G$ and $H = \pi[G] \subseteq K_J$. Clearly

$$w(H) \leqslant w[K_J] = |J| \leqslant \alpha_0.$$

Now $\pi_J|[S]$ is a homeomorphism, and $S \subseteq \overline{\langle G, \mathcal{T}_1\rangle} = \mathrm{cl}_K\langle G, \mathcal{T}_1\rangle$. From $\pi_J[S] \subseteq \mathrm{cl}_K\, \pi_J[G] = \mathrm{cl}_K\, H = \overline{H}$ then follows $w(H) = w(\overline{H}) \geqslant w(S) = \alpha_0$, so finally $w(H) = \alpha_0$, as required. $\square$

Corollary 2.8 (Remus [48, 2.9]). *Let G be an infinite discrete maximally almost periodic group. Then some* $\delta \leqslant |G|$ *satisfies* $\mathcal{B}_\delta(G) \neq \emptyset$.

Remarks 2.9. (a) Since a compact Hausdorff topology admits no strictly coarser Hausdorff topology, the reader might question how Theorem 2.7 can be valid in the case that the hypothesized group $\langle G, \mathcal{T}_1\rangle$ is compact. This objection is not valid, since for such groups G the hypothesis $|G| < w(G)$ is incompatible with Theorem 2.6(a); indeed, every compact Hausdorff space X satisfies $w(X) \leqslant |X|$ (cf. [30, 3.1.21]).

(b) Attempts to discard the hypothesis $\alpha_0 \geqslant |G|$ in Theorem 2.7 are thwarted for two reasons. (1) We show in Theorem 3.15 that there are groups G such that $\mathcal{B}(G) \neq \emptyset$, yet $\mathcal{B}_\alpha(G) = \emptyset$ for every $\alpha < |G|$; and (2) we show in Theorem 3.12 that for every infinite Abelian group G (say with $|G| = \gamma \geqslant \omega$) there is $\mathcal{T}_1 \in \mathcal{B}_{2^\gamma}(G)$ such that every $\mathcal{T}_0 \in \mathcal{B}_{\alpha_0}(G)$ with $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies $\alpha_0 \geqslant \gamma$.

Theorem 2.10. *Let* $\mathcal{T}_i$ $(i = 0, 1)$ *be group topologies on a group G with* $\mathcal{T}_0 \in PK(G)$.

(a) *If* $\mathcal{A}_i$ $(i = 0, 1)$ *is a local base at* 1 *for* $\mathcal{T}_i$, *then the family* $\mathcal{A} := \{A_0A_1\colon A_i \in \mathcal{A}_i\}$ *is a local base at* 1 *for* $\mathcal{T}_0 \wedge \mathcal{T}_1$.

(b) *If* $\chi\langle G, \mathcal{T}_i\rangle = \alpha_i$ *then* $\chi\langle G, \mathcal{T}_0 \wedge \mathcal{T}_1\rangle \leqslant \alpha_0 \cdot \alpha_1$.

Proof. (a) It suffices to show that $\mathcal{A}$ is a local base at 1 for a group topology on G. The four required properties given by [35, 4.5] are readily verified, using the fact (see for example [50, 9.22(b)]) that $\langle G, \mathcal{T}_0\rangle$ is a SIN-group in the sense that for every $U \in \mathcal{A}_0$ there is $V \in \mathcal{A}_0$ such that $V \subseteq \bigcap_{x\in G} xUx^{-1}$.

(b) is immediate from (a). □

Remarks 2.11. (a) For a result stronger and more general than Theorem 2.10(a), due to S. Dierolf and W. Roelcke, see [51, 4.11]. This dissertation contains also (pp. 43–44) an example showing that if in Theorem 2.10 above the topology $\mathcal{T}_0 \in PK(G)$ is replaced by an arbitrary group topology, the resulting assertion can fail.

(b) Given a family $\mathfrak{t}$ of group topologies on a group G, the topology $\vee\mathfrak{t}$ generated by $\bigcup\mathfrak{t}$ is a group topology on G; we write $\vee_{top}\,\mathfrak{t} = \vee_{gtop}\,\mathfrak{t}$. Since $\vee_{top}\,\mathfrak{t}$ is the topology induced on (the diagonal copy of) G by the product space $\prod_{t\in\mathfrak{t}} \langle G, t\rangle$, we have $\vee_{top}\,\mathfrak{t} = \vee_{gtop}\,\mathfrak{t} \in PK(G)$ if $\mathfrak{t} \subseteq PK(G)$; thus $PK(G)$ is a complete lattice. It is worthwhile to take note, however, of the following fact, which helps to explain our interest in Section 4 in group topologies which do not satisfy the Hausdorff separation property: $\mathcal{B}(G)$ need not be a lattice, and $PK(G)$ need not be a sublattice of the lattice of topologies on G. For an example to this effect fix distinct $p, q \in \mathbb{P}$ and let $\mathcal{T}_p$ and $\mathcal{T}_q$ be respectively the p-adic and q-adic topology on the group $G = \mathbb{Z}$. (A base at 0 in $\mathcal{T}_p$ is all sets of the form $\{kp^n\colon k \in \mathbb{Z}\}$ $(0 < n \in \mathbb{Z})$.) Then $\mathcal{T}_p, \mathcal{T}_q \in \mathcal{B}(\mathbb{Z})$, but from Theorem 2.10(a) it follows that a subset U of $\mathbb{Z}$ which contains both a $\mathcal{T}_p$- and a $\mathcal{T}_q$-neighborhood of 0 satisfies $U = \mathbb{Z}$. Thus $\mathcal{T}_p$ and $\mathcal{T}_q$ have no common lower bound in $\mathcal{B}(\mathbb{Z})$, and $\mathcal{T}_p \wedge_{gtop} \mathcal{T}_q = \{\emptyset, \mathbb{Z}\} \in PK(\mathbb{Z})\backslash\mathcal{B}(\mathbb{Z})$; every co-finite subset of $\mathbb{Z}$ is $\mathcal{T}_p$-open and $\mathcal{T}_q$-open, so the inclusion $\mathcal{T}_p \wedge_{top} \mathcal{T}_q \supseteq \mathcal{T}_p \wedge_{gtop} \mathcal{T}_q$ is proper.

The above-cited example is from Kowalsky [41, 36E] (where $\vee$ and $\wedge$ correspond to our $\wedge$ and $\vee$, respectively). See also Remus [45, pp. 18–21] for a detailed examination of this and related phenomena.

Notation 2.12. A group G with its discrete topology $\mathcal{P}(G)$ is denoted G_d or $\langle G, \mathcal{T}_d\rangle$. A group G with its anti-discrete topology $\{\emptyset, G\}$ is denoted G_{ad} or $\langle G, \mathcal{T}_{ad}\rangle$.

Given an Abelian group $G = \langle G, \mathcal{T}\rangle$, the set of continuous homomorphisms from G to the circle group $\mathbb{T}$ is denoted $\hat{G}$ or $\langle G, \mathcal{T}\rangle\hat{}$. Thus in particular

$\widehat{G_d} = \langle G, \mathcal{T}_d\rangle\hat{} = \mathrm{Hom}(G, \mathbb{T})$

for every Abelian group G.

In Theorems 2.13–2.16 we cite from the literature several results we need later.

Theorem 2.13 (cf. [35, §16]). *Every Abelian group G of infinite rank such that $|G| = \gamma$ contains a direct sum as follows: $G \supseteq \bigoplus_{\xi<\gamma} G_\xi$ with each G_ξ cyclic, $|G_\xi| > 1$.*

Theorem 2.14 (Kakutani [40]). *Every infinite Abelian group H satisfies $|\mathrm{Hom}(H, \mathbb{T})| = 2^{|H|}$.*

Parts of the following result may be viewed as a refinement (in the Abelian context) of the fact that every totally bounded group G satisfies $\mathrm{b}G = \overline{G}$.

Theorem 2.15 [22]. *Let G be an Abelian group. For H a point-separating subgroup of* $\mathrm{Hom}(G, \mathbb{T})$ *let $\mathcal{T}_H$ be the topology induced on G by H. Then*

(a) $\mathcal{T}_H \in \mathcal{B}(G)$;

(b) $\langle G, \mathcal{T}_H\rangle\hat{} = H$; *and*

(c) *if H and I are different point-separating subgroups of* $\mathrm{Hom}(G, \mathbb{T})$, *then $\mathcal{T}_H \neq \mathcal{T}_I$.*

Conversely, every $\mathcal{T} \in \mathcal{B}(G)$ satisfies: $H = \langle G, \mathcal{T}\rangle\hat{}$ separates points of G, and $\mathcal{T} = \mathcal{T}_H$.

Finally: a subgroup H of $\mathrm{Hom}(G, \mathbb{T})$ *separates points of G if and only if H is dense in* $\mathrm{Hom}(G, \mathbb{T})$ *in the usual compact group topology (inherited from $\mathbb{T}^G$).*

For a group G we denote by $LA(\mathrm{b}G_d)$ the lattice of closed, normal subgroups of the Bohr compactification $\mathrm{b}G_d$ of the discrete group G_d, and by Σ_0 the dual object (defined as in [36], for example) of $\mathrm{b}G_d$. The relations we need between these structures and the lattice $PK(G)$ are given in the following theorem.

Theorem 2.16. *Let G be a group.*

(a) (Remus [45, 3.7]; [46, Lemma 2]) *There is a lattice anti-isomorphism ψ from $PK(G)$ onto $LA(\mathrm{b}G_d)$ such that every $\mathcal{T} \in PK(G)$ satisfies $\mathrm{b}G_d/\psi(\mathcal{T}) = \mathrm{b}\langle G, \mathcal{T}\rangle$.*

(b) (Remus [48, 2.3]) *There is an order-isomorphism f from $PK(G)$ onto a subset of $\mathcal{P}(\Sigma_0)$ such that every totally bounded $\mathcal{T} \in PK(G)$ satisfies $|f(\mathcal{T})| = w\langle G, \mathcal{T}\rangle$.*

Notation 2.17. Let α, β and γ be infinite cardinals. Then

(a) $C(\gamma,\beta)$ means: there is a chain $\mathcal{C} \subseteq \mathcal{P}(\gamma)$ such that $|\mathcal{C}| = \beta$;

(b) $E(\gamma,\beta,\alpha)$ means: there is a chain $\mathcal{C} \subseteq \mathcal{P}(\gamma)$ such that $|\mathcal{C}| = \beta$ and each $C \in \mathcal{C}$ satisfies $|C| = \alpha$.

The assertions of Theorem 2.18 are proved in Baumgartner [4]. See also [15, §1] for commentary and additional references.

Theorem 2.18 (Baumgartner [4]). (a) $C(2^\omega, 2^{(2^\omega)})$ *fails in some models of ZFC*;

(b) $C(2^\kappa, 2^{(\kappa^+)})$ *for all* $\kappa \geqslant \omega$;

(c) $C(\kappa,\kappa^+)$ *for all* $\kappa \geqslant \omega$; *and*

(d) *if GCH is assumed, then* $C(\kappa, 2^\kappa)$ *for all* $\kappa \geqslant \omega$.

Corollary 2.19. *Let* α *and* γ *be infinite cardinals with* $\alpha \leqslant 2^\gamma$. *Then* $E(2^\gamma, \alpha^+, \alpha)$ *holds.*

Proof. If the chain $\mathcal{C} \subseteq \mathcal{P}(\alpha)$ witnesses $C(\alpha,\alpha^+)$, then $\{C \times \alpha\colon\ C \in \mathcal{C}\}$ witness $E(\alpha\times\alpha, \alpha^+, \alpha)$. From $|\alpha \times \alpha| = \alpha \leqslant 2^\gamma$ then follows $E(2^\gamma,\alpha^+,\alpha)$. □

It is noted in [15, 1.10] that if α, β and γ are cardinals with $\alpha \geqslant \omega$ such that $E(\gamma,\beta,\alpha)$, then $\beta \leqslant 2^\alpha$. From this remark it follows, since a topological space $\langle X, \mathcal{T}\rangle$ with $w(X,\mathcal{T}) \leqslant \alpha$ satisfies $|\mathcal{T}| \leqslant 2^\alpha$, that if X is a set and $\mathcal{C}$ is a chain of topologies on X such that $w(X,\mathcal{T}) \leqslant \alpha$ for each $\mathcal{T} \in \mathcal{C}$ (with $\alpha \geqslant \omega$), then $|\mathcal{C}| \leqslant 2^{2^\alpha}$. We show now that for Tychonoff topologies a better inequality is available.

Theorem 2.20. *Let* α *and* β *be cardinals with* $\alpha \geqslant \omega$, *let* X *be a set, and let* $\mathcal{C}$ *be a chain of Tychonoff topologies on* X *such that* $|\mathcal{C}| = \beta$ *and* $w(X,\mathcal{T}) \leqslant \alpha$ *for each* $\mathcal{T} \in \mathcal{C}$. *Then* $\beta \leqslant 2^\alpha$.

Proof. Clearly $|X| \leqslant 2^\alpha$. Suppose $\beta = (2^\alpha)^+$, for $\mathcal{T} \in \mathcal{C}$ choose $D(\mathcal{T}) \subseteq X$ such that $|D(\mathcal{T})| \leqslant \alpha$ and $\mathcal{D}(\mathcal{T})$ is $\mathcal{T}$-dense in X, and note from $|[X]^{\leqslant\alpha}| \leqslant 2^\alpha$ that there are a (fixed) $D \subseteq X$ and $\mathcal{C}' \subseteq \mathcal{C}$ such that $|\mathcal{C}'| = \beta$ and $D(\mathcal{T}) = D$ for all $\mathcal{T} \in \mathcal{C}'$. For $\mathcal{T} \in \mathcal{C}'$ there is an embedding $e_\mathcal{T}\colon \langle X,\mathcal{T}\rangle \to [0,1]^\alpha$, and since $|([0,1]^\alpha)^D| = 2^\alpha < \beta$ there are distinct $\mathcal{T}, \mathcal{T}' \in \mathcal{C}'$ (say with $\mathcal{T} \subsetneq \mathcal{T}'$) such that $e_\mathcal{T}|D = e_{\mathcal{T}'}|D$. The functions $e_\mathcal{T}, e_{\mathcal{T}'}$ are $\mathcal{T}'$-continuous on X and agree on the $\mathcal{T}'$-dense set D, so $e_\mathcal{T} = e_{\mathcal{T}'}$ and hence $\mathcal{T} = \mathcal{T}'$, a contradiction. □

3. Chains in $\mathcal{B}(G)$ when $|G/G'| = |G|$

For groups G as in the title of this section we relate the existence of chains in $\mathcal{B}(G)$ of length β to condition $C(2^\gamma,\beta)$, and we relate the existence of chains in $\mathcal{B}_\alpha(G)$ of length β to condition $E(2^\gamma,\beta,\alpha)$. The following consequence of Theorem 2.16(b) is basic.

Theorem 3.1. *Let* α, β *and* γ *be infinite cardinals and let* G *be a group such that* $|G| = \gamma$. *If there is a chain* $\mathcal{C} \subseteq \mathcal{B}_\alpha(G)$ *such that* $|\mathcal{C}| = \beta$, *then* $E(2^\gamma,\beta,\alpha)$ *holds.*

Proof. Since the dual object Σ_0 of bG_d satisfies $|\Sigma_0| \leqslant 2^\gamma$ (Remus [48, 2.5]), this is immediate from Theorem 2.16(b). □

In our work [15, 3.4, 4.4] we showed that if G is an Abelian group or a free group with $|G| = \gamma \geqslant \omega$, then $\mathcal{B}(G)$ contains a chain $\mathcal{C}$ with $|\mathcal{C}| = \beta$ if and only if $C(2^\gamma, \beta)$ holds. Now in Theorem 3.3 we generalize this statement.

The following lemma is an immediate consequence of [48, 2.7].

Lemma 3.2. *Let $\langle G, \mathcal{T}\rangle$ be a topological group and let $N \lhd G$. Let $\mathcal{T}_q$ be the (possibly non-Hausdorff) quotient topology on G/N and let $\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_1 \in PK(G/N)$ satisfy $\widetilde{\mathcal{S}}_i \supseteq \mathcal{T}_q$, $\widetilde{\mathcal{S}}_0 \neq \widetilde{\mathcal{S}}_1$. Let $\mathcal{S}_i$ be the initial topology on G with respect to the canonical homomorphism $\phi: G \to G/N$. Then $\mathcal{T} \vee \mathcal{S}_0 \neq \mathcal{T} \vee \mathcal{S}_1$.*

Theorem 3.3. *Let α, β and γ be infinite cardinals, and let G be a (discrete) maximally almost periodic group such that $|G| = |G/G'| = \gamma \geqslant \omega$.*

(A) *The following conditions are equivalent:*
  (a) $C(2^\gamma, \beta)$;
  (b) *there is a chain $\mathcal{C} \subseteq \mathcal{B}(G)$ such that $|\mathcal{C}| = \beta$.*

(B) *Let $\log\gamma \leqslant \alpha \leqslant 2^\gamma$. Then either $\mathcal{B}_\delta(G) = \emptyset$ for all $\delta \leqslant \alpha$, or the following conditions are equivalent:*
  (a) $E(2^\gamma, \beta, \alpha)$;
  (b) *there is a chain $\mathcal{C} \subseteq \mathcal{B}_\alpha(G)$ such that $|\mathcal{C}| = \beta$.*

Proof. (This proof is based in part on the proof of Theorem 2.13 of [48].) That (b) ⇒ (a) in (A) is obvious; that (b) ⇒ (a) in (B) is a restatement of Theorem 3.1. For (a) ⇒ (b) in (A), use Theorem 2.7 to find $\delta \leqslant \gamma$ and $\mathcal{T} \in \mathcal{B}_\delta(G)$; the (not necessarily Hausdorff) quotient topology $\mathcal{T}_q$ on G/G' then satisfies $w\langle G/G', \mathcal{T}_q\rangle \leqslant \delta$. Using $|\langle G/G', \mathcal{T}_q\rangle\hat{}\,| \leqslant \delta < 2^\gamma$ (cf. [48, 2.4]) and Theorem 2.14 it is easy to see from (a) that there is a chain $\mathcal{H}$ of point-separating subgroups of $\mathrm{Hom}(G/G', \mathbb{T})$ such that $|\mathcal{H}| = \beta$ and each $H \in \mathcal{H}$ satisfies $H \supseteq \langle G/G', \mathcal{T}_q\rangle\hat{}$. (For details, based on the relation $|\mathrm{Hom}(G/G', \mathbb{T})/\langle G/G', \mathcal{T}_q\rangle\hat{}\,| = 2^\gamma$, see [15, 3.4].) The topology $\widehat{\mathcal{S}_H}$ induced by H on G/G' satisfies $\widehat{\mathcal{S}_H} \supseteq \mathcal{T}_q$, and (with $\mathcal{S}_H$ the initial topology on G with respect to the canonical homomorphism from G onto G/G') the map $H \to \mathcal{T} \vee \mathcal{S}_H$ from $\mathcal{H}$ into $\mathcal{B}(G)$ is one-to-one by Lemma 3.2. The proof of (A) is complete. To prove (a) ⇒ (b) in (B), it being assumed that some $\delta \leqslant \alpha$ satisfies $\mathcal{B}_\delta(G) \neq \emptyset$, repeat the proof of (a) ⇒ (b) in (A) with two modifications: (1) Choose $\mathcal{T} \in \mathcal{B}_\delta(G)$ so that not only $\delta \leqslant \gamma$ but also $\delta \leqslant \alpha$, and (2) use (a) to arrange $|H| = \alpha$ for each $H \in \mathcal{H}$. Each $H \in \mathcal{H}$ satisfies

$$w\langle G, \mathcal{S}_H\rangle = w\langle G/G', \widehat{\mathcal{S}_H}\rangle = |H| = \alpha,$$

and hence

$$w\langle G, \mathcal{T} \vee \mathcal{S}_H\rangle \leqslant w\langle G, \mathcal{T}\rangle + w\langle G, \mathcal{S}_H\rangle = \delta + \alpha = \alpha$$

by Theorem 2.5(b), while $w\langle G, \mathcal{T} \vee \mathcal{S}_H\rangle \geqslant \alpha$ follows from

$$(\mathcal{T} \vee \mathcal{S}_H)_q \supseteq \mathcal{T}_q \vee (\mathcal{S}_H)_q = \mathcal{T}_q \vee \widehat{\mathcal{S}_H} = \widehat{\mathcal{S}_H}. \qquad \square$$

Remark 3.4. We first announced Theorem 3.3 in [12]. A result which improves, extends and subsumes Theorem 3.3 has been announced and proved by Dikranjan [26,27] and co-authors [5] (see Section 1 above).

Remark 3.5. A routine argument, as in [6, 5.2] or [30, 1.5.1], for example, shows that if $\alpha < \log\gamma$ or $\alpha > 2^\gamma$ then every group G with $|G| = \gamma$ satisfies $\mathcal{B}_\alpha(G) = \emptyset$. Thus in Theorem 3.3(B) the condition $\log\gamma \leqslant \alpha \leqslant 2^\gamma$ cannot be omitted. It was shown by Dikranjan [26, p. 141]; [27, Lemma 2.1] that for a maximally almost periodic group G one has $\mathcal{B}_\alpha(G) \neq \emptyset$ if and only if $\alpha \in [\gamma(G), \Gamma(G)]$; see also in this connection [5, Lemma 7.3]. In these statements $\gamma(G)$ is defined to be $\min\{\kappa\colon \mathcal{B}_\kappa(G) \neq \emptyset\}$, and $\Gamma(G)$ is the weight of the finest totally bounded group topology on G.

Corollary 3.6. *Let α, β and γ be infinite cardinals such that $\log\gamma \leqslant \alpha \leqslant 2^\gamma$, and let G be a group such that $|G| = \gamma$ and either G is Abelian or G is a free group. Then the following conditions are equivalent:*

(a) $E(2^\gamma, \beta, \alpha)$;

(b) *there is a chain $\mathcal{C} \subseteq \mathcal{B}_\alpha(G)$ such that $|\mathcal{C}| = \beta$.*

Proof. That G admits a totally bounded group topology of weight $\log\gamma$ is given by [6, 4.4] when G is Abelian and by [48, 2.15] when G is a free group. Thus Theorem 3.3(B) applies. □

Corollary 3.7. *If α, β and γ are as in Corollary 3.6, then there is a chain $\mathcal{C} \subseteq \mathcal{B}_\alpha(G)$ such that $|\mathcal{C}| = \alpha^+$.*

Proof. From Corollaries 2.19 and 3.6. □

Remark 3.8. The condition $|\mathcal{C}| = \alpha^+$ in Corollary 3.7 cannot in general be strengthened in ZFC to $|\mathcal{C}| = 2^\alpha$. Indeed fix $\gamma \geqslant \omega$, choose G with $|G| = \gamma$, and set $\alpha = 2^\gamma$. Then α, γ and G are as in Corollary 3.6, so if there is a chain $\mathcal{C} \subseteq \mathcal{B}_\alpha(G)$ with $|\mathcal{C}| = 2^\alpha$ then $E(2^\gamma, \beta, \alpha)$ holds by Corollary 3.6 and hence in particular $C(2^\gamma, 2^{2^\gamma})$ holds; but as indicated above in Theorem 2.18 this condition fails even for $\gamma = \omega$ in some models of ZFC.

It is tempting to strive for a direct proof of the implication (b) ⇒ (a) of Theorem 3.3(B), avoiding reference to the dual object Σ_0 and the order-preserving embedding $f\colon PK(G) \to \mathcal{P}(\Sigma_0)$ of [48, 2.5], by responding positively to the following natural question [11, Question 2].

Given a set X and a chain $\mathcal{C}$ of (Hausdorff) topologies on X with $w(X, \mathcal{T}) = \alpha$ for each $\mathcal{T} \in \mathcal{C}$, can one choose for each $\mathcal{T} \in \mathcal{C}$ a base $\mathcal{B}(\mathcal{T})$ for $\mathcal{T}$ such that $|\mathcal{B}(\mathcal{T})| = \alpha$ and $\mathcal{B}(\mathcal{T}) \subseteq \mathcal{B}(\mathcal{T}')$ whenever $\mathcal{T}, \mathcal{T}' \in \mathcal{C}$ with $\mathcal{T} \subseteq \mathcal{T}'$?

When X is an Abelian group and $\mathcal{C} \subseteq \mathcal{B}_\alpha(X)$, Theorem 2.15 furnishes a positive response to this question. Nevertheless the answer is "No" in general, as we shall see.

Theorem 3.9. *Let $\gamma \geqslant \omega$. For every Abelian group G with $|G| = \gamma$ there is a chain $\mathcal{C}$ of Hausdorff group topologies on G with $|\mathcal{C}| = \gamma^+$ and $w\langle G, \mathcal{T}\rangle = \gamma$ for each $\mathcal{T} \in \mathcal{C}$ such that one cannot choose a base $\mathcal{B}(\mathcal{T})$ for $\mathcal{T}$ so that simultaneously* (1) $|\mathcal{B}(\mathcal{T})| = \gamma$ *for each $\mathcal{T} \in \mathcal{C}$ and* (2) $\mathcal{B}(\mathcal{T}) \subseteq \mathcal{B}(\mathcal{T}')$ *whenever $\mathcal{T}, \mathcal{T}' \in \mathcal{C}$ with $\mathcal{T} \subseteq \mathcal{T}'$. One may arrange further that the discrete topology $\mathcal{T}_d$ is in $\mathcal{C}$, and that $\mathcal{C}\backslash\{\mathcal{T}_d\} \subseteq \mathcal{B}_\gamma(G)$.*

Proof. With $H = \mathrm{Hom}(G, \mathbb{T})$ we have $|H| = 2^\gamma$ by Theorem 2.6(b). Now recursively, using $2^\gamma \geqslant \gamma^+$ and $|G| = \gamma$, choose a γ^+-sequence $\mathcal{H} = \{H(\xi)\colon \xi < \gamma^+\}$ of subgroups of H such that $H(0)$ separates points of G, $\eta < \xi < \gamma^+$ implies $H(\eta) \subseteq H(\xi)$ and $H(\xi) \neq H(\eta)$, and each $H(\xi)$ satisfies $|H(\xi)| = \gamma$. By Theorem 2.15 and [6, 4.3] the map $H(\xi) \to \mathcal{T}_{H(\xi)}$ is an order-isomorphism from the chain $\mathcal{H}$ into $\mathcal{B}_\gamma(G)$.

Now let $\mathcal{T}$ be any group topology for G such that $w\langle G, \mathcal{T}\rangle = \gamma$ and each $\mathcal{T}_{H(\xi)} \subseteq \mathcal{T}$. (For example, let $\mathcal{T} = \mathcal{T}_d$.) To see that the chain $\mathcal{C} := \{\mathcal{T}_{H(\xi)}\colon \xi < \gamma^+\} \cup \{\mathcal{T}\}$ is as required it is enough to show that if $\mathcal{A}(\xi)$ is a base for $\mathcal{T}_{H(\xi)}$ then

$$\left|\bigcup_{\xi<\gamma^+} \mathcal{A}(\xi)\right| \geqslant \gamma^+$$

(for then no base $\mathcal{A}$ for $\mathcal{T}$ with $|\mathcal{A}| \leqslant \gamma$ can contain each $\mathcal{A}(\xi)$). But this is obvious: For each $\xi < \gamma^+$ there is

$$U(\xi) \in \mathcal{T}_{H(\xi+1)}\backslash\mathcal{T}_{H(\xi)} = \mathcal{T}_{H(\xi+1)}\backslash \bigcup_{\eta\leqslant\xi} \mathcal{T}_{H(\eta)},$$

so there is $B(\xi) \in \mathcal{A}(\xi+1)\backslash \bigcup_{\eta\leqslant\xi} \mathcal{A}(\eta)$. □

We have noted already in (the proof of) Corollary 3.6 that for every Abelian group G with $|G| = \gamma$ there is $\mathcal{T} \in \mathcal{B}_{\log\gamma}(G)$; indeed as shown in [6], using [22], one may choose $\mathcal{T} = \mathcal{T}_H$ with H a dense subgroup of the compact group $\mathrm{Hom}(G, \mathbb{T}) \subseteq \mathbb{T}^G$ with $|H| = \log\gamma$. This result makes natural the following question, which to our knowledge has not been addressed in the literature.

Question 3.10. Let G be an Abelian group with $|G| = \gamma \geqslant \omega$, and let $\mathcal{T} \in \mathcal{B}(G)$. Must there exist $\mathcal{S} \in \mathcal{B}_{\log\gamma}(G)$ such that $\mathcal{S} \subseteq \mathcal{T}$?

We give a strong negative answer to this question.

Lemma 3.11. *Let G_i $(i = 0, 1)$ be totally bounded topological groups and $h\colon G_1 \twoheadrightarrow G_0$ a continuous surjective homomorphism. Then $w(G_0) \leqslant w(G_1)$.*

Proof. The Weil completion [55] $\overline{G_i}$ of G_i satisfies $w(\overline{G_i}) = w(G_i)$. It is well known that the function h, being uniformly continuous, extends to a continuous surjection $\bar{h}\colon \overline{G_1} \twoheadrightarrow \overline{G_0}$ (see in this connection for example [7, III§3, Remarque following Proposition 6]). The result then follows from this general topological statement: A continuous surjection between compact Hausdorff spaces cannot raise weight (cf. [30, 3.1.22]). □

It follows in particular from Lemma 3.11 that if G is a group and $\mathcal{T}_i \in \mathcal{B}(G)$ $(i = 0, 1)$ with $\mathcal{T}_0 \subseteq \mathcal{T}_1$, then $w\langle G, \mathcal{T}_0\rangle \leqslant w\langle G, \mathcal{T}_1\rangle$. When the condition $\mathcal{T}_1 \in \mathcal{B}(G)$ is omitted the resulting statement can fail; for an example it is enough to choose any group G with a totally bounded group topology $\mathcal{T}_0$ such that $w\langle G, \mathcal{T}_0\rangle > |G|$ and to take for $\mathcal{T}_1$ the discrete topology on G.

Theorem 3.12. *Let γ and α be cardinals such that $\omega \leqslant \gamma \leqslant \alpha \leqslant 2^\gamma$. For every Abelian group G with $|G| = \gamma$ there is $\mathcal{T} \in \mathcal{B}_\alpha(G)$ with this property:*

every Hausdorff group topology $\mathcal{S}$ on G such that $\mathcal{S} \subseteq \mathcal{T}$ satisfies $w\langle G, \mathcal{S}\rangle \geqslant \gamma$. $(*)$

Proof. For such α and γ there is a point-separating subgroup H of $\mathrm{Hom}(G, \mathbb{T})$ such that $|H| = \alpha$, and the topology $\mathcal{T} = \mathcal{T}_H$ (notation as in Theorem 2.15) satisfies $\mathcal{T} \in \mathcal{B}_\alpha(G)$. Thus $\mathcal{B}_\alpha(G) \neq \emptyset$, and when $\gamma = \omega$ any $\mathcal{T} \in \mathcal{B}_\alpha(G)$ is as required; in what follows we take $\gamma > \omega$.

We claim first that it is enough to show that some subgroup G' of G satisfies (the analogue of) $(*)$. Suppose $G' \subseteq G$ and there is $\mathcal{T}' \in \mathcal{B}_\alpha(G')$ such that every $\mathcal{S}' \in \mathcal{B}(G')$ with $\mathcal{S}' \subseteq \mathcal{T}'$ satisfies $w\langle G', \mathcal{T}'\rangle \geqslant \gamma$, and let $H' = \langle G', \mathcal{T}'\rangle\hat{}$; then $|H'| = \alpha$ and $\mathcal{T}' = \mathcal{T}_{H'}$ by Theorem 2.15. For each $h' \in H'$ choose $h \in \mathrm{Hom}(G, \mathbb{T})$ such that $h|G' = h'$, and choose further a point-separating set F of homomorphisms from G/G' to $\mathbb{T}$ such that $|F| \leqslant |G/G'| \leqslant \gamma$; let H be the subgroup of $\mathrm{Hom}(G, \mathbb{T})$ generated by

$$\{h\colon\ h' \in H'\} \cup \{f \circ \phi\colon\ f \in F\}$$

(with $\phi : G \to G/G'$ the usual homomorphism) and define $\mathcal{T} = \mathcal{T}_H$. Then $\mathcal{T} \in \mathcal{B}(G)$, and since $|H| = \alpha + \gamma = \alpha$ we have $w\langle G, \mathcal{T}\rangle = \alpha$ by Theorem 2.15—that is, $\mathcal{T} \in \mathcal{B}_\alpha(G)$. If $\mathcal{S} \in \mathcal{B}(G)$ and $\mathcal{S} \subseteq \mathcal{T}$ then every $k \in \langle G, \mathcal{S}\rangle\hat{}$ satisfies $k|G' \in H'$, so $\mathcal{S}|G' \subseteq \mathcal{T}|G' = \mathcal{T}'$; thus $w\langle G, \mathcal{S}\rangle \geqslant w\langle G', \mathcal{S}|G'\rangle \geqslant \gamma$. The claim is proved.

From $\gamma > \omega$ and Theorem 2.13 follows the existence of a subgroup G' of G of the form

$$G' = \bigoplus_{\xi < \gamma} G_\xi$$

with each G_ξ cyclic, $|G_\xi| > 1$. We choose disjoint subsets A_i $(i = 0, 1)$ of γ such that $\gamma = A_0 \cup A_1$ and $|A_i| = \gamma$, and we write $G' = G'_0 \oplus G'_1$ with

$$G'_i = \bigoplus_{\xi \in A_i} G_\xi.$$

The compact dual group $\widehat{G'}$ of the discrete group G' satisfies

$$\widehat{G'} = \widehat{G'_0} \times \widehat{G'_1} = \prod_{\xi \in A_0} \widehat{G_\xi} \times \prod_{\xi \in A_1} \widehat{G_\xi}$$

(cf. [35, 23.21]) with each $\widehat{G_\xi}$ a compact metrizable group. For $\xi < \gamma$ we choose a finite or countable dense subgroup D_ξ of $\widehat{G_\xi}$. (When G_ξ is finite we must choose $D_\xi = \widehat{G_\xi}$;

when $G_\xi = \mathbb{Z}$ we may take for D_ξ the torsion subgroup of $\hat{\mathbb{Z}} = \mathbb{T}$.) Then defining $E_i = \bigoplus_{\xi\in A_i} D_\xi$ $(i = 0, 1)$ we have: $|E_i| = \gamma$ and $E_0 \times E_1$ is dense in $\widehat{G'} = \widehat{G'_0} \times \widehat{G'_1}$. Since $\gamma \leqslant \alpha \leqslant 2^\gamma = |\widehat{G'_1}|$ there is a subgroup E'_1 of $\widehat{G'_1}$ such that $E_1 \subseteq E'_1$ and $|E'_1| = \alpha$. The topology $\mathcal{T}'$ induced on G' by $E_0 \times E'_1$ is a Hausdorff topology (since $E_0 \times E'_1$ is dense in $\widehat{G'}$) and satisfies $w\langle G', \mathcal{T}'\rangle = |E_0 \times E'_1| = \alpha$. Suppose finally that $\mathcal{S}'$ is a Hausdorff group topology for G' such that $\mathcal{S}' \subseteq \mathcal{T}'$, and using Theorem 2.15 let F be that subgroup of $\mathrm{Hom}(G', \mathbb{T}) = \widehat{G'_0} \times \widehat{G'_1}$ which induces $\mathcal{S}'$. From Theorem 2.15 we have $F \subseteq E_0 \times E'_1$ and $w\langle G', \mathcal{S}'\rangle = |F|$, so to show $w\langle G', \mathcal{S}'\rangle \geqslant \gamma$ it suffices to show $|F| \geqslant \gamma$. Since $\mathcal{S}'$ is a Hausdorff topology the group F is dense in $\widehat{G'_0} \times \widehat{G'_1}$ (2.15), so the projection $\pi_0 : E_0 \times E'_1 \to E_0$ satisfies: $\pi_0[F]$ is dense in $E_0 = \bigoplus_{\xi\in A_0} D_\xi$. That $d(E_0) \geqslant \gamma$ is obvious: If $S \subseteq E_0$ and $|S| < \gamma$ then there is $\xi \in A_0$ such that $x_\xi = 1_\xi$ for all $x \in S$, and then $\pi_\xi^{-1}(D_\xi \backslash \{1\})$ is a nonempty open subset of E_0 missing S. It then follows that $|F| \geqslant |\pi_\xi[F]| \geqslant d(E_0) \geqslant \gamma$, as required. □

Remark 3.13. Let us say, in consonance with usage suggested by [9, p. 195], that a topological group $\langle G, \mathcal{T}\rangle$ is a *van der Waerden* group if $\langle G, \mathcal{T}\rangle$ is a compact group on which every homomorphism to a compact group is continuous. Using this terminology it is a theorem of van der Waerden [54], exposed at some length in [20] in the special case $G = SO(3, \mathbb{R})$, that every compact, connected, semi-simple Lie group is a van der Waerden group. A compact group G is a van der Waerden group if and only if its topology is the only (hence, the finest) totally bounded group topology on G (cf. [24, pp. 39–40]), so every such group satisfies $|\mathcal{B}(G)| = 1$. The existence of such groups shows that the hypothesis "$|G| = |G/G'| = \gamma$" of Theorem 3.3 cannot be replaced by "$|G| = \gamma$" in the two implications (a) ⇒ (b) of Theorem 3.3, but leaves open the following question, which is suggested by Corollary 3.6 and its proof: Let G be a (discrete) maximally almost periodic group such that $|G| = |G/G'| = \gamma \geqslant \omega$; does G admit a totally bounded group topology of weight $\log\gamma$ (that is, is $\mathcal{B}_{\log\gamma}(G) \neq \emptyset$)? We answer this question in the negative in Theorem 3.15 but the proof requires some preparation.

Theorem 3.14. *Let $\{\langle G_i, \mathcal{T}_i\rangle : i \in I\}$ be a set of van der Waerden groups with each $|Z(G_i)| = 1$, let $\mathcal{T}$ be the product topology on $G := \prod_{i\in I} G_i$, and let H be a subgroup of G such that $H \supseteq \bigoplus_{i\in I} G_i$. Then every $\mathcal{S} \in \mathcal{B}(H)$ satisfies $\mathcal{S} \supseteq \mathcal{T}|H$.*

Proof. It is enough to show for (fixed) $i \in I$ that the projection $\pi_i : \langle H, \mathcal{S}\rangle \to \langle G_i, \mathcal{T}_i\rangle$ is continuous.

We note first that for $x \in H\backslash\ker(\pi_i)$ there is $y(x) \in H$ such that $x \cdot y(x) \neq y(x) \cdot x$. Indeed for $i \neq j \in I$ take $y(x)_j = 1_j$, and for $y(x)_i$ choose any $p \in G_i$ such that $x_i \cdot p \neq p \cdot x_i$; the existence of such p follows from $x_i \notin Z(G_i)$. The hypothesis $H \supseteq \bigoplus_{j\in I} G_j$ ensures that the point $y(x)$ so defined satisfies $y(x) \in H$.

Now for $x \in H\backslash\ker(\pi_i)$ let $c_H(y(x))$ denote the centralizer in H of $y(x)$; that is, let

$$c_H(y(x)) = \{h \in H : h \cdot y(x) = y(x) \cdot h\}.$$

The groups $c_H(y(x))$ are $\mathcal{S}$-closed in H, so from the relation

$$\ker(\pi_i) = \bigcap\{c_H(y(x))\colon\ x \in H\backslash\ker(\pi_i)\}$$

—a consequence of the relation $x \notin c_H(y(x)$—it follows that $\ker(\pi_i)$ is $\mathcal{S}$-closed in H. Hence $H/\ker(\pi_i)$, endowed with the quotient topology $\mathcal{S}_q$, is a (Hausdorff) topological group. Since algebraically $H/\ker(\pi_i)$ is the van der Waerden group G_i we have $\mathcal{S}_q = \mathcal{T}_i$, so $\pi_i : \langle H, \mathcal{S}\rangle \to \langle G_i, \mathcal{T}_i\rangle$ is continuous, as required. □

Theorem 3.15. *For every $\gamma \geqslant \mathfrak{c}$ there is a group G such that*
 (i) *G is divisible and G_d is a maximally almost periodic group*;
 (ii) $|G| = |G/G'| = \gamma$; *and*
 (iii) *every $\mathcal{S} \in \mathcal{B}(G)$ satisfies $w\langle G, \mathcal{S}\rangle \geqslant \gamma$.*

Proof. Let L be a compact, connected Lie group with trivial center. It is then immediate from Bourbaki [8, p. 101, Proposition 2] that $L = L'$. Let A be a divisible Abelian group such that $|A| = \gamma$, and define $H = \bigoplus_\gamma L$ and $G = H \times A$. Then G is divisible, and G_d is maximally almost periodic since L_d and A_d are so. From $|H| = \gamma \cdot \mathfrak{c} = \gamma$ and $|A| = \gamma$ follows $|G| = \gamma$ and $|G/G'| = |A| = \gamma$, so it remains to show (iii).

Given $\mathcal{S} \in \mathcal{B}(G)$ we have $\mathcal{S}|H \in \mathcal{B}(H)$ so by Theorem 3.14 the product topology $\mathcal{T}$ on L^γ satisfies $\mathcal{S}|H \supseteq \mathcal{T}|H$. Then from Lemma 3.11 and the $\mathcal{T}$-density of H in L^γ follows

$$w\langle G, \mathcal{S}\rangle \geqslant w\langle H, \mathcal{S}|H\rangle \geqslant w\langle H, \mathcal{T}|H\rangle = w\langle L^\gamma, \mathcal{T}\rangle = \gamma,$$

as required. □

For a specific example as in Theorem 3.15 one may choose $L = SO(3, \mathbb{R})$, $A = \bigoplus_\gamma \mathbb{Q}$.

It is shown in the preprint [5, 7.13] that for every $\gamma > \omega$ there is a group G of finite exponent satisfying conditions (ii) and (iii) of Theorem 3.15 above; see also [26, p. 145] for an earlier announcement.

We show in Theorem 3.17 that for suitably restricted van der Waerden groups G_i and for $H = \bigoplus_{i \in I} G_i$, the conclusion of Theorem 3.14 can strengthened.

Lemma 3.16. *Every compact, algebraically simple Lie group is a van der Waerden group.*

Proof. The connected component C of such a group G satisfies either $C = \{1\}$ or $C = G$, so G is totally disconnected or connected. In the first of these cases, since every compact totally disconnected group has a base at the identity consisting of open-and-closed normal subgroups (cf. [35, 7.7]), G is finite, hence discrete, hence a van der Waerden group. In the second case G is a compact, connected, (algebraically) simple Lie group and hence is a van der Waerden group (see Remark 3.13). □

Theorem 3.17. *Let $\{\langle G_i, \mathcal{T}_i\rangle\colon\ i \in I\}$ be a set of compact, algebraically simple, non-Abelian Lie groups, let $\mathcal{T}$ be the product topology on $G := \prod_{i\in I} G_i$, and let $H = \bigoplus_{i\in I} G_i$. Then $\mathcal{T}|H$ is the only totally bounded group topology on H (in symbols: $\mathcal{B}(H) = \{\mathcal{T}|H\}$).*

Proof. Given $\mathcal{S} \in \mathcal{B}(H)$, from 3.14 we have $\mathcal{S} \supseteq \mathcal{T}|H$. To show $\mathcal{S} \subseteq \mathcal{T}|H$, it is enough to show that every $\mathcal{S}$-continuous homomorphism h from H to a compact group K is $\mathcal{T}|H$-continuous. By the Peter–Weyl–van Kampen theorem (cf. [35, 22.14]) each such K embeds algebraically and topologically into a product of the form $\prod_{1\leqslant n<\omega} \mathfrak{U}(n)^{\alpha_n}$ with $\mathfrak{U}(n)$ the group of $n \times n$ unitary matrices over the complex field, so it is enough to treat the case $h\colon H \to \mathfrak{U}(n)$. It follows from (the proof of) [29, 7.3.11] that the $\mathcal{S}$-closed group $\ker(h)$ has algebraically the form $\ker(h) = (\bigoplus_{i\in J} G_i) \times \{1_{I\setminus J}\}$, so the group

$$L := H/\ker(h) \simeq \left(\bigoplus_{i\in I\setminus J} G_i\right) \times \{1_J\} \subseteq G$$

is (isomorphic to) a subgroup of $\mathfrak{U}(n)$. Let $\mathcal{U}_0$ and $\mathcal{T}_0$ denote the topologies induced on L by $\mathfrak{U}(n)$ and $\langle G, \mathcal{T}\rangle$, respectively, and let $\overline{\langle L, \mathcal{U}_0\rangle} = \overline{L}^{\mathfrak{U}(n)}$ and $\overline{\langle L, \mathcal{T}_0\rangle} = \prod_{i\in I\setminus J} G_i$ be the corresponding Weil completions. Since $\mathcal{U}_0 \supseteq \mathcal{T}_0$ by Theorem 3.14, the identity function $\mathrm{id}\colon \langle L, \mathcal{U}_0\rangle \twoheadrightarrow \langle L, \mathcal{T}_0\rangle$ extends to a continuous homomorphism

$$\overline{\mathrm{id}}\colon \overline{\langle L, \mathcal{U}_0\rangle} \twoheadrightarrow \prod_{i\in I\setminus J} G_i.$$

It is well known that every closed subgroup, and every (Hausdorff) quotient group of a Lie group, is again a Lie group (cf. [53, 2.12.16, 2.9.6]), so in particular the group $\overline{\langle L, \mathcal{U}_0\rangle}$ and its quotient $\overline{\langle L, \mathcal{U}_0\rangle}/\ker(\overline{\mathrm{id}})$, which is topologically isomorphic to $\prod_{i\in I\setminus J} G_i$, is a Lie group. If $|I\setminus J| \geqslant \omega$ then $\overline{\langle L, \mathcal{U}_0\rangle}/\ker(\overline{\mathrm{id}})$ is a nondiscrete Lie group in which every neighborhood of 1 contains a nontrivial subgroup, contrary to [44, 2.2.14]. Thus $I\setminus J$ is finite. Since (as is easily verified) each product of finitely many van der Waerden groups is itself a van der Waerden group, and since by Lemma 3.16 each of the groups $\langle G_i, \mathcal{T}_i\rangle$ $(i \in I)$ is a van der Waerden group, the group $\langle L, \mathcal{T}_0\rangle$ is a van der Waerden group. It is then clear that $h\colon \langle H, \mathcal{T}|H\rangle = \langle L, \mathcal{T}_0\rangle \oplus \ker(h) \to \mathfrak{U}(n)$ is continuous as required; indeed given U open in $\mathfrak{U}(n)$ the function $h_0 := h|L$ satisfies $h_0^{-1}(U) \in \mathcal{T}_0$, so $h^{-1}(U)$, which is $((h_0^{-1}(U)) \times (\Pi_{i\in I\setminus J} G_i)) \cap H$, is open in $\langle H, \mathcal{T}|H\rangle$. □

Remarks 3.18. (a) It is a theorem of Goto [33, Corollary 3] that every proper normal subgroup of a simple, connected Lie group G lies in the center of G. Thus the hypothesized condition on the groups G_i $(i \in I)$ in Theorem 3.17 is equivalent to the condition that each of those groups is either a finite, non-Abelian simple group or a compact, connected, simple Lie group with trivial center.

(b) In work independent of the present paper, the authors of [5, 7.14] have shown a special case of Theorem 3.17 above: For an arbitrary cardinal $\gamma \geqslant \omega$ and with A the usual alternating group of degree 5 in the discrete topology, the only totally bounded group topology on $\bigoplus_\gamma A$ is the restriction to $\bigoplus_\gamma A$ of the product topology on A^γ.

(c) It is shown by Ajtai, Havas and Komlós [1] that every infinite Abelian group G admits a Hausdorff group topology $\mathcal{T}$ which is not maximally almost periodic. It is interesting to note that the analogous assertion fails for many infinite non-Abelian groups. For examples to this effect let $\{F_i\colon\ i \in I\}$ be any set (not necessarily faithfully indexed) of finite groups with $|I| \geqslant \omega$ and with $1 = |Z(F_i)| < |F_i|$ for each $i \in I$, give each F_i the discrete topology, and let $\mathcal{T}$ be the product topology on $G := \prod_{i\in I} F_i$. Then just as in the proof of Theorem 3.14, reference to van der Waerden groups now being suppressed, one sees that for every subgroup H of G satisfying $H \supseteq \bigoplus_{i\in I} F_i$ and for every Hausdorff group topology $\mathcal{S}$ on H, each epimorphism $\pi_i : \langle H, \mathcal{S}\rangle \twoheadrightarrow \langle F_i, \mathcal{T}_i\rangle$ is continuous. Thus the points of H are separated by $\mathcal{S}$-continuous homomorphisms into compact groups, i.e., $\langle H, \mathcal{S}\rangle$ is necessarily a maximally almost periodic topological group.

4. On the existence of minimally almost periodic group topologies

The results of this section are all due to the second-listed co-author and are taken from the unpublished thesis [49, Kapitel 4]. Here we construct (in general, non-Hausdorff) group topologies which we use in Section 5 to find long chains of Hausdorff group topologies which are not totally bounded.

We remind the reader: A topological group $\langle G, \mathcal{T}\rangle$ is minimally almost periodic if and only if $|\mathsf{b}(G)| = 1$—that is, if every $\mathcal{T}$-continuous $h \in \mathrm{Hom}(G, K)$ with K compact satisfies $h(x) = 1_K$ for all $x \in G$.

The following result is basic to our argument.

Lemma 4.1. *Let G be a group with normal subgroup N such that G/N has a minimally almost periodic group topology $\mathcal{T} \neq \mathcal{T}_{ad}$, and let $\mathcal{U}$ be the initial topology on G induced by the canonical homomorphism $\phi: G \to \langle G/N, \mathcal{T}\rangle$. Then*

(i) *$\langle G,\mathcal{U}\rangle$ is a minimally almost periodic topological group*;

(ii) *$\langle G,\mathcal{U}\rangle \neq G_{ad}$; and*

(iii) *$\chi\langle G,\mathcal{U}\rangle = \chi\langle G/N, \mathcal{T}\rangle$.*

Proof. Clearly $\mathcal{T} = \mathcal{U}_q$, so (ii) and (iii) are obvious. To prove (i) let $h \in \mathrm{Hom}(\langle G,\mathcal{U}\rangle, K)$ with K compact and h continuous. Then $N \subseteq \ker(h)$ and the homomorphism $\bar{h}: G/N \to K$ such that $\bar{h} \circ \phi = h$ is $\mathcal{T}$-continuous on G/N, so $h(x) = \bar{h}(xN) = 1_K$ for every $x \in G$, as required. □

The following theorem paves the way for Theorem 4.5, the principal result of this section.

Theorem 4.2 (Ajtai, Havas and Komlós [1]). *Let $G = \mathbb{Z}$ or $G = \mathbb{Z}(p^\infty)$ ($p \in \mathbb{P}$) or $G = \bigoplus_\omega \mathbb{Z}(p)$ ($p \in \mathbb{P}$) or $G = \bigoplus_{n<\omega} \mathbb{Z}(p_n)$ ($p_n \in \mathbb{P}$) with $p_n < p_{n+1}$ (all $n < \omega$). Then G admits a Hausdorff minimally almost periodic group topology.*

Lemma 4.3. *Every group G with $|G/G'| \geqslant \omega$ has a quotient isomorphic to one of the groups listed in Theorem* 4.2.

Proof. Clearly we may assume that G is an infinite Abelian group. We consider two cases.

Case 1. G is a torsion group. Let $G = \bigoplus_{p\in\mathbb{P}} G_p$ be the usual resolution of G into its p-primary components (cf. [35, Appendix A]; [31, §8)]. If each G_p is finite then there are infinitely many $p \in \mathbb{P}$ such that $1 < |G_p| < \omega$ with G_p a direct sum of cyclic groups, so G has a quotient of the form $G/N = \bigoplus_{n<\omega} \mathbb{Z}(p_n)$ with $p_n < p_{n+1}$ (all $n < \omega$). Suppose then that some G_p is infinite, and note that $G_p = G/\oplus\{G_q\colon q \in \mathbb{P},\ q \neq p\}$. Since G_p is an infinite Abelian p-group, there is by [31, 32.3] a subgroup A of G_p of the form $A = \bigoplus_{i\in I} \mathbb{Z}(p^{r_i})$ such that G_p/A is divisible. If $G_p = A$ then G_p has a quotient of the form $G/N = \bigoplus_\omega \mathbb{Z}(p)$; if $G_p \neq A$ then G/A is the direct sum of groups of the form $\mathbb{Z}(p^\infty)$, hence has $\mathbb{Z}(p^\infty)$ as a quotient.

Case 2. G is not a torsion group. Let T be the torsion subgroup of G, set $H = G/T$, and let B be a maximal independent subset of H. Then $H/\langle B\rangle$ is a torsion group, so if $|H/\langle B\rangle| \geqslant \omega$ the desired conclusion follows from Case 1. We assume therefore that $|H/\langle B\rangle| < \omega$. If B is infinite then from

$$H/\langle B\rangle \simeq \big(H/2\langle B\rangle\big)/\big(\langle B\rangle/2\langle B\rangle\big) \simeq \big(H/2\langle B\rangle\big)/\Big(\bigoplus_{|B|} \mathbb{Z}(2)\Big)$$

it follows that $H/2\langle B\rangle$ is an infinite torsion group, so again Case 1 applies; if B is finite then H is finitely generated and from the familiar structure theorem for finitely generated Abelian groups ([35, A.27]; [31, 15.5)] it follows that $\mathbb{Z}$ is a quotient of H. □

Lemma 4.4. *Let $\langle G, \mathcal{T}\rangle$ be a countable Abelian Hausdorff topological group. Then there is a Hausdorff group topology $\mathcal{S}$ for G such that $\mathcal{S} \subseteq \mathcal{T}$ and $\chi(G, \mathcal{S}) = \omega$.*

Proof. For $1 \neq x \in G$ let U_x be a $\mathcal{T}$-open neighborhood of 1 such that $x \notin U_x$, and let $\mathcal{S}$ be the smallest group topology for G containing each of the sets U_x $(1 \neq x \in G)$. □

Now we can state the main result of this section.

Theorem 4.5. *Every group G such that $|G/G'| \geqslant \omega$ admits a minimally almost periodic group topology $\mathcal{T}$ such that $\mathcal{T} \neq \mathcal{T}_{ad}$ and $\chi(G, \mathcal{T}) = \omega$.*

Proof. This is immediate from Lemmas 4.1, 4.3, 4.4 and Theorem 4.2. □

Remark 4.6. It is natural to inquire when looking at Theorem 4.5 whether every infinite Abelian group admits a Hausdorff minimally almost periodic group topology. The answer to this question is "No"; indeed the authors have noted in [10, 3J] that for every infinite cardinal γ there is an example G to this effect with $|G| = \gamma$ and with G an Abelian torsion group of bounded order. To the authors' best knowledge there exists at present no characterization or classification of those Abelian groups which admit a Hausdorff minimally almost periodic group topology. For some information on groups of this kind, the interested reader may consult Remus [47].

5. Chains in $\mathcal{N}(G)$ when $|G/G'| = |G|$

Here for groups G with $|G/G'| = |G| = \gamma \geqslant \omega$ we use the results of Section 4 to construct an order-isomorphism from the partially ordered set $\mathcal{B}(G)$ into $\mathcal{N}(G)$. This furnishes in Theorem 5.3 for such groups G a chain $\mathcal{C} \subseteq \mathcal{N}_\alpha(G)$ with $|\mathcal{C}| = \beta$, provided condition $E(2^\gamma, \beta, \alpha)$ is assumed.

Though our first lemma is a special case of [25, Lemma 1], we include a proof in the interest of completeness.

Lemma 5.1. *Let $\mathcal{S}_i$ $(i = 0, 1)$ be group topologies on a group G such that $\mathcal{S}_0 \subsetneq \mathcal{S}_1$. If there is an $\mathcal{S}_1$-dense subset D of G such that $\mathcal{S}_0|D = \mathcal{S}_1|D$, then $\mathcal{S}_0 = \mathcal{S}_1$.*

Proof. It suffices to show that every $\mathcal{S}_1$-neighborhood U of 1 satisfies $\mathcal{S}_0 - \operatorname{int} U \neq \emptyset$. Given U, let V be a symmetric $\mathcal{S}_1$-neighborhood of 1 such that $V^2 \subseteq U$, and let $W \in \mathcal{S}_0$ satisfy $W \cap D = V \cap D$. From $W \in \mathcal{S}_1$ follows

$$W \subseteq \overline{W}^{\mathcal{S}_1} = \overline{W \cap D}^{\mathcal{S}_1} = \overline{V \cap D}^{\mathcal{S}_1} = \overline{V}^{\mathcal{S}_1} \subseteq V^2 \subseteq U,$$

so $\emptyset \neq \mathcal{S}_0 - \operatorname{int} W = W \subseteq \mathcal{S}_0 - \operatorname{int} U$. □

Theorem 5.2 [49, 5.1]. *Let $\langle G, \mathcal{T}\rangle$ be a minimally almost periodic group with $\mathcal{T} \neq \mathcal{T}_{ad}$, and for $\mathcal{S} \in \mathcal{B}(G)$ define $\widetilde{\mathcal{S}} = \mathcal{S} \vee \mathcal{T}$. Then*

(a) *the map $\mathcal{S} \to \widetilde{\mathcal{S}}$ is an order-isomorphism from the partially ordered set $\mathcal{B}(G)$ into $\mathcal{N}(G)$; and*

(b) *if in addition $\chi\langle G, \mathcal{T}\rangle = \omega$ then every $\mathcal{S} \in \mathcal{B}_\alpha(G)$ $(\alpha \geqslant \omega)$ satisfies $\widetilde{\mathcal{S}} \in \mathcal{N}_\alpha(G)$.*

Proof. (a) From $\widetilde{\mathcal{S}} \supseteq \mathcal{S} \in \mathcal{B}(G)$ it follows that each of the group topologies $\widetilde{\mathcal{S}}$ satisfies the Hausdorff separation axiom. If $\widetilde{\mathcal{S}} \in \mathcal{B}(G)$ then from $\mathcal{T} \subseteq \widetilde{\mathcal{S}}$ it would follow that the minimally almost periodic topology $\mathcal{T}$ satisfies $\mathcal{T} \in PK(G)$, so that $\mathcal{T} = \mathcal{T}_{ad}$. The proof that $\widetilde{\mathcal{S}} \in \mathcal{N}(G)$ whenever $\mathcal{S} \in \mathcal{B}(G)$ is complete.

For the isomorphism statement in (a) it is enough to show that the map $\mathcal{S} \to \widetilde{\mathcal{S}}$ is strictly monotone in the sense that

$$\text{if } \mathcal{S}_i \in \mathcal{B}(G) \ (i = 0, 1) \text{ with } \mathcal{S}_0 \subseteq \mathcal{S}_1 \text{ and } \mathcal{S}_0 \neq \mathcal{S}_1, \text{ then } \widetilde{\mathcal{S}}_0 \neq \widetilde{\mathcal{S}}_1. \qquad (*)$$

(Indeed in any upper semi-lattice L a map $L \to L$ of the form $s \to \tilde{s} = s \vee t$ (for fixed $t \in L$) is injective whenever it is strictly monotone, as $\tilde{s_0} = \tilde{s_1}$ gives $\widetilde{s_0 \vee s_1} = \tilde{s_0} \vee \tilde{s_1} = \tilde{s_0} = \tilde{s_1}$.)

To prove $(*)$, let $\mathcal{S}_i \in \mathcal{B}(G)$ $(i = 0, 1)$ satisfy $\mathcal{S}_0 \subseteq \mathcal{S}_1$ and $\widetilde{\mathcal{S}}_0 = \widetilde{\mathcal{S}}_1$, and let $\mathcal{A}_0$, $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively be local bases for $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{T}$ at 1. Since $\mathcal{S}_1 \wedge \mathcal{T}$ is a minimally almost periodic, pre-compact group topology on G we have $\mathcal{S}_1 \wedge \mathcal{T} = \mathcal{T}_{ad}$. Now from Theorem 2.10(a) the set $\mathcal{A} := \{A_1 \cdot A_2\colon\ A_i \in \mathcal{A}_i\}$ is a local base for $\mathcal{S}_1 \wedge \mathcal{T}$ at 1, so

$\mathcal{S}_1 \wedge \mathcal{T} = \mathcal{T}_{ad}$ gives $A_1 \cdot A_2 = G$ for all $A_i \in \mathcal{A}_i$ $(i = 1, 2)$. It is then immediate that the "diagonal" $\Delta = \{\langle p, p\rangle\colon p \in G\}$ is dense in $\langle G \times G, \mathcal{S}_1 \times \mathcal{T}\rangle$: Indeed if $\langle p_1, p_2\rangle \in G \times G$ and $A_i = A_i^{-1} \in \mathcal{A}_i$ then to find $p \in G$ such that $\langle p, p\rangle \in (p_1A_1) \times (p_2A_2)$ it suffices to choose $a_i \in A_i$ so that $a_1a_2^{-1} = p_1^{-1}p_2$ and to set $p = p_1a_1 = p_2a_2$.

Now for $i = 0, 1$ the diagonal map

$$d_i : \langle G, \widetilde{\mathcal{S}_i}\rangle \to \langle G \times G, \mathcal{S}_i \times \mathcal{T}\rangle$$

given by $p \to \langle p, p\rangle$ is an algebraic and topological embedding, so from $\widehat{\mathcal{S}_0} = \widehat{\mathcal{S}_1}$ follows $\mathcal{S}_0 \times \mathcal{T}|\Delta = \mathcal{S}_1 \times \mathcal{T}|\Delta$. Since Δ is $\mathcal{S}_1 \times \mathcal{T}$-dense in $G \times G$ and $\mathcal{S}_0 \times \mathcal{T} \subseteq \mathcal{S}_1 \times \mathcal{T}$ we have $\mathcal{S}_0 \times \mathcal{T} = \mathcal{S}_1 \times \mathcal{T}$ from Lemma 5.1, as required.

(b) Identifying $\langle G, \widetilde{\mathcal{S}}\rangle$ algebraically and topologically with the dense subset Δ of $\langle G \times G, \mathcal{S} \times \mathcal{T}\rangle$ as above, using Theorem 2.5(a) we have

$$\chi\langle G, \widetilde{\mathcal{S}}\rangle = \chi\langle G \times G, \mathcal{S} \times \mathcal{T}\rangle = \chi\langle G, \mathcal{S}\rangle + \chi\langle G, \mathcal{T}\rangle = \alpha + \omega = \alpha,$$

as required. □

Theorem 5.3. *Let α, β and γ be infinite cardinals, and let G be a (discrete) maximally almost periodic group such that $|G| = |G/G'| = \gamma \geqslant \omega$.*

(A) *The following conditions are equivalent:*
 (a) $C(2^\gamma, \beta)$;
 (b) *there is a chain $\mathcal{C} \subseteq \mathcal{B}(G)$ such that $|\mathcal{C}| = \beta$;*
 (c) *there is a chain $\mathcal{C} \subseteq \mathcal{N}(G)$ such that $|\mathcal{C}| = \beta$.*

(B) *Let $\log\gamma \leqslant \alpha \leqslant 2^\gamma$. Then either $\mathcal{B}_\delta(G) = \emptyset$ for all $\delta \leqslant \alpha$, or* (a) *and* (b) *are equivalent and each implies* (c):
 (a) $E(2^\gamma, \beta, \alpha)$;
 (b) *there is a chain $\mathcal{C} \subseteq \mathcal{B}_\alpha(G)$ such that $|\mathcal{C}| = \beta$;*
 (c) *there is a chain $\mathcal{C} \subseteq \mathcal{N}_\alpha(G)$ such that $|\mathcal{C}| = \beta$.*

Proof. The two equivalences (a) ⇔ (b) are from Theorem 3.3, restated here for emphasis. By Theorem 4.5 there is a minimally almost periodic group topology $\mathcal{T} \neq \mathcal{T}_{ad}$ on G such that $\chi\langle G, \mathcal{T}\rangle = \omega$, so the two implications (b) ⇒ (c) follow from Theorem 5.2. That (c) ⇒ (a) in (A) is obvious. □

We have been unable to determine whether the implication (c) ⇒ (a) in (B) holds in Theorem 5.3. Specifically, we pose this question.

Question 5.4. Let α, β and γ be infinite cardinals such that $\log\gamma \leqslant \alpha \leqslant 2^\gamma$. Let G be a group with $|G| = \gamma$ and with a chain $\mathcal{C} \subseteq \mathcal{N}_\alpha(G)$ such that $|\mathcal{C}| = \beta$. Does $E(2^\gamma, \beta, \alpha)$ hold?

We indicated above in (the proof of) Corollary 3.6 that every Abelian group G with $|G| = \gamma \geqslant \omega$, and every free group G with $|G| = \gamma \geqslant \omega$, satisfy $\mathcal{B}_{\log\gamma}(G) \neq \emptyset$; therefore such groups satisfy all hypotheses of Theorem 5.3. We have been unable to answer Question 5.4 even for such groups G.

Remark 5.5. In the absence of the maximally almost periodic hypothesis of Theorems 3.3 and 5.3, the existence of a long chain in $\mathcal{N}(G)$ does not imply the existence of one in $\mathcal{B}(G)$. Indeed we show now that $\mathcal{B}(G) = \emptyset$ is possible.

Theorem 5.6. *Let $\gamma \geqslant \omega$. There is a group G with $|G| = |G/G'| = \gamma$ and $\mathcal{B}(G) = \emptyset$ with the following properties.*

(A) *For all $\beta \geqslant \omega$ the following conditions are equivalent.*
  (a) $C(2^\gamma, \beta)$;
  (b) *there is a chain $\mathcal{C} \subseteq \mathcal{N}(G)$ such that $|\mathcal{C}| = \beta$.*

(B) *If $\log\gamma \leqslant \alpha \leqslant 2^\alpha$ and $E(2^\gamma, \beta, \alpha)$ holds, then there is a chain $\mathcal{C} \subseteq \mathcal{N}_\alpha(G)$ such that $|\mathcal{C}| = \beta$.*

Proof. Let $n \in \mathbb{N}$ with $n \geqslant 2$ and let $GL(n,\mathbb{Q})$ and $SL(n,\mathbb{Q})$ denote respectively the general linear and the special linear n-dimensional matrix groups with entries in $\mathbb{Q}$. It is known (see for example Hein [34, Kapitel 1, §2]) that $(GL(n,\mathbb{Q}))' = SL(n,\mathbb{Q})$, $Z(GL(n,\mathbb{Q})) = \mathbb{Q}^*$ (the multiplicative group of nonzero rational numbers), and $GL(n,\mathbb{Q})/SL(n,\mathbb{Q}) = \mathbb{Q}^*$.

Now define $G = \bigoplus_\gamma GL(n,\mathbb{Q})$. Then

$$G/G' = \bigoplus_\gamma GL(n,\mathbb{Q}) / \bigoplus_\gamma SL(n,\mathbb{Q}) = \bigoplus_\gamma \mathbb{Q}^*,$$

so $|G| = |G/G'| = \gamma$. Since the (discrete) group $GL(n,\mathbb{Q})$ is not maximally almost periodic (cf. [35, 22.22(g)]), surely G_d is not maximally almost periodic; that is, $\mathcal{B}(G) = \emptyset$.

Now given $\beta \geqslant \omega$, the implication (b) $\Rightarrow$ (a) of (A) is clear. To prove (a) $\Rightarrow$ (b), and to prove (B), use $\bigoplus_\gamma \mathbb{Q}^* = Z(G) \subseteq G$ and note from Theorem 5.3 above that there is a chain $\mathcal{C}' \subseteq \mathcal{N}(\bigoplus_\gamma \mathbb{Q}^*)$ such that $|\mathcal{C}'| = \beta$ (and also $\mathcal{C}' \subseteq \mathcal{N}_\alpha(\bigoplus_\gamma \mathbb{Q})$ if $E(2^\gamma, \beta, \alpha)$ holds). For $\mathcal{T} \in \mathcal{C}'$ the topology $\mathcal{S}(\mathcal{T}) := \{xU\colon x \in G,\ U \in \mathcal{T}\}$ satisfies $\mathcal{S}(\mathcal{T})|\bigoplus_\gamma \mathbb{Q}^* = \mathcal{T}$ and $\mathcal{S}(\mathcal{T}) \in \mathcal{N}(G)$ (and with $\mathcal{S}(\mathcal{T}) \in \mathcal{N}_\alpha(G)$ if $\mathcal{T} \in \mathcal{N}_\alpha(\bigoplus_\gamma \mathbb{Q}^*)$), so $\mathcal{C} := \{\mathcal{S}(\mathcal{T})\colon \mathcal{T} \in \mathcal{C}'\}$ is a chain as required. □

6. Totally bounded group topologies on certain non-Abelian groups

The results of this section, with 6.1–6.10 all from the unpublished thesis of Remus [49, Section 3.2 of Kapitel 3], are presented here for use in Section 7. The principal computation is in Theorem 6.4: If $\mathcal{T}_i$ $(i = 0, 1)$ are totally bounded group topologies on a group G with $\mathcal{T}_0 \subseteq \mathcal{T}_1$ and $\overline{\langle G, \mathcal{T}_1\rangle}$ connected and $w\langle G, \mathcal{T}_0\rangle = \alpha_0 < \alpha_1 = w\langle G, \mathcal{T}_1\rangle$, then the "topological interval" $[\mathcal{T}_0, \mathcal{T}_1]$ (defined as in Section 1) satisfies $|[\mathcal{T}_0, \mathcal{T}_1]| = 2^{\alpha_1}$.

Throughout this and the next section, given a topological group G, we denote by $\mathcal{F}(G)$ the set of compact, connected normal subgroups of G.

Lemma 6.1. *Let G be a topological group and $N \lhd G$. The connected component C of N satisfies $C \lhd G$.*

Lemma 6.2 [39, 4(1)]. *If $M \lhd N \lhd G$ with M, N and G compact and connected, then $M \lhd G$.*

Lemma 6.3. *Let G be an infinite compact group and H a closed normal subgroup. Then*

(a) ([16, 6.1]) $w(G) = w(H) + w(G/H)$;

(b) ([38, 3.2]) *if H is a totally disconnected subgroup of the connected component of G, then $w(G) = w(G/H)$.*

Theorem 6.4. *Let $\langle G, \mathcal{T}_1\rangle$ be a totally bounded topological group with $w\langle G, \mathcal{T}_1\rangle = \alpha_1 > \omega$ and with $\overline{\langle G, \mathcal{T}_1\rangle}$ connected. Then every totally bounded group topology $\mathcal{T}_0$ on G such that $\mathcal{T}_0 \subseteq \mathcal{T}_1$ and $w\langle G, \mathcal{T}_0\rangle = \alpha_0 < \alpha_1$ satisfies $|[\mathcal{T}_0, \mathcal{T}_1]| = 2^{\alpha_1}$.*

Proof. The anti-isomorphism $\psi : PK(G) \twoheadrightarrow LA(\mathrm{b}G_d)$ of 2.16(b) satisfies $\mathrm{b}\langle G, \mathcal{S}\rangle = \mathrm{b}G_d/\psi(\mathcal{S})$ for $\mathcal{S} \in PK(G)$. In what follows we write $\psi(\mathcal{T}_i) = N_i$ $(i = 0, 1)$; suitably restricted, ψ then gives an anti-isomorphism from the "interval" $[\mathcal{T}_0, \mathcal{T}_1]$ of totally bounded group topologies on G between $\mathcal{T}_0$ and $\mathcal{T}_1$ onto the "interval" $[N_1, N_0]$ of closed, normal subgroups of $\mathrm{b}G_d$ between N_1 and N_0. The map

$$\phi : [N_1, N_0] \twoheadrightarrow \bigl[\{1\}, N_0/N_1\bigr] \subseteq LA\bigl(\overline{\langle G, \mathcal{T}_1\rangle}\bigr)$$

given by $\phi(N) = N/N_1$ is an order-isomorphism. We write $M_0 = \phi(N_0)$ and we show $|[\{1\}, M_0]| = 2^{\alpha_1}$. For notational convenience we write $G = \langle G, \mathcal{T}_1\rangle$ and $\overline{G} = \overline{\langle G, \mathcal{T}_1\rangle}$, and we denote by C the connected component of M_0.

Since $w(G) = w(\overline{G}) = \alpha_1$ and each $M \in LA(\overline{G})$ is closed in $\overline{G}$ we have $|LA(\overline{G})| \leqslant 2^{\alpha_1}$. We claim

(1) $w(C) = \alpha_1$ and

(2) $|\mathcal{F}(C)| = 2^{\alpha_1}$.

(1) The group M_0/C is totally disconnected, so from $C \lhd \overline{G}$ and $\overline{G}/M_0 = (\overline{G}/C)/(M_0/C)$ (cf. 6.1 and [35, 5.35]) we have $w(\overline{G}/M_0) = w(\overline{G}/C)$ (using Lemma 6.3(b)). From $\overline{G}/M_0 = \overline{\langle G, \mathcal{T}_0\rangle}$ then follows $w(\overline{G}/C) = \alpha_0$, so $w(C) = \alpha_1$ by Lemma 6.3(a). Thus (1) is proved. To prove (2) we cite the proof of Theorem 6.5 in our work [16]: There is a continuous epimorphism h from C onto a product $K = \prod_{i \in I} K_i$ of compact connected metrizable groups with $|I| = w(C)$. (The proof uses a standard structure theorem from the theory of compact connected groups—see for example Price [44, 6.5.6]). When the connected component $Z_0(C)$ of the center of C satisfies $w(Z_0(C)) < w(C)$ the groups K_i may be chosen (non-Abelian) Lie groups, and when $w(Z_0(C)) = w(C)$ one may choose $K_i = \mathbb{T}$ for all $i \in I$.) In the present case from (1) we have $|I| = \alpha_1$. For $F \in \mathcal{F}(K)$ let C_F be the connected component of $h^{-1}(F)$. Then $C_F \lhd C$ (by Lemma 6.1) and $h[C_F] = F$ by [35, 7.12]; thus the map $F \to C_F$ is one-to-one from $\mathcal{F}(K)$ into $\mathcal{F}(C)$. Clearly $|\mathcal{F}(K)| \geqslant 2^{\alpha_1}$, so (2) is proved; finally from Lemma 6.2 follows $\mathcal{F}(C) \subseteq [\{1\}, M_0]$, so $2^{\alpha_1} \geqslant |LA(\overline{G})| \geqslant |[\{1\}, M_0]| \geqslant |\mathcal{F}(C)| = 2^{\alpha_1}$, as required. $\square$

Now in Theorems 6.5 and 6.7 we give two consequences of Theorem 6.4.

Theorem 6.5. *If $\langle G, \mathcal{T}\rangle$ is a totally bounded group such that $w\langle G, \mathcal{T}\rangle = 2^{|G|}$ and $\overline{\langle G, \mathcal{T}\rangle}$ is connected, then* $|\{\mathcal{S} \in \mathcal{B}(G)\colon\ \mathcal{S} \subseteq \mathcal{T}\}| = 2^{2^{|G|}}$.

Proof. Take $\mathcal{T} = \mathcal{T}_1$. By Theorem 2.7 there is $\mathcal{T}_0 \in \mathcal{B}_{|G|}$ such that $\mathcal{T}_0 \subseteq \mathcal{T}_1$, and Theorem 6.4 gives $|[\mathcal{T}_0, \mathcal{T}]| = |[\mathcal{T}_0, \mathcal{T}_1]| = 2^{w\langle G,\mathcal{T}_1\rangle} = 2^{2^{|G|}}$. □

Given a compact group $\langle G, \mathcal{T}\rangle$ with $w(G, \mathcal{T}) = \alpha$, we let $Ps(G, \mathcal{T})$ (respectively $CPs(G, \mathcal{T})$) be the partially ordered set of group topologies on G which are finer than $\mathcal{T}$ and are pseudocompact (respectively are pseudocompact and connected). For each cardinal γ set

$$Ps_\gamma(G) = \{\mathcal{U} \in Ps(G, \mathcal{T})\colon\ w(G, \mathcal{U}) = \gamma\}, \quad \text{and}$$
$$CPs_\gamma(G) = \{\mathcal{U} \in CPs(G, \mathcal{T})\colon\ w(G, \mathcal{U}) = \gamma\}.$$

From Lemma 3.11 it follows that each $\mathcal{U} \in Ps_\gamma(G, \mathcal{T})$ satisfies $\alpha \leqslant w(G, \mathcal{U}) \leqslant 2^{|G|}$.

Lemma 6.6. *Let $\langle G, \mathcal{T}\rangle$ be a nondegenerate compact, connected group. Then $Ps(G, \mathcal{T}) = CPs(G, \mathcal{T})$.*

Proof. Being connected, the compact group $\langle G, \mathcal{T}\rangle$ is divisible [42]. That every $\mathcal{S} \in Ps(G, \mathcal{T})$ is connected is given by this result of Wilcox [57]: every pseudocompact, divisible group is connected. □

As usual the (a) *width* the (b) *height* and the (c) *depth* of a partially ordered set P are defined to be the supremum of the cardinality of those subsets of P which are respectively (a) an anti-chain (b) well-ordered and (c) anti-well-ordered. If there is an anti-chain $A \subseteq P$ such that $|A| = width(P)$ then we say that *width(P) is assumed*, and similarly for *height(P)* and *depth(P)*.

Theorem 6.7. *Let $\langle G, \mathcal{T}\rangle$ be a compact, connected group such that $w(G, \mathcal{T}) = \alpha$ with $\mathrm{cf}(\alpha) > \omega$, and let $\alpha \leqslant \gamma \leqslant 2^{|G|}$. Define $\bar{\gamma} = \min\{\gamma^+, 2^{|G|}\}$. Then*

(a) $|Ps(G, \mathcal{T})| = |CPs(G, \mathcal{T})| = 2^{2^{|G|}}$;
(b) $|Ps_\gamma(G, \mathcal{T})| = |CPs_\gamma(G, \mathcal{T})| = 2^{\gamma\cdot|G|}$;
(c) *$width(Ps_\gamma(G, \mathcal{T})) = width(CPs_\gamma(G, \mathcal{T})) = 2^{\gamma\cdot|G|}$ and these widths are assumed;*
(d) *$height(Ps_\gamma(G, \mathcal{T})) = height(CPs_\gamma(G, \mathcal{T})) = \bar{\gamma}$ and these heights are assumed; and*
(e) *$depth(Ps_\gamma(G, \mathcal{T})) = depth(CPs_\gamma(G, \mathcal{T})) = \gamma$ and these depths are assumed.*

Proof. That $Ps_\gamma(G, \mathcal{T}) = CPs_\gamma(G, \mathcal{T})$ is immediate from Lemma 6.6. That $2^{\gamma\cdot|G|}$, $\bar{\gamma}$ and γ are upper bounds as asserted in (b)–(e) can be shown as in the first part of the proof of [48, Theorem 2.13]. Now take $\mathcal{T} = \mathcal{T}_0$. It is shown in our work [16, 6.6(b)] that there is a pseudocompact group topology $\mathcal{T}_1$ on G such that $\mathcal{T}_0 \subseteq \mathcal{T}_1$ and $w(G, \mathcal{T}_1) = 2^{|G|}$. Each topology in $[\mathcal{T}_0, \mathcal{T}_1]$ is the continuous image of the pseudocompact topology $\mathcal{T}_1$, hence is pseudocompact. The verification that the cited upper bounds in (b)–(e) are assumed is then routine, using (the proof of) Theorem 6.4 for the interval $[\mathcal{T}_0, \mathcal{T}_1]$ and the formula given in Theorem 6.10 below. Statement (a) is the special case $\gamma = 2^{|G|}$ of (b). □

Remarks 6.8. (a) Theorem 6.7 shows in effect that Theorem 5.8(b) of [16] remains valid for compact, connected groups G such that $\mathrm{cf}(w(G)) > \omega$.

(b) It is immediate from Notation 2.3 that for each topology $\mathcal{S} \in [\mathcal{T}_0, \mathcal{T}_1]$ as in Theorem 6.4 the identity function $\mathrm{id}: \langle G, \mathcal{T}_1\rangle \to \langle G, \mathcal{S}\rangle$ extends continuously to a surjection $\overline{\mathrm{id}}: \overline{\langle G, \mathcal{T}_1\rangle} \twoheadrightarrow \overline{\langle G, \mathcal{S}\rangle}$. Thus each of the completions $\overline{\langle G, \mathcal{S}\rangle}$ with $\mathcal{S} \in [\mathcal{T}_0, \mathcal{T}_1]$ is a compact, connected topological group.

(c) We have no example proving that the hypothesis "$\overline{\langle G, \mathcal{T}_1\rangle}$ is connected" in Theorem 6.4. is necessary. This suggests the following question, to which we anticipate that the answer is "Yes".

Question 6.9. Let G be a group and let $\mathcal{T}_i \in \mathcal{B}_{\alpha_i}(G)$ $(i = 0, 1)$ with $\alpha_1 > \alpha_0 \geqslant \omega$. Must $|[\mathcal{T}_0, \mathcal{T}_1]| = 2^{\alpha_1}$? Must $|\{\mathcal{S} \in \mathcal{B}(G)\colon\ \mathcal{S} \subseteq \mathcal{T}_1\}| = 2^{\alpha_1}$?

(That $|[\mathcal{T}_0, \mathcal{T}_1]| \geqslant \alpha_1$ is shown already in [49, Satz 3.25] using the theory of unitary representations; indeed there is a chain $\mathcal{C} \subseteq [\mathcal{T}_0, \mathcal{T}_1]$ such that $|\mathcal{C}| = \alpha_1$. Further, it is shown there that for every cardinal γ with $\alpha_0 \leqslant \gamma < \alpha_1$ one has $|\mathcal{B}_\gamma(G) \cap [\mathcal{T}_0, \mathcal{T}_1]| \geqslant \gamma^+$. Recently the present authors [17] gave a different proof of these same results, using an argument based on the method of Theorem 2.7 above.)

In connection with Question 6.9 it is silly to ask whether every group $\langle G, \mathcal{T}\rangle$ with $\mathcal{T} \in \mathcal{B}_\alpha(G)$ and $\alpha > \omega$ satisfies $|\{\mathcal{S} \in \mathcal{B}(G)\colon\ \mathcal{S} \subseteq \mathcal{T}\}| = 2^\alpha$. Indeed if $\mathcal{T}$ is a compact topology then $\{\mathcal{S} \in \mathcal{B}(G)\colon\ \mathcal{S} \subseteq \mathcal{T}\} = \{\mathcal{T}\}$.

In the statement and proof of the next theorem, which was used in Theorem 6.7 and will be needed also in Section 7, we use notation as in the proof of Theorem 6.4.

Theorem 6.10. *Every $F \in \mathcal{F}(K)$ satisfies $w\langle G, (\phi \circ \psi)^{-1}(C_F)\rangle = w(K/F) + w\langle G, \mathcal{T}_0\rangle$.*

Proof. Since $\overline{G}/C_F$ is the compact completion of $\langle G, (\phi \circ \psi)^{-1}(C_F)\rangle$, from Theorem 2.5 we have $w\langle G, (\phi \circ \psi)^{-1}(C_F)\rangle = w(\overline{G}/C_F)$. Since $(\overline{G}/C_F)/(C/C_F) = \overline{G}/C$ (cf. [35, 5.35]), Lemma 6.3(a) gives $w(\overline{G}/C_F) = w(C/C_F) + w(\overline{G}/C)$. Now $w\langle G, \mathcal{T}_0\rangle = w(\overline{G}/C)$ (see the proof of Theorem 6.4), so $w(\overline{G}/C_F) = w(C/C_F) + w\langle G, \mathcal{T}_0\rangle$. Since $h^{-1}(F)/C_F$ is totally disconnected and $(C/C_F)/(h^{-1}(F)/C_F) = C/h^{-1}(F)$ we have $w(C/C_F) = w(C/h^{-1}(F))$ by Lemma 6.3(b). Hence $w(C/C_F) = w(K/F)$, and $w\langle G, (\phi \circ \psi)^{-1}(C_F)\rangle = w(K/F) + w\langle G, \mathcal{T}_0\rangle$ follows. □

In the interest of completeness we now give a result, possibly known and closely related to a part of the argument given in the proof of Lemma 6.6, which we have not found stated explicitly in the literature.

Theorem 6.11. *Let $\mathcal{T}_i$ $(i = 0, 1)$ be pseudocompact group topologies on a group G with $\mathcal{T}_0 \subseteq \mathcal{T}_1$. Then $\langle G, \mathcal{T}_0\rangle$ is connected if and only if $\langle G, \mathcal{T}_1\rangle$ is connected.*

Proof. Surely if $\langle G, \mathcal{T}_1\rangle$ is connected then its continuous image $\langle G, \mathcal{T}_0\rangle$ is connected. For the converse let $\langle G, \mathcal{T}_0\rangle$ be connected. Then the completion $\overline{\langle G, \mathcal{T}_0\rangle}$ is connected, hence divisible [42]. Now as in [42], given $p \in \overline{\langle G, \mathcal{T}_1\rangle}$, set $p_0 = p$ and recursively find $p_{n+1} \in \overline{\langle G, \mathcal{T}_1\rangle}$ such that $p_{n+1}^{n!} = p_n$. Then $\{p_n\colon n \in \mathbb{N}\}$ generates a divisible Abelian subgroup A_p of $\overline{\langle G, \mathcal{T}_1\rangle}$ such that $p \in A_p$. The groups $\mathrm{cl}_{\overline{\langle G, \mathcal{T}_1\rangle}}(A_p)$ are then compact, divisible Abelian groups, hence are connected [35, 24.25]. Hence their union, which is $\overline{\langle G, \mathcal{T}_1\rangle}$, is connected. Since $\langle G, \mathcal{T}_1\rangle$ is pseudocompact the group $\overline{\langle G, \mathcal{T}_1\rangle}$ is its Stone–Čech compactification (cf. [23]), so the connectedness of $\overline{\langle G, \mathcal{T}_1\rangle}$ gives the connectedness of $\langle G, \mathcal{T}_1\rangle$. □

In connection with Theorem 6.11 it should be noted that a connected, pseudocompact group need not be divisible. Thus a divisible, compact group can have a dense, connected, nondivisible subgroup. See [57] for an example.

7. Chains of topological group topologies for certain groups

Here we extend the class of groups G for which the conclusions of Theorems 3.3 and 5.3 are valid.

Theorem 7.1. *Let α, β and γ be infinite cardinals and let $\langle G, \mathcal{T}\rangle$ be a totally bounded group with $|G| = \gamma$ such that $w\langle G, \mathcal{T}\rangle = 2^\gamma$ and $\overline{\langle G, \mathcal{T}\rangle}$ is connected.*

(A) *The following conditions are equivalent:*
  (a) *$C(2^\gamma, \beta)$ holds;*
  (b) *there is a chain $\mathcal{C} \subseteq \mathcal{B}(G)$ such that $|\mathcal{C}| = \beta$ and $\bigcup \mathcal{C} \subsetneq \mathcal{T}$.*

(B) *Let $\gamma \leqslant \alpha \leqslant 2^\gamma$. Then the following conditions are equivalent:*
  (a) *$E(2^\gamma, \beta, \alpha)$ holds;*
  (b) *there is a chain $\mathcal{C} \subseteq \mathcal{B}_\alpha(G)$ such that $|\mathcal{C}| = \beta$ and $\bigcup \mathcal{C} \subsetneq \mathcal{T}$.*

Proof. That (b) ⇒ (a) in (A) is obvious, while (b) ⇒ (a) in (B) is given by Theorem 3.1. To prove (a) ⇒ (b) in (A) we write $\mathcal{T}_1 = \mathcal{T}$ and $\alpha_1 = 2^\gamma$, we choose $\mathcal{T}_0 \in \mathcal{B}_\gamma(G)$ with $\mathcal{T}_0 \subseteq \mathcal{T}_1$ as given by Theorem 2.7, and we will sharpen the proof of Theorem 6.4 to find a chain $\mathcal{C}^* \subseteq [N_1, N_0]$ with $|\mathcal{C}^*| = \beta$; then $\mathcal{C} := \{\psi^{-1}(N)\colon N \in \mathcal{C}^*\}$ is as required. Indeed it is enough to note with C and $h\colon C \to K = \prod_{i\in I} K_i$ as in Theorem 6.4 with $|I| = 2^\gamma$ that if $\mathcal{A}$ is a chain in $\mathcal{P}(I)$ with $|\mathcal{A}| = \beta$ then

$$\mathcal{A}^* := \left\{ \prod_{i\in I\setminus A} K_i \times \{1_i\}_{i\in A}\colon\ A \in \mathcal{A}\right\}$$

is a chain in $\mathcal{F}(K)$ anti-isomorphic to $\mathcal{A}$, so $\mathcal{C}^* := \{\phi^{-1}(C_F)\colon\ F \in \mathcal{A}^*\}$ is a chain as required.

To prove (a) ⇒ (b) in (B) choose $\mathcal{A}$ as above with $|A| = \alpha$ for each $A \in \mathcal{A}$ and for $A \in \mathcal{A}$ set

$$F(A) = \prod_{i\in I\setminus A} K_i \times \{1_{i\in A}\}.$$

From $K/F(A) = \prod_{i \in A} K_i$ follows $w(K/F(A)) = \alpha$, so Theorem 6.10 gives $w\langle G, (\phi \circ \psi)^{-1}(C_{F(A)})\rangle = \alpha$ for each $A \in \mathcal{A}$. □

For use later we remark that in Theorem 7.1(A) and (B) the constructed chains $\mathcal{C}$ satisfy: $\mathcal{T}_0 \subseteq \mathcal{S} \subseteq \mathcal{T}_1$ for each $\mathcal{S} \in \mathcal{C}$.

Theorem 7.2. *Let α, β and α_0 be infinite cardinals and let $\langle G, \mathcal{T}_0\rangle$ be a compact connected group with $w\langle G, \mathcal{T}_0\rangle = \alpha_0$. Suppose that either* $\mathrm{cf}(\alpha_0) > \omega$ *or the connected component $Z_0(C)$ of the center C of G satisfies $w(Z_0(C)) = \alpha_0 > \omega$.*

(A) *the following conditions are equivalent:*
   (a) *$C(2^{2^{\alpha_0}}, \beta)$ holds;*
   (b) *there is a chain $\mathcal{C}$ of connected, pseudocompact group topologies on G such that $|\mathcal{C}| = \beta$ and $\mathcal{T}_0 \subseteq \bigcap \mathcal{C}$.*

(B) *Let $\alpha_0 \leqslant \alpha \leqslant 2^{2^{\alpha_0}}$. Then the following conditions are equivalent:*
   (a) *$E(2^{2^{\alpha_0}}, \beta, \alpha)$ holds;*
   (b) *there is a chain $\mathcal{C}$ of connected, pseudocompact group topologies on G such that $|\mathcal{C}| = \beta$, $\mathcal{C} \subseteq \mathcal{B}_\alpha(G)$, and $\mathcal{T}_0 \subseteq \bigcap \mathcal{C}$.*

Proof. We have $|G| = 2^{\alpha_0}$ from Theorem 2.15. It then follows from our work [16, 6.6(b)], writing $\alpha_1 = 2^{|G|} = 2^{2^{\alpha_0}}$, that there is a pseudocompact group topology $\mathcal{T}_1$ on G such that $\mathcal{T}_1 \supseteq \mathcal{T}_0$ and $w\langle G, \mathcal{T}_1\rangle = \alpha_1$. Since $\overline{\langle G, \mathcal{T}_1\rangle}$ is connected by Theorem 6.11, the required statements follow from the proofs of Theorems 6.4 and 7.1—indeed in (A) and (B) the chain $\mathcal{C}$ may be chosen so that $\mathcal{C} \subseteq [\mathcal{T}_0, \mathcal{T}_1]$. □

The construction given in the proof of the following theorem is closely related to an argument used in [18,21,16] to produce pseudocompact group topologies.

Theorem 7.3. *Let F be a compact metric group with $|F| > 1$ and let $\mathcal{T}_0$ be the product topology on F^ω. Then there is a totally bounded group topology $\mathcal{T}_1$ on F^ω such that $\mathcal{T}_1 \supseteq \mathcal{T}_0$ and $w(F^\omega, \mathcal{T}_1) = 2^{\mathfrak{c}}$.*

Proof. Let $W = \beta(\omega)\backslash\omega$ and define $\phi: F^\omega \to F^W$ by $\phi(f)(p) = \bar{f}(p)$; here for $f \in F^\omega$ the symbol $\bar{f}$ denotes that continuous function $\bar{f}: \beta(\omega) \to F$ such that $f \subseteq \bar{f}$.

We claim that graph(ϕ) is dense in $F^\omega \times F^W$. Indeed given $C = \{k_i: i \leqslant n\}$ and $\{p_i: i \leqslant n\}$, faithfully indexed subsets of ω and W respectively, and given $u_i \in F_{k_i}$ and $v_i \in F_{p_i}$ for $i \leqslant n$, we can find $f \in F^\omega$ such that $f(h_i) = u_i$ and $\phi(f)(p_i) = v_i$. Clearly there are pairwise disjoint subsets A_i $(i \leqslant n)$ of ω such that $A_i \in p_i$; it is enough to define $f \in F^\omega$ by

$$f(k) = \begin{cases} u_i & \text{if } k = k_i \in C, \\ v_i & \text{if } k \in A_i \backslash C \in p_i; \end{cases}$$

then $\phi(f) = \bar{f}(p_i) = v_i$, as required. The claim is proved.

Since $|W| = 2^{\mathfrak{c}}$ (cf. [32, 9.2], for example) we have $w(F^\omega \times F^W) = 2^{\mathfrak{c}}$; the completion of the group graph(ϕ) is $F^\omega \times F^W$, so Theorem 2.5 gives $w(\mathrm{graph}(\phi)) = 2^{\mathfrak{c}}$.

To complete the proof we define $\mathcal{T}_1$ on F^ω so that the isomorphism $\langle F^\omega, \mathcal{T}_1\rangle \to \text{graph}(\phi)$ is a homeomorphism into $F^\omega \times F^W$. That $\mathcal{T}_1 \supseteq \mathcal{T}_0$ follows from continuity of the projection function $\pi : F^\omega \times F^W \to F^\omega$ with respect to the product topologies. □

Theorem 7.4. *Let F be a compact, connected Lie group with trivial center and let $\mathcal{T}_0$ be the usual product topology on F^ω. Then*

(a) *$\mathcal{T}_0$ is the only pseudocompact group topology on F^ω;*

(b) *$|\mathcal{B}(F^\omega)| = |\{\mathcal{S} \in \mathcal{B}(F^\omega)\colon \mathcal{S} \supseteq \mathcal{T}_0\}| = 2^{2^{\mathfrak{c}}}$; and*

(c) *there are chains $\mathcal{C} \subseteq \mathcal{B}(F^\omega)$ and $\mathcal{C}' \subseteq \mathcal{B}(F^\omega)$ with $\mathcal{T}_0 \subseteq \bigcap \mathcal{C}$ and $\mathcal{T}_0 \subseteq \bigcap \mathcal{C}'$ such that $|\mathcal{C}| = (2^{\mathfrak{c}})^+$ and $|\mathcal{C}'| = 2^{(\mathfrak{c}^+)}$.*

Proof. (a) From Theorem 3.14 it follows that every $\mathcal{S} \in \mathcal{B}(F^\omega)$ satisfies $\mathcal{S} \supseteq \mathcal{T}_0$. The required statement then follows from [18, 3.1]: A compact metric group topology admits no proper pseudocompact group refinement.

(b) By Theorem 7.3 there is $\mathcal{T}_1 \in \mathcal{B}(F^\omega)$ such that $\mathcal{T}_1 \supseteq \mathcal{T}_0$ and $w\langle F^\omega, \mathcal{T}_1\rangle = 2^{\mathfrak{c}}$ with $\langle F^\omega, \mathcal{T}_1\rangle$ dense in $F^\omega \times F^{2^{\mathfrak{c}}} \simeq F^{2^{\mathfrak{c}}}$. Since $\overline{\langle F^\omega, \mathcal{T}_1\rangle} = F^{2^{\mathfrak{c}}}$ is connected, we have $|[\mathcal{T}_0, \mathcal{T}_1]| = 2^{2^{\mathfrak{c}}}$ by Theorem 6.4 and hence $|\{\mathcal{S} \in \mathcal{B}(F^\omega)\colon \mathcal{S} \supseteq \mathcal{T}_0\}| \geqslant 2^{2^{\mathfrak{c}}}$; that $|\mathcal{B}(F^\omega)| \leqslant 2^{2^{\mathfrak{c}}}$ is immediate from $|F^\omega| = \mathfrak{c}$.

(c) We have $C(2^{\mathfrak{c}}, (2^{\mathfrak{c}})^+)$ and $C(2^{\mathfrak{c}}, 2^{(\mathfrak{c}^+)})$ from Theorems 2.18(c) and 2.18(b), respectively. The topology $\mathcal{T}_1$ on F^ω given in (b) has $w\langle F^\omega, \mathcal{T}_1\rangle = 2^{\mathfrak{c}}$ and $\overline{\langle F^\omega, \mathcal{T}_1\rangle}$ connected, so according to (the remark following) Theorem 7.1 there are chains $\mathcal{C}$ and $\mathcal{C}'$ of the indicated cardinalities as required, each element $\mathcal{S}$ of which indeed satisfies even $\mathcal{T}_0 \subseteq \mathcal{S} \subseteq \mathcal{T}_1$. □

Acknowledgement

We have profited from an attentive, detailed reading by the referee. Among other contributions he has simplified a couple of our proofs, uncovered a gap (now corrected) in the statement of one of our lemmas, noted several typographical errors, and augmented our list of references. We are grateful for his help.

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