Transitive bislim geometries of gonality 3, part I: The geometrically homogeneous cases

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Abstract

We consider point–line geometries having three points on every line, having three lines through every point (bislim geometries), and containing triangles. We classify such geometries under the hypothesis of the existence of a collineation group acting transitively on the point set. In the first part of this work, we introduce the local structure at a point and consider some cases where this local space already determines the geometry.

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1. Introduction

Bislim geometries are also known as $n_3$ configurations. These have been studied for a long time, see the survey papers [3,7], and the beautiful paper of Coxeter [2]. Two aspects have been extensively studied in particular, and these are enumeration and realization. Results in both topics usually heavily depend on the help of a computer. In [6], the first author started a comprehensive theory about realizations, but also about more general embeddings of bislim geometries. In [8], we presented a classification of all flag transitive partial linear bislim geometries with triangles (we say that the geometry has gonality 3 if it is partial linear and contains triangles). This roughly characterizes the examples described by Coxeter in [2] arising from a hexagonal tiling of the Euclidean plane (the honeycomb geometry) by factoring out with respect to a group. Only one example does not arise in this way, and can hence be seen as a sporadic case, and

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this is Desargues’ configuration (on 10 points). Another example, the so-called Möbius–Kantor configuration (on 8 points), does arise as a quotient of that tiling, but not in a standard way (i.e., not in a way described by Coxeter in [2]). The latter example is in fact included in an infinite class of point transitive bislim geometries containing triangles and constructed as follows: the point set is the set of integers modulo $n$; the lines are all translates of the 3-set $\{0, 1, 3\}$. This simple construction leads us to try to classify all point transitive bislim geometries of gonality 3. It turns out that many other infinite classes arise containing some special geometries that were noticed by people before, but, as far as we know, not in such a general systematic context.

There are several corollaries of our result. First of all, one can translate the conditions to a classification of a class of transitive trivalent graphs. The classification of symmetric trivalent graphs (i.e., trivalent graphs with an automorphism group acting transitively on the ordered edges) is an intriguing problem that attracts many graph and group theorists, since the beautiful observation of Tutte [4,5] on the order of the vertex stabilizer in such a graph, see e.g. [1]. Our result shows that the graphs that contain no 4-circuits, but do contain 6-circuits, are manageable, even if only vertex transitivity (in the non-bipartite case) or transitivity on one bipartition class (in the bipartite case) are required.

Also, one can remark that most classes of examples contain a “universal” one from which the others can be deduced as quotients. But not all classes are like that. What is the deeper reason here? More insight could be illuminating for lifting the hypothesis on the existence of triangles; we comment on this later on.

The method that we use is completely different from the flag transitive case in [8]. In fact, we sometimes explicitly assume that the geometries under consideration are not flag transitive. But with a little more effort, our proof would imply an alternative one for the flag transitive case. What we do is subdivide the problem into cases depending on some local structure of the given geometry, by which we mean the geometry induced on the points collinear to a given point. This seems to be the right way to approach these geometries. If a collineation group acts transitively on the point set, then all the local structures are isomorphic (the geometry is geometrically point homogeneous). In this first part, we consider the geometrically point homogeneous case. Although there are in principle 77 possibilities for the local structure of a geometrically point homogeneous bislim geometry containing triangles, only a few survive the geometrical point homogeneity assumption. Once this is noted, the classification of the point transitive ones boils down to ad hoc methods for getting control over the various cases. Some of these methods only need to use the geometrical point homogeneity condition, and are precisely these cases that we treat in the present paper. In part II, we treat the remaining cases (and show that the geometrical point homogeneity condition is not strong enough to replace point transitivity). All this requires in our opinion some beautiful geometric and permutation group theoretic reasonings.

We also note that some classes are not explicitly classified, but reduced to a class of graphs, or of (factor) groups (of a given “universal” group). It will be clear that a further specification is out of reach.

Furthermore, it is worthwhile to note that the geometric point homogeneity condition is strong enough to filter out only those local structures that appear in point transitive bislim geometries of gonality 3. But for some local structures, there exist geometric point homogeneous examples which are not point transitive, and for other local structures, all geometric point homogeneous ones are also point transitive. Also, for some local structures, all geometric point homogeneous examples arise from a universal example on considering quotients, and for other local structures this is not the case (while all point transitive examples do arise from a universal one).
The paper is organized as follows. In the next section we introduce notation. In Section 3 we describe all geometries involved in our Main Result 1, and then we can state this theorem. The proof will be given in Section 5. We end the paper with a proposition (in Section 6) stating how one can distinguish the isomorphism classes of quotients of the honeycomb geometry. In part II, we introduce some more examples, state our Main Result II (the point transitive case) and complete the proof.

2. Preliminaries

A point–line geometry $\Gamma = (\mathcal{P}, \mathcal{L}, l)$ consists of two disjoint sets $\mathcal{P}$ (the point set) and $\mathcal{L}$ (the line set), together with a symmetric incidence relation $l$ between $\mathcal{P}$ and $\mathcal{L}$. The graph with vertex set $\mathcal{P} \cup \mathcal{L}$, where two vertices are adjacent if they represent an incident point–line pair, is called the incidence graph of $\Gamma$, and is also denoted by $\Gamma$ (since this graph unambiguously determines the geometry and vice versa), and we use graph theoretic notation. For instance, if $n$ is any natural number, then $\Gamma_n(x)$ denotes the set of vertices at distance $n$ from the vertex $x$. The incidence graph is a bipartite graph. Every automorphism of that graph fixing the two bipartition classes is a collineation of the geometry. Also, if the graph is connected, then we say that the geometry is connected. In many cases in the literature, connectivity is part of the definition of a geometry, and that is the case here, too. Hence from now on, we include connectivity in the definition of a point–line geometry. A point–line geometry where every line carries exactly three points is called slim. If also every point is incident with three lines, then the geometry is called bislim. The dual of a geometry is obtained by interchanging the point and line set; the incidence graph remains unchanged. A duality is an automorphism of the incidence graph interchanging the two bipartition classes.

The gonality of a geometry is half of the girth of its incidence graph. In this paper, we are only concerned with geometries having gonality distinct from 2 (the so-called partial linear spaces, because two points determine at most one line); in fact we will assume gonality 3 all the time (this means that the geometry has triangles). If a geometry $\Gamma$ admits a collineation group $G$ acting transitively on the point set, then we say that the pair $(\Gamma, G)$ is point transitive, or that $G$ acts point transitively on $\Gamma$. A flag is an incident point–line pair, or, equivalently, an edge of the incidence graph. The pair $(\Gamma, G)$ is flag transitive if $G$ acts transitively on the set of flags of $\Gamma$.

We will also use some obvious notation from incidence geometry like $ab$ is the line incident with the points $a$ and $b$, if it exists and is unique. We extend this notation to $abc$ to say that the points $a, b, c$ are incident with a common line and to denote that unique line (we sometimes express this by saying that the line $abc$ exists).

Let $\Gamma' = (\mathcal{P}, \mathcal{L}, l)$ be a bislim geometry of gonality 3. Let $x$ be any point of $\Gamma$ and $L$ any line incident with $x$. Let $x_1, x_2$ be the two other points incident with $L$, and let $L_1, L_2$ be the two other lines incident with $x$. The points on $L_i, i = 1, 2$, different from $x$ will be denoted by $y_i$ and $z_i$. The local structure at the point $x$ is the subgeometry $\Gamma_x$ of $\Gamma$ with point set $x \cup \Gamma'_2(x)$ and line set the elements of $\Gamma'_1(x) \cup \Gamma'_3(x)$ incident with 2 or 3 of these points. Remark that this subgeometry is not necessarily bislim (in fact, it is only bislim if it coincides with $\Gamma$ itself!). Denote the set of lines of $\Gamma_x$ not through $x$ by $\Gamma'_x$. If $\Gamma'_x$ is isomorphic to some geometry $\Gamma''$, for all points $x$, then we say that $\Gamma$ is geometrically point homogeneous and point-locally $\Gamma''$. Similarly for geometrically line homogeneity and line-locally geometries (for instance, a geometrically line homogeneous bislim geometry of gonality 3 is a bislim geometry of gonality 3 the dual of which is geometrically point homogeneous). If a geometry is point-locally $\Gamma''$ and line-locally
also \( \Gamma' \), then we say that \( \Gamma \) is locally \( \Gamma' \), or \( \Gamma \) has local structure \( \Gamma' \), and \( \Gamma \) is geometrically homogeneous.

A 1-cover of a bislim geometry \( \Gamma = (\mathcal{P}, \mathcal{L}, l) \) is a bislim geometry \( \tilde{\Gamma} = (\tilde{\mathcal{P}}, \tilde{\mathcal{L}}, \tilde{l}) \) together with a (necessarily surjective) incidence preserving mapping \( \theta : \tilde{\mathcal{P}} \to \mathcal{P}; \tilde{\mathcal{L}} \to \mathcal{L} \) such that the three points on any line \( \tilde{L} \) of \( \tilde{\Gamma} \) are mapped onto the three points of \( \tilde{L}^\theta \), and dually for the three lines through any point \( \tilde{x} \) of \( \tilde{\Gamma} \). Clearly, the local structure of \( \tilde{\Gamma} \) at a point \( \tilde{x} \) can abstractly be viewed as a subgeometry of the local structure of \( \Gamma \) at the point \( \tilde{x}^\theta \). Now, if for all points and lines \( \tilde{A} \) of \( \tilde{\Gamma} \), the local structure at \( \tilde{A} \) is mapped under \( \theta \) bijectively onto the local structure of \( \Gamma \) at \( \tilde{A}^\theta \), then we say that \( \tilde{\Gamma} \) is a 1\( \frac{1}{2} \)-cover with covering epimorphism \( \theta \). We now have the usual definition of universal 1\( \frac{1}{2} \)-cover \( \Gamma \) of \( \Gamma \) with universal covering epimorphism \( \theta \): for every 1\( \frac{1}{2} \)-cover \( \theta : \tilde{\Gamma} \to \Gamma \), there exists a cover \( \tilde{\theta} : \tilde{\Gamma} \to \tilde{\Gamma} \) such that \( \tilde{\theta} = \theta \theta \), and then \( \tilde{\theta} \) is a 1\( \frac{1}{2} \)-cover. Finally we say that \( \Gamma \) is 1\( \frac{1}{2} \)-connected if for every 1\( \frac{1}{2} \)-cover the covering epimorphism is an isomorphism. Clearly for every 1\( \frac{1}{2} \)-connected bislim geometry, the identity defines a universal 1\( \frac{1}{2} \)-cover, and every universal 1\( \frac{1}{2} \)-cover is 1\( \frac{1}{2} \)-connected. Also, if some geometrically point homogeneous bislim geometry \( \Gamma \) with given local structure is a 1\( \frac{1}{2} \)-cover of every geometrically point homogeneous bislim geometry with that same given local structure, then \( \tilde{\Gamma} \) is clearly the universal 1\( \frac{1}{2} \)-cover of every member and \( \tilde{\Gamma} \) is 1\( \frac{1}{2} \)-connected.

3. Examples of geometrically point homogeneous bislim geometries with gonality 3

For a list of local structures, refer to Appendix. The local structure with number \( n \) of that list will be referred to as \( \text{LS}(n) \). Later on, we will prove that this list is complete, see Lemma 5.1 below.

3.1. A family associated with trivalent graphs of girth at least 4

Let there be given an arbitrary (finite or infinite) connected 3-regular (or trivalent, or cubic) graph \( G = G(V, E) \), where \( V \) is the set of vertices and \( E \) the set of edges. We define a geometry \( \Gamma := \Gamma_G \) in the following way. To every ordered edge \((v, w)\) of \( G \) we attach a point \( P_{(v,w)} \) and a line \( L_{(v,w)} \). If \( v \in V \) is adjacent to \( w, w_1 \) and \( w_2 \), then the point \( P_{(v,w)} \) and the line \( L_{(v,w)} \) are incident with the lines \( L_{(v,w_1)}, L_{(v,w_2)}, L_{(v,w)} \), and with the points \( P_{(v,w_1)}, P_{(v,w_2)}, P_{(v,w)} \), respectively. It is easily seen that the geometry \( \Gamma \) is bislim and has gonality 3. Indeed, with above notation, \( \{P_{(v,w)}, P_{(v,w_1)}, P_{(v,w_2)}\} \) is a triangle of \( \Gamma \) with sides \( L_{(v,w)}, L_{(v,w_1)} \) and \( L_{(v,w_2)} \). Also, if \( G(V, E) \) does not contain triangles, then \( \Gamma_G \) is locally \( \text{LS}(1) \). If, on the other hand, \( G(V, E) \) contains a triangle \( \{a, b, c\} \), then the local structure in \( P_{(a,b)} \) is isomorphic either to \( \text{LS}(5) \) (if the edge \( \{a, b\} \) of \( G \) is contained in a unique triangle) or to \( \text{LS}(13) \) (otherwise). Indeed, the triangle \( \{a, b, c\} \) of \( G \) induces the extra triangle \( \{P_{(a,b)}, P_{(b,c)}, P_{(c,a)}\} \) in \( \Gamma_G \). If \( G(V, E) \) contains at least one triangle, then it is easy to see that \( \Gamma_G \) is geometrically point homogeneous if and only if \( G(V, E) \) is the complete graph on 4 vertices. In the latter case, \( \Gamma_G \) is isomorphic to the Coxeter geometry, introduced by Coxeter [2] and named after him in [8]. This geometry is flag transitive and has local structure \( \text{LS}(13) \).

If \( T \) is a trivalent tree, then it is clear that \( \Gamma_T \) is a 1\( \frac{1}{2} \)-cover of every example \( \Gamma_G \) with \( G \) a trivalent graph without triangles. Also, \( \Gamma_T \) is 1\( \frac{1}{2} \)-connected (this can be shown using the observation that the geometric realization of \( T \) is simply connected as a topological space — or equivalently \( T \) is simply connected as a simplicial complex).
3.2. A wild example

Let $D$ be the dual of the geometry defined by the vertices and edges of a complete graph $K_4$ on 4 vertices. Let $F$ be a countable (or finite) collection of mutually disjoint geometries isomorphic to $D$ and let $B$ be a set of triples of points of distinct members of $F$ that partitions the union $\mathcal{U}$ of the point sets of all the members of $F$ such that the bipartite graph $G(\mathcal{F} \cup B, E)$, with $\{f, b\} \in E$, for $f \in F$ and $b \in B$, if $b$ contains a point of $f$, is connected. Define the geometry $\Gamma_{\mathcal{F}, B}$ as follows. The point set is $\mathcal{U}$, and the line set is the union of the line sets of all members of $\mathcal{F}$, together with all members of $B$. It is clear that the local structure in each point is isomorphic to $\text{LS}(4)$. However, the local structure of a line in some member of $\mathcal{F}$ is isomorphic to $\text{LS}(10)$, while the local structure of an element of $B$ is isomorphic to $\text{LS}(0)$.

Just like in the examples of Section 3.1, the geometry $\Gamma_{\mathcal{F}, B}$, with $G(\mathcal{F} \cup B, E)$ a bipartite $(3, 6)$-valent tree, is a $1\frac{1}{2}$-connected $1\frac{1}{2}$-cover of every example $\Gamma_{\mathcal{F}, B}$.

3.3. Quotients of the honeycomb geometry

Let $\mathbb{E}$ be the real Euclidean plane, and consider the tiling $T$ of $\mathbb{E}$ in regular hexagons (a honeycomb). The skeleton of this honeycomb is in fact a bipartite graph which divides the vertices into two classes that we will designate as black and white. We define the honeycomb geometry $S_\infty$ as the geometry with points the black vertices and lines the white vertices, and where incidence is adjacency.

Let $W(\tilde{A}_2)$ be the full collineation group of $S_\infty$, or equivalently, the group of isometries of $\mathbb{E}$ preserving the honeycomb tiling $T$ and stabilizing the set of black vertices (which is the Weyl group of type $\tilde{A}_2$, whence the notation).

Let $G$ be a subgroup of $W(\tilde{A}_2)$ such that for every vertex $v$ of $T$, the graph theoretic distance between two distinct vertices of the orbit $v^G$ is at least 6. Then the quotient geometry $S_\infty/G$ defined in the obvious way by identifying the elements in the same orbit, is a bislim geometry which is a partial linear space. If the graph theoretic distance between two distinct vertices of the orbit $v^G$ is at least 8, for each vertex $v$, then the local structure of $S_\infty/G$ is $\text{LS}(13)$.

Note that the point set of $S_\infty$ can be identified with the set of pairs $(i, j)$ of integers, and the lines are the 3-subsets $\{(i, j), (i + 1, j), (i, j + 1)\}$, for all integers $i, j$.

Very explicitly, we can now define the following quotient geometries.

(HC1) Let $r, s$ be two integers with $0 \leq s \leq r$ and $r^2 + rs + s^2 \geq 7$. We define a geometry $S_{(r, s)}$ as follows. The points are the equivalence classes of ordered pairs $(i, j)$, with $i, j$ integers and with respect to the equivalence relation $\sim$ defined as $(i, j) \sim (i', j')$ if $(i - i', j - j') = (kr, ks)$, for some integer $k$. We denote by $(i, j)/ \sim$ the equivalence class containing $(i, j)$. The lines of the geometry are the 3-sets $\{(i, j)/ \sim, (i + 1, j)/ \sim, (i, j + 1)/ \sim\}$, for all integers $i, j$.

(HC2) Let $a, c$ be integers with $a > 0$, $0 < c < a$ and for every integer linear combination of $(a, 0)$ and $(c, d)$, say $(r, s)$, $r^2 + rs + s^2 \geq 7$. The points of the geometry $M_{(a, 0), (c, d)}$ are the equivalence classes of ordered pairs $(i, j)$, with $i, j$ integers, with respect to the equivalence relation $\approx$, defined as $(i, j) \approx (i', j')$ if $(i - i', j - j') = (ka + \ell c, \ell d)$, for some integers $k$ and $\ell$. With notation similar to that in the previous example, the lines of the geometry are the 3-sets $\{(i, j)/ \approx, (i + 1, j)/ \approx, (i, j + 1)/ \approx\}$, for all integers $i, j$. 
(HC3a) Let \( r \) be an integer with \( r \geq 2 \). The points of the geometry \( S_{(r)}^* \) are the equivalence classes of ordered integer pairs \((i, j)\) with respect to the equivalence relation \( \sim \) defined as \((i, j) \sim (i', j')\) if either \((i - i', j - j') = (-2kr, 4kr)\), for some integer \( k \), or \((i + i' + r + j, j' - j - 2r) = (-2kr, 4kr)\), for some integer \( k \). One checks that this is indeed an equivalence relation (in particular, it is symmetric!). The lines are again, with notation similar to that used before, the 3-sets \( \{(i, j)/ \sim, (i + 1, j)/ \sim, (i, j + 1)/ \sim\}\), for all integers \( i, j \).

(HC3b) Let \( r \) be an integer with \( r \geq 1 \). The points of the geometry \( S_{(r)}^{**} \) are the equivalence classes of ordered integer pairs \((i, j)\) with respect to the equivalence relation \( \sim^{**} \) defined as \((i, j) \sim^{**} (i', j')\) if either \((i - i', j - j') = (-2kr + k, 4k)\), for some integers \( k \), or \((i + i' + r + j, j' - j - 2r) = (-2kr + k, 4k)\), for some integer \( k \). One checks that this is indeed an equivalence relation (in particular, it is symmetric!). The lines are again, with notation similar to that used before, the 3-sets \( \{(i, j)/ \sim^{**}, (i + 1, j)/ \sim^{**}, (i, j + 1)/ \sim^{**}\}\), for all integers \( i, j \).

(HC4a) Let \( r, s \) be two integers with \( r \geq 2 \) and \( s \geq 3 \). The points of the geometry \( M_{(r), (s, 0)}^* \) are the equivalence classes of ordered integer pairs \((i, j)\) with respect to the equivalence relation \( \approx \) defined as \((i, j) \approx (i', j')\) if either \((i - i', j - j') = (-k(2r + 1), 2k(2r + 1))\), for some integers \( k \), \( \ell \), or \((i + i' + r + j, j' - j - 2r) = (-k(2r + 1), 2k(2r + 1))\), for some integers \( k \), \( \ell \). One again checks that this is indeed an equivalence relation. The lines are, again, with notation similar to that used before, the 3-sets \( \{(i, j)/ \approx, (i + 1, j)/ \approx, (i, j + 1)/ \approx\}\), for all integers \( i, j \).

(HC4b) Let \( r, s \) be two integers with \( r \geq 1 \) and \( s \geq 3 \). The points of the geometry \( M_{(r), (s, 0)}^{**} \) are the equivalence classes of ordered integer pairs \((i, j)\) with respect to the equivalence relation \( \approx^{**} \) defined as \((i, j) \approx^{**} (i', j')\) if either \((i - i', j - j') = (-k(2r + 1) + k, 2k(2r + 1))\), for some integers \( k \), \( \ell \), or \((i + i' + r + j, j' - j - 2r) = (-k(2r + 1) + k, 2k(2r + 1))\), for some integers \( k \), \( \ell \). One checks that this is indeed an equivalence relation. The lines are again, with notation similar to that used before, the 3-sets \( \{(i, j)/ \approx^{**}, (i + 1, j)/ \approx^{**}, (i, j + 1)/ \approx^{**}\}\), for all integers \( i, j \).

One easily checks that all these geometries are bislim with gonality 3. The geometries in (HC1) and (HC3a), (HC3b) are infinite, while the others are finite.

In most of the above cases, the local structure of the geometry is equal to \( LS(13) \). But in some parameter cases also \( LS(n) \) occurs, for \( n \in \{24, 35, 51, 58, 73, 77\} \), in a geometrically homogeneous bislim geometry (see the Main Result 1 below), and also \( LS(m) \) occurs, for \( m \in \{25, 33, 48, 60\} \), in other quotients. Indeed, one checks that \( M_{(1), (3, 0)}^{**} \) is a bislim geometry with nine points and nine lines and the local structures in points and lines are \( LS(33) \) and \( LS(60) \) (note that there are exactly three bislim partial linear spaces with nine points, and the other two appear in the Main Result under (iv) and (viii)). Also, \( M_{(1), (n, 0)}^{**}, n \geq 4 \), is a geometry in which some points and lines have local structure \( LS(13) \) (not occurring when \( n = 4 \), \( LS(25) \) and \( LS(48) \). Likewise for \( S_{(1)}^{**} \), which we conjecture to be the (universal) \( 1^1/2 \)-cover of \( M_{(1), (n, 0)}^{**}, n \geq 5 \) (and this would provide an example of a universal \( 1^1/2 \)-cover in the case where we have different local structures).

Also, note that the geometries under (HC2) are not all pairwise non-isomorphic. In Section 6 we will address the question of assigning unique parameters to each isomorphism class in (HC2).
We will also prove there that, given these geometries with those parameters, all geometries mentioned in this section are pairwise non-isomorphic, i.e., the geometry determines in each case uniquely the parameters.

4. Statement of Main Result

In the present paper we will prove:

Main Result 1. If $\Gamma = (\mathcal{P}, \mathcal{L}, 1)$ is a geometrically point homogeneous bislim geometry of gonality 3 which is point-locally LS(n), $1 \leq n \leq 77$, then $n \in \{1, 4, 5, 13, 24, 34, 35, 51, 58, 73, 77\}$. In particular, we have the following characterizations.

(i) If $n = 1$, then there are always lines with local structure LS(1), and there are always points incident with three such lines. Also, if $\Gamma$ is not geometrically homogeneous, then there are lines and local structure LS(0), and there are points incident with a unique such line. Furthermore, if $\Gamma$ has the property that, whenever $\{x_1, x_2, x_3\}$ is a triangle, then for all $i, j, k$ such that $\{i, j, k\} = \{1, 2, 3\}$, there exists another triangle $\{a, b, c\}$ such that $x_1ibc$ and $a\ell x_3$ then $\Gamma \cong \Gamma_3$, with $G$ a cubic graph of girth $\geq 4$. These are the examples of Section 3.1. All these examples are geometrically homogeneous.

(ii) If $n = 4$, then $\Gamma$ is isomorphic to $\Gamma_{\mathcal{F}, \mathcal{B}}$, where $\mathcal{F}$ is a countable collection of geometries isomorphic to the dual of $K_4$ and $\mathcal{B}$ is a set of triples of points of distinct members of $\mathcal{F}$ that partitions the union of the point sets of all the members of $\mathcal{F}$ such that the bipartite graph $\mathcal{G}(\mathcal{F} \cup \mathcal{B}, E)$, with $\{f, b\} \in E$, for $f \in \mathcal{F}$ and $b \in \mathcal{B}$, if $b$ contains a point of $f$, is connected. These are the examples of Section 3.2.

(iii) If $n = 13$, then $\Gamma$ is isomorphic to a quotient of the honeycomb geometry, which is the universal $1_{2\frac{1}{2}}$-cover of $\Gamma$ (and which is also the only $1_{2\frac{1}{2}}$-connected bislim geometry with this local structure). In particular, $\Gamma$ is isomorphic either to

- $\mathcal{S}_{(r, s)}$, where $0 \leq s \leq r$ and $r^2 + rs + s^2 \geq 12$, or to

- $\mathcal{M}_{(a, 0), (c, d)}$, with $a, d > 0$, $0 \leq c < a$, such that for every integer linear combination of $(a, 0)$ and $(c, d)$, say $(r, s)$, one has $r^2 + rs + s^2 \geq 12$, or to $\mathcal{S}_{(r)}$, with $r \geq 2$, or to $\mathcal{M}_{(r), (s, 0)}$, with $r \geq 2$ and $s \geq 4$.

(iv) If $n = 24$, then $\Gamma$ is isomorphic to a quotient of the honeycomb geometry. In particular, $\Gamma$ is isomorphic either to

- $\mathcal{S}_{(3, 0)}$, or to

- $\mathcal{M}_{(3, 0), (1-d, 2d+1)}$ or $\mathcal{M}_{(3, 0), (1-d, 2d+1)}$ or $\mathcal{M}_{(3, 0), (1-d, 2d)}$ or $\mathcal{M}_{(3, 0), (1-d, 2d)}$, with $d \geq 2$, or to

- $\mathcal{M}_{(r), (3, 0)}^*$ or $\mathcal{M}_{(r), (3, 0)}^{**}$ with $r \geq 2$.

The geometry $\mathcal{S}_{(3, 0)}$ is the unique $1_{2\frac{1}{2}}$-connected member of this family and is hence the universal $1_{2\frac{1}{2}}$-cover of $\Gamma$.

(v) If $n = 34$, then $\Gamma$ is isomorphic to the Desargues configuration.

(vi) If $n = 35$, then $\Gamma \cong \mathcal{M}_{(3, 0), (0, 3)}$ is the Pappus configuration.

(vii) If $n = 51$, then either $\Gamma \cong \mathcal{S}_{(2, 1)}$, or $\Gamma \cong \mathcal{M}_{(n, 0), (2, 1)}$, with $n \geq 10$. In any case, $\mathcal{S}_{(2, 1)}$ is the universal $1_{2\frac{1}{2}}$-cover of $\Gamma$, and it is also the only $1_{2\frac{1}{2}}$-connected member of this family.

(viii) If $n = 58$, then $\Gamma \cong \mathcal{M}_{(3, 0), (1, 3)} \cong \mathcal{M}_{(9, 0), (2, 1)}$.

(ix) If $n = 73$, then $\Gamma \cong \mathcal{M}_{(4, 0), (1, 2)} \cong \mathcal{M}_{(8, 0), (2, 1)}$ is the Möbius–Kantor configuration.

(x) If $n = 77$, then $\Gamma \cong \mathcal{M}_{(7, 0), (2, 1)} \cong \text{LS(77)}$ is the Fano plane.
Remark 4.1. From the theorem it follows that, whenever the local structure of a geometrically point homogeneous bislim geometry contains, as an abstract geometry, the geometry $\text{LS}(13)$ (by deleting some lines in the local structure) as a subgeometry, then it is 1-covered by the honeycomb geometry. We are not aware of a direct proof of that fact, but it would be somewhat interesting and tempting to find one.

Remark 4.2. In case of local structure $\text{LS}(5)$, we cannot draw any reasonable conclusion from the point homogeneity. Only if the geometry is point transitive we will be able to characterize (see part II).

5. Proof of Main Result 1

The proof of the Main Result has three main parts. First, we classify all possible local structures of bislim geometries of gonality 3 in general. Then we eliminate those structures that cannot arise as local structure of a geometrically point homogeneous bislim geometry. Finally, we consider each remaining local structure separately in detail.

5.1. Enumeration of all possible local structures

We will not prove the next lemma, which is a tedious but easy exercise. We give some comments below.

Lemma 5.1. If $\Gamma$ is an arbitrary bislim geometry of gonality 3, and $x$ is any point or line of $\Gamma$, then the local structure $\Gamma_x$ is isomorphic to one of the 78 configurations listed in Appendix.

Remark 5.2. Local structure $\text{LS}(0)$ can indeed appear, notwithstanding the condition on the gonality. But, of course, there must be at least one point with local structure different from $\text{LS}(0)$.

Remark 5.3. A possible strategy for the proof of the lemma is to subdivide according to the number of transversals (a transversal in the local structure $\Gamma_x$ is a line of $\Gamma_3(x)$ incident with three points of $\Gamma_2(x)$). This number can be 0, 1, 2, 3 or 4. Then one counts the number of all possible local structures with $i$ transversals, say this is $N_i$, $i \in \{0, 1, 2, 3, 4\}$. One also counts the number of local structures isomorphic to the ones in the list of Appendix. The sum of all these is equal to the sum of the $N_i$, $i \in \{0, 1, 2, 3, 4\}$ and the lemma is proved.

5.2. Elimination of some local structures

Let $\Gamma$ be a bislim geometry of gonality 3 with a homogeneous local structure on its points. Let $x$ be some point of $\Gamma$. Then $\Gamma_x$ is one of the local structures of Appendix, but not $\text{LS}(0)$ (because this would imply that $\Gamma$ has no triangles). We now use the notation summarized in Fig. 1 for the elements in the local structure $\text{LS}(1)$ up to $\text{LS}(77)$. It should be used in this way for all pictures in Appendix. For instance, in $\text{LS}(38)$, the lines not through $x$ are $x_1y_1y_2$ and $x_2z_1$.

We will prove:

Lemma 5.4. If $\Gamma$ is a geometrically point homogeneous bislim geometry of gonality 3, then for every point $x$ of $\Gamma$, the local structure $\Gamma_x$ is isomorphic to $\text{LS}(n)$, with $n \in \{1, 4, 5, 13, 24, 34, 35, 51, 58, 73, 77\}$.
Proof. We will look at all possible local structures and derive contradictions in the appropriate cases. Concerning terminology, we will talk about the “third point” on a line $M$ and the “third line” through a point $u$ to refer to the unique element of the geometry $\Gamma$ on $M$ and through $u$, respectively, but not contained in the local structure in question.

We first look at the local structures without transversals, i.e., local structures $\text{LS}(1)$ up to $\text{LS}(35)$. But $\text{LS}(n)$, with $n \in \{2, 6, 7, 9, 14, 15, 16, 17, 18, 22, 25, 26, 27, 28, 31, 32, 33\}$, contains a point $u \in \{x_1, x_2, y_1, y_2, z_1, z_2\}$ such that $\Gamma_u$ does contain a transversal. This contradicts the geometric point homogeneity of $\Gamma$.

Local structure $\text{LS}(3)$ has two lines $x_1y_1$ and $x_1z_2$ in $\Gamma_x$. To obtain local structure $\text{LS}(3)$ in the point $y_1$, the “third line” through this point has to intersect the line $x_1z_2$ in its “third point”. But then $\Gamma_{x_1}$ cannot be isomorphic to $\text{LS}(3)$.

Local structure $\text{LS}(8)$ has three lines $x_1y_1$, $x_2y_2$ and $x_2z_1$ in $\Gamma_x$. To obtain local structure $\text{LS}(8)$ in the point $x_2$ we need that $x_1$ is collinear with either the “third point” $a$ on $x_2z_1$ or the “third point” $b$ on $x_2y_2$. In both cases this line can be the line $x_1y_1$ or the “third line” through $x_1$. So we consider four different cases. First we look at the case where $x_1y_1$ is a line. The line $x_2x_2$ is the only line through $x$ belonging to $\Gamma_{x_2}$. This is also the case for the point $x_2$ since if not then $y_2$ should be collinear with $a$, but then $\Gamma_{x_2} \not\cong \text{LS}(8)$. It follows that $z_2$ is collinear with the “third point” $b$ on the line $x_2y_2$. Considering $\Gamma_{z_2}$ it is easily seen that no line through $x$ belongs to $\Gamma_{z_2}$.

Hence $b$ is collinear with a point, say $c$, on the line through $z_2$ different from $L_2$ and $z_2b$. In $\Gamma_{z_2}$ we already have that $xx_2$ and $z_2b$ are lines of $\Gamma_{z_2}$. But then either the line $z_2c$ and the “third line” through $y_2$ intersect or the line $bc$ intersects the “third line” through $y_2$. Considering the local structure in $z_2$ the first case cannot occur since there would be two lines through $y_2$ in $\Gamma_{z_2}$. But in the second case we get a contradiction considering $\Gamma_{b}$. Hence this case cannot occur. Secondly we look at the case where $x_1y_1b$ is a line. The line $x_2x_2$ is the only line through $x$ belonging to $\Gamma_{x_2}$. This is also the case for the point $x_2$ since if not then $y_2$ should be collinear with $a$, leading to $\Gamma_{x_2} \not\cong \text{LS}(8)$. It follows that $z_2$ is collinear with the “third point” $b$ on the line $x_2y_2$. Considering $\Gamma_{z_2}$ it is easily seen that no line through $x$ belongs to $\Gamma_{z_2}$. Hence $b$ is collinear with a point, say $c$, on the line through $z_2$ different from $L_2$ and $z_2b$. But then $c$ should be equal to the point $x_1$ or $y_1$, which is impossible. So, also this case cannot occur. Next we consider the case where $x_1b$ is a line different from the line $x_1y_1$. Since $x_1y_1$ is the only line through $x$ belonging to $\Gamma_{x_1}$, since $x_2b$ is a line belonging to $\Gamma_{x_1}$ and since $x_1y_1$ and $x_2b$ are non-parallel lines in $\Gamma_{x_1}$, there are two lines through $x_2$ belonging to $\Gamma_{x_1}$. It is easy to see that this is impossible. Hence also this case cannot
occur. Finally we consider the case where \( x_1 a \) is a line different from the line \( x_1 y_1 \). To obtain local structure \( \text{LS}(8) \) in the point \( x_1 \) we need two lines through \( x_2 \) belonging to \( \Gamma^l_{x_1} \) (analogously to the previous case). This is impossible and hence also this case cannot occur.

If \( \Gamma_x = \text{LS}(10) \) and \( \Gamma_{x_2} \cong \text{LS}(10) \), then \( x_2a, x_2b \) and \( ab \) are lines, with \( a \) and \( b \) the “third points” on \( x_1 y_1 \) and \( x_1 y_2 \), respectively. This now contradicts \( \Gamma_{x_1} \cong \text{LS}(10) \).

For \( \text{LS}(11) \) we consider \( \Gamma_{x_2} \). No line through \( x \) belongs to \( \Gamma^l_{x_2} \) and it follows that the two lines through \( x_1 \) different from \( L \) should belong to \( \Gamma^l_{x_2} \). So \( x_2a \) and \( x_2b \) are two different lines with \( a \) the “third point” on the line \( x_1 y_1 \) and \( b \) the “third point” on the line \( x_1 y_2 \). This contradicts \( \Gamma^l_{x_1} \cong \text{LS}(11) \).

If \( \Gamma_x = \text{LS}(12) \), then we consider \( \Gamma_{x_2} \). No line through \( x \) belongs to \( \Gamma^l_{x_2} \) and it follows that the two lines through \( x_1 \) different from \( L \) should belong to \( \Gamma^l_{x_2} \). So \( x_2a \) and \( x_2b \) are two different lines with \( a \) the “third point” on the line \( x_1 y_1 \) and \( b \) the “third point” on the line \( x_1 y_2 \). But then \( \Gamma^l_{x_1} \cong \text{LS}(12) \).

If \( \Gamma_x = \text{LS}(19) \) and \( \Gamma_{x_1} \cong \text{LS}(19) \), then either \( x_2 z_1 a \) and \( x_2 z_2 b \) are lines or \( x_2 z_1 b \) and \( x_2 z_2 a \) are lines, with \( a \) and \( b \) the “third point” on \( x_1 y_1 \) and \( x_1 y_2 \), respectively. To obtain \( \Gamma_{y_2} \cong \text{LS}(19) \) in the first case, we need that \( y_2 ac \) is the “third line” through \( y_2 \) and that \( b \) is collinear with \( c \). But then it is impossible that \( \Gamma_{y_2} \cong \text{LS}(19) \). In the second case it is impossible that \( \Gamma_{y_2} \cong \text{LS}(19) \).

For \( \text{LS}(20) \) we consider \( \Gamma_{x_1} \). It follows that \( y_1 y_2 a, x_1 y_1 b \) and \( x_1 a \) are lines, with \( a \) the “third point” on \( x_2 z_1 \) and \( b \) the “third point” on \( x_2 y_2 \). But now \( \Gamma_{y_2} \not\cong \text{LS}(20) \).

Consider \( \text{LS}(21) \). In \( \Gamma_{x_1} \), the point \( x \) is collinear with only one other point – namely, \( y_1 \) – in \( \Gamma_{y_2}(x_1) \), hence — as can be seen in \( \Gamma_x \), that point must be collinear with two points of \( \Gamma_{y_2}(x_1) \). Hence \( y_1 a \) is a line, with \( a \) a point on the “third line” through \( x_1 \). This implies that \( y_1 y_2 a \) is a line, leading to \( \Gamma_{y_2} \not\cong \text{LS}(21) \).

If \( \Gamma \) is point-locally \( \text{LS}(23) \), then in \( \Gamma_{x_2} \), we see that either \( x_1 y_1 a \) or \( x_1 y_2 a \) is a line, with \( a \) the “third point” on the line \( x_2 z_1 \). In the first case \( x_1 y_2 b \) and \( ab \) are lines, with \( b \) a point on the “third line” through \( x_2 \), leading to \( \Gamma_{x_1} \not\cong \text{LS}(23) \). Similarly, in the second case, \( x_1 y_1 b \) and \( ab \) being lines leads to \( \Gamma_{x_1} \not\cong \text{LS}(23) \).

We rule out \( \text{LS}(29) \) in a completely similar way.

Let, in \( \text{LS}(30) \), \( a, b \) be the points on the “third line” through \( x_1 \) and let \( c \) be the “third point” on \( x_1 y_1 \). Considering \( \Gamma_{x_1} \), we see, similarly to before, that \( y_1 y_2 a, x_2 z_1 a, x_2 y_2 c \) and \( bc \) are lines. Now looking at \( \Gamma_a \) it follows that \( z_1 z_2 b \) is a line. But now \( \Gamma_{z_1} \) cannot be isomorphic to \( \text{LS}(30) \).

**Now assume that \( \Gamma_x \) contains transversals.**

If we consider \( \Gamma_{x_1} \) in the case \( \Gamma_x = \text{LS}(n), n \in \{36, 38, 39, 42, 61, 63, 66, 68, 69, 70, 74, 75, 76\} \), then we see that, from what is already induced by \( \Gamma_x \) in \( \Gamma_{x_1} \), the latter cannot be isomorphic to \( \text{LS}(n) \).

Considering likewise \( \Gamma_{y_2} \) rules out \( \text{LS}(n) \), for \( n \in \{37, 41, 43, 44, 60, 64, 65, 71\} \), and similarly using \( \Gamma_{y_1} \) rules out \( \text{LS}(m) \), with \( m \in \{40, 45, 47, 52, 53, 55, 56, 59, 62, 67\} \). And consideration of \( \Gamma_{y_2} \) rules out \( \text{LS}(k) \), with \( k \in \{46, 48, 54\} \).

If \( \Gamma_x = \text{LS}(49) \), then we consider \( \Gamma_{z_2} \). No line through \( x \) belongs to \( \Gamma^l_{z_2} \) and it follows that the two lines through \( y_2 \) should belong to \( \Gamma^l_{z_2} \) one of which is a transversal. Clearly, this is impossible.

If \( \Gamma_x = \text{LS}(50) \), then we consider \( \Gamma_{x_2} \). The lines \( x z_1 \) and \( x z_2 \) are lines of \( \Gamma^l_{x_2} \). The point \( x \) in \( \Gamma_{x_2} \) plays the same role as the point \( x_2 \) in \( \Gamma_x \). It follows that the two lines through \( x_1 \) should belong to \( \Gamma^l_{x_2} \) one of which is a transversal. This is impossible.

In the case \( \Gamma_x = \text{LS}(72) \) we consider the point \( x_1 \). No line through \( x \) belongs to \( \Gamma^l_{x_1} \). Hence, the two lines through \( x_2 \) should be transversals in \( \Gamma_{x_1} \), which is impossible.
5.3. Analysis of bislim geometries with some specific local structures

5.3.1. Local structures LS(34), LS(35), LS(58), LS(73) and LS(77)

In the next lemma we take a look at those local structures that give rise to a unique geometry. We use the same terminology and notation as in the proof of Lemma 5.4 above.

Lemma 5.5. If \( \Gamma \) is a bislim geometry of gonality 3 which is point-locally one of LS(34), LS(35), LS(58), LS(73) or LS(77), then \( \Gamma \) is uniquely determined.

Proof. We start with LS(34). Let \( a \) be the “third point” on \( x_2z_1 \) and let \( b \) be the “third point” on \( x_2z_2 \). In \( \Gamma_{x_2} \) we deduce that either \( x_1y_1b, x_1y_2a \) and \( ab \), or \( x_1y_1a, x_1y_2b \) and \( ab \) are lines. In the former case we get a contradiction in \( \Gamma_{y_1} \). In the latter case the local structure in \( y_1 \) gives rise to the lines \( abc \) and \( z_1z_2c \), with \( c \) the “third point” on the line \( y_1y_2 \). We obtain the Desargues geometry.

Consider now LS(35). Looking at \( \Gamma_{x_1} \), it is easily seen that \( y_1y_2a, x_2z_1a, x_2y_2b \) and \( z_1z_2b \) are lines, with \( a \) the “third point” on the line \( x_1z_2 \) and \( b \) the “third point” on \( x_1y_1 \). We obtain the Pappus geometry.

If \( \Gamma_{x_1} \cong \text{LS}(58) \), then let \( a \) and \( b \) be the “third points” on the lines \( x_1z_1 \) and \( x_2z_1 \), respectively. The isomorphisms \( \Gamma_{x_1} \cong \Gamma_{x_1} \) and \( \Gamma_{x_2} \cong \Gamma_{x_2} \) imply that \( y_2a, x_2z_2a \), and \( y_1z_2b \) and \( aby_2 \), respectively, are lines. We obtain a unique bislim geometry on 9 points and 9 lines, which is easily seen to be isomorphic to \( M(9,0),(2,1) \).

For \( \Gamma_{x_1} \cong \text{LS}(73) \), let \( a \) be the “third point” on the line \( x_1z_1 \). Considering \( \Gamma_{x_1} \), we see that \( x_2y_2a \) and \( y_1z_2a \) are lines. We get the Möbius–Kantor geometry \( M(8,0),(2,1) \).

The local structure LS(77) is itself the Fano plane.

The lemma is proved. □

For the definition and several constructions of the Desargues geometry, the Pappus geometry and the Möbius–Kantor geometry we refer to [6–8].

This leaves us with geometric point homogeneous bislim geometries with local structure isomorphic to one of LS(1), LS(4), LS(5), LS(13), LS(24) and LS(51).

5.3.2. Local structure LS(1)

Let \( \Gamma \) be point-locally LS(1). Note that being point-locally LS(1) is equivalent with saying that every point of \( \Gamma \) belongs to a unique triangle. Clearly the LS in the line \( L \) is isomorphic to LS(1). The unique triangle containing the point \( x_2 \) cannot have a second vertex on \( L \) and so we conclude that all lines through \( x_2 \) play the same role as \( L \) and therefore have local structure LS(1). If \( \Gamma \) is not geometrically homogeneous, then without loss of generality we may assume that the local structure in \( L_2 \) is not LS(1). But since there are only three points on \( L_2 \), and every point is contained in a unique triangle, the line \( L_2 \) is contained in at most one triangle. If it is contained in exactly one triangle, then it has local structure LS(1), contradicting our hypothesis. Hence it has local structure LS(0). It is clear that the points \( x, y_2 \) and \( z_2 \) are incident with a unique such line, since the two other lines through each of these points are contained in the unique triangle of these respective points.
Now suppose that \( \Gamma \) has the property that, whenever \( \{x_1, x_2, x_3\} \) is a triangle, then for all \( i, j, k \) such that \( \{i, j, k\} = \{1, 2, 3\} \), there exists another triangle \( \{a, b, c\} \) such that \( x_i b c \) and \( a x_j x_k \) (we call this the triangle property). We associate a directed graph \( \mathcal{G}_\Delta \) to \( \Gamma \) as follows. The vertices of the graph \( \mathcal{G}_\Delta \) are the triangles of \( \Gamma \). A vertex \( v = \{p_1, p_2, p_3\} \) is adjacent to a vertex \( w = \{q_1, q_2, q_3\} \) if \( v \neq w \) and if one of the points \( p_1, p_2, p_3 \) is incident with one of the lines \( q_1 q_2, q_2 q_3, q_3 q_1 \). Note that this is symmetric exactly because of the triangle property. Since a triangle has three vertices, we obtain a cubic graph. It is easily seen that this graph cannot have a clique of size 3 – hence the girth is at least 4 – and that the construction in Section 3.1 is opposite to the one given here.

Hence (i) of the Main Result 1 is proved.

We now present an example of a point-locally \( \text{LS}(1) \) geometry which is not geometrically line homogeneous. By the above, our example must contain at least one line in which the \( \text{LS} \) is \( \text{LS}(0) \).

Let \( T \) be a (\( n \) infinite) directed tree with one vertex \( v \) having indgree 3 and outdegree 2, and with all other vertices having indgree 3 and outdegree 3. In each vertex \( x \neq v \) we fix an arbitrary pairing of the incoming edges with the outgoing edges and we call this pairing opposition. In \( v \), we fix an arbitrary pairing of the outgoing edges with two arbitrary incoming edges (and call this again opposition). The unique incoming edge in \( v \) that is not paired this way will be denoted by \( e \). We define the following geometry \( \Gamma = (P, L, I) \). The point set is a copy of the set of directed edges completed with a symbol \( \infty \), and the line set is a second copy of the set of directed edges. Let \( Z \) be a point or line corresponding to the directed edge \( (x, y) \). Then \( Z \) is incident with the lines or points, respectively, corresponding to the two non-opposite incoming edges in \( x \), or corresponding to the (usually two) non-opposite outgoing edges in \( y \), respectively, and with the line or point, respectively, corresponding to the edge \( (x, y) \). If \( Z \) is a line, and there is only one non-opposite outgoing edge in \( y \), then we declare \( Z \) to be incident with \( \infty \) as well. This way, all elements are incident with exactly three others, except for \( \infty \). It is easy to see, and left for the reader to check, that the local structure in all points different from \( \infty \) is isomorphic to \( \text{LS}(1) \).

Now we take two additional copies \( \Gamma'' = (P', L', I') \) and \( \Gamma''' = (P'', L'', I'') \) of \( \Gamma \) and denote the points corresponding to \( \infty \) by \( \infty' \) and \( \infty'' \), respectively. We define a new geometry \( \Delta \) as the disjoint union of \( \Gamma', \Gamma'' \) and \( \Gamma''' \), completed with an additional (new) line \( L_\infty \), which we denote incident with the three points \( \infty, \infty', \) and \( \infty'' \). Then it is clear that \( \Delta \) is bislim and point-locally \( \text{LS}(1) \), while all lines are locally \( \text{LS}(1) \), except for the line \( L_\infty \), in which the \( \text{LS} \) is isomorphic to \( \text{LS}(0) \).

Of course, by considering trees with more vertices having outdegree 2, we can build bislim geometries which are point-locally \( \text{LS}(1) \) and which have an arbitrary number of lines with local structure \( \text{LS}(0) \). But we remark that no finite such geometry \( \Gamma \) can exist. Indeed, on the one hand, the number of points of such a geometry must be equal to the number of lines, but on the other hand, the map that assigns to each point \( x \) the unique line of \( \Gamma_x \) is clearly injective, and hence bijective; but every line in the image is locally \( \text{LS}(1) \).

### 5.3.3. Local structure \( \text{LS}(4) \)

Let \( x \) be a point of the geometry \( \Gamma \) with geometrically point homogeneous local structure \( \text{LS}(4) \). According to previous notation, we let \( x_1 y_1 \) and \( x_2 z_1 \) be the lines of \( \Gamma_x \) in \( \Gamma_3(x) \), and \( y_2, z_2 \) are the other points in \( \Gamma_2(x) \), with \( x y_2 z_2 \) a line of \( \Gamma \). Considering \( \Gamma_{x_1} \), we see that \( x_2 \) is collinear with the “third point” \( u_1 \) on the line \( x_1 y_1 \). But if \( x_2 y_1 \) is distinct from \( x_2 z_1 \), then we cannot have local structure \( \text{LS}(4) \) in \( x_2 \), hence \( x_2 z_1 u_1 \) is a line. The subgeometry defined by \( x, x_1, x_2, y_1, z_1, u_1 \) and the lines \( x x_1 x_2, x y_1 z_1, u_1 x_1 y_1 \) and \( u_1 x_2 z_1 \) is isomorphic to the dual of
K_4. Moreover, the points y_2 and z_2 are not contained in a common triangle of \( \Gamma \) as this would imply, looking in \( \Gamma_{z_2} \), that x and z_2 are collinear with a point distinct from y_2. Hence we see that \( \Gamma_L \) is isomorphic to LS(0), and that every point is contained in a unique such line. Also, \( \Gamma_L \) is easily seen to be isomorphic to LS(10). Removing all lines with local structure LS(0) from \( \Gamma \), we obtain a disjoint union of a family \( \mathcal{F} \) of geometries isomorphic to the dual of \( K_4 \).

Now (ii) of the Main Result 1 is clear.

### 5.3.4. Local structure LS(13)

It is shown in [8] that every bislim geometry with \( \Gamma_x \cong LS(13) \), for all points \( x \) of \( \Gamma \), is covered by the honeycomb geometry \( S_\infty \), in the sense of a \( 1\frac{1}{2} \)-covering. Hence \( \Gamma \) is a quotient geometry of \( S_\infty \) with respect to some automorphism group of \( S_\infty \), which is the group of all deck transformations.

Hence we have to classify all collineation groups \( G \) such that the quotient of \( S_\infty \) with respect to \( G \) is a bislim geometry which is point-locally LS(13).

We introduce some notation.

We may identify the points and lines of \( S_\infty \) with the vertices of the above mentioned honeycomb tiling of the real Euclidean plane \( \mathbb{E} \) in regular hexagons. Let \( e \) be a vertex corresponding to a line of \( S_\infty \), and let \( a, d, f \) be the points incident with \( e \) (and hence the vertices adjacent to \( e \)). Let \( b \) be the unique point of \( S_\infty \) contained in a triangle together with \( a \) and \( f \) (hence \( a, f, b \) are vertices of the same hexagon in the tiling) and let \( c \) be the vertex corresponding with the line \( ab \) of \( S_\infty \). Denote by \( h \) the center of the hexagon containing \( a, b, f \). Let \( W(\hat{A}_2) \) be, as before, the full collineation group of \( S_\infty \). To avoid confusion, we will call the lines of the Euclidean plane \( \mathbb{E} \) Euclidean lines and abbreviate this to \( \mathbb{E}-\text{linese} \).

It is easily seen that each element of \( W(\hat{A}_2) \) is conjugate to one of the following:

- (T) A translation with translation vector \( \overrightarrow{ax} \), for some point \( x \) of \( S_\infty \).
- (Rf) A reflection about the E-line \( ac \), which is then called the (reflection) axis of the reflection.
- (Rt) A rotation of 120° clockwise or counterclockwise about \( a, c \) or \( h \).
- (G) A glide reflection, i.e., the product of a reflection and a translation in the direction of the reflection axis. Here, we have two possibilities. First, the axis is the E-line \( ac \), and the translation vector is in \( 3\mathbb{Z}\overrightarrow{ac} \) (type 1); secondly, the axis contains the mid-points of the intervals \([a, e]\) and \([b, c]\), and the translation vector belongs to \((3\mathbb{Z} + \frac{3}{2})\overrightarrow{ac} \) (type 2).

But if \( G \) contains a reflection or a rotation, then the quotient geometry is not bislim anymore, since at least two elements incident with a common one are identified.

Suppose now that \( G \) contains a glide reflection. Then either all glide reflections in \( G \) have the same axis, or there are two glide reflections with distinct axes. In the first case, either the only translations in \( G \) are parallel to the axis of the glide reflection (Case (i)), or there are translations in different directions; but then the composition of a glide reflection with a translation in another direction produces a glide reflection with a different axis (parallel to the given one) — a contradiction. If we have glide reflections with different axes, then either all these axes are parallel (Case (ii)) or there are two non-parallel axes; but then the composition of the corresponding glide reflections produces a rotation — a contradiction. If \( G \) does not contain glide reflections, then it consists either of parallel translations (Case (iii)), or of translations in more than one direction (Case (iv)).

With the above notation, we choose a basis in \( \mathbb{E} \) as follows: we take the point \( d \) as the origin, the first basis vector is \( \overrightarrow{df} \); the second one is \( \overrightarrow{da} \).

**First suppose Case (i).** Without loss of generality we may assume that the axis of all glide reflections is the E-line either through \((0, 0)\) and \((-1, 2)\) — and then the (smallest) associated
translation vector of the glide reflection is equal to \((-r, 2r)\), for some positive integer \(r\) — or through \((1/4, 0)\) and \((0, 1/2)\) — and then the (smallest) associated translation vector is equal to \((-r - 1/2, 2r + 1)\), for some positive integer \(r\). It is easy to see that in both cases \(r \geq 2\), otherwise we identify points in such a way that we disturb the local structure \(\text{LS}(13)\). Identifying the points and lines of \(\mathcal{S}_\infty\) in the same orbit, the first possibility gives rise to example (HC3a), and the second to (HC3b). Hence we obtain (iiiic) of the Main Result 1.

Consider now Case (ii). If all glide reflections are of type 1, then without loss of generality we may assume that one glide reflection \(h\) has axis the E-line through \((0, 0)\) and \((-1, 2)\) — and then the (smallest) associated translation vector of the glide reflection is equal to \((-r, 2r)\), for some positive integer \(r\). Suppose now that \(h\) is a glide reflection of the same type with axis \(2x + y = s\) parallel to the axis of \(g\) \((2x + y = 0)\), \(s > 0\) and \(s\) minimal. The (smallest) associated translation vector of this glide reflection is then also equal to \((-r, 2r)\), \(r > 0\). Composition of these two glide reflections gives a translation with vector \(\overrightarrow{v} = (s - 2r, 4r)\). By adding \((2r, -4r)\) (which, viewed as a translation, belongs to \(G\)), we have that the translation with vector \((s, 0)\) belongs to \(G\). We obtain now the example \(\mathcal{M}_{(r),(s,0)}^*\) of (HC4a).

If all glide reflections are of type 2, then without loss of generality we may assume that one glide reflection \(g\) has axis the E-line through \((1/4, 0)\) and \((0, 1/2)\) — and then the (smallest) associated translation vector of the glide reflection is equal to \((-r - 1/2, 2r + 1)\), for some positive integer \(r\). Suppose now that \(h\) is a glide reflection of the same type with axis \(2x + y = s + 1/2\) parallel to the axis of \(g\) \((2x + y = 1/2)\), \(s\) positive integer and \(s\) minimal. The (smallest) associated translation vector of this glide reflection is then also equal to \((-r - 1/2, 2r + 1)\), \(r > 0\). Composition of these two glide reflections gives a translation with vector \(\overrightarrow{v} = (s - 2r - 1, 4r + 2)\). By adding \((2r + 1, -4r - 2)\) (which, viewed as a translation, belongs to \(G\)), we have that the translation with vector \((s, 0)\) belongs to \(G\). We obtain the example \(\mathcal{M}_{(r),(s,0)}^{**}\) of (HC4b).

This gives rise to (iiiid) of the Main Result 1.

If there are two glide reflections of different type, then the corresponding minimal vectors have to be equal or opposite, since otherwise we can multiply the one with the biggest vector with the (inverse) square of the other to obtain a glide reflection with shorter translation vector. But the translation vector of a glide reflection of type 1 is conjugate to an even multiple of \((-1/2, 1)\); for type 2 this is an odd multiple of \((-1/2, 1)\), a contradiction.

Consider now Case (iii). It is clear that, up to conjugacy, we can choose the minimal translation in \(G\) to have vector \((r, s)\), with \(0 \leq s \leq r\) and \(r^2 + rs + s^2 \geq 12\). The latter condition is necessary and sufficient for points at graph theoretic distance 6 not to get identified, for otherwise the quotient geometry is not point-locally \(\text{LS}(13)\). It is easy to see that we obtain \(\mathcal{S}_{(r,s)}\), see (HC1) and (iiiia) of the Main Result 1.

Finally, consider Case (iv). Here, \(G\) defines a sublattice of the lattice \(\mathbb{Z}(1, 0) + \mathbb{Z}(0, 1)\). It is easy to see that a basis can be chosen that contains a point \((a, 0)\), with \(a > 0\). The second basis vector \((c, d)\) can always be chosen such that \(d > 0\), and by combining with \((a, 0)\), we may assume that \(0 \leq c < a\). Now \(a, c, d\) satisfy the conditions in (HC2) remarking that the Euclidean distances between the identified vertices \((0, 0)\) and \((ka + \ell c, \ell d)\), \(k, \ell \in \mathbb{Z}\), do exceed \(\sqrt{12}\). Hence we obtain \(\mathcal{M}_{(a,0),(c,d)}\). We have thus found (iiiib) of the Main Result 1.

Clearly, the honeycomb geometry is a \(1\frac{1}{2}\)-cover of every geometry which is point-locally \(\text{LS}(13)\). (iii) of the Main Result 1 is now proved.
5.3.5. Local structure \(LS(24)\)

Let \(\Gamma\) be a geometrically point homogeneous bislim geometry which is point-locally \(LS(24)\), and let \(x\) be a point of \(\Gamma\). With previous notation, the lines of \(\Gamma_x\) are \(x_1y_1\), \(x_2z_1\), \(x_1z_2\) and \(y_1y_2\). The point \(x\) is a vertex of exactly four triangles: \(\triangle xx_1y_1\), \(\triangle xx_2z_1\), \(\triangle xx_1z_2\) and \(\triangle yy_1y_2\). With respect to \(x\), the triangle \(\triangle xx_1y_1\) has the characteristic property that its vertices different from \(x\) are exactly those points in \(\Gamma_x^2(x)\) that are contained in another triangle containing \(x\). If we call \(\triangle xx_1y_1\) therefore special for \(x\), then we claim that \(\triangle xx_1y_1\) is special for all its vertices. Indeed, we show this for \(x_1\), the proof for \(y_1\) being completely similar. By definition, the special triangle for \(x_1\) with vertices in \([x_1] \cup \Gamma_2(x_1)\) is either \(\triangle xx_1y_1\) or \(\triangle xx_1z_2\). Suppose the latter is special for \(x_1\).

Considering \(\Gamma_{x_1}\), it then follows that \(z_2\) is collinear with the “third point” \(z'_1\) on the line \(x_1y_1\) of \(\Gamma\), and that \(x_2\) is collinear with the “third point” \(y'_2\) on \(x_1z_2\). Clearly we have \(z_2x_1 \neq z_2z'_1 \neq z_2x\) and \(x_2y'_2 \neq x_2x\). Also notice that \(y'_2\) is not collinear with \(x\).

Suppose by way of contradiction that \(x_2z_1y'_2\) is a line. Since there are now two lines \(x_2z_1\) and \(x_1y'_2\) meeting the two lines \(x_2z_1\) and \(x_2x_1\), and since there is a unique point \(z_1\) collinear with both \(x\) and \(x_2\) (and not on \(x_2x_1\)), comparing \(\Gamma_{x_2}\) with \(LS(24)\) implies that \(z'_1\) is collinear with \(x_2\) (because clearly \(y_1\) is not collinear with \(x_2\)). That gives an extra line in \(\Gamma_{x_1}\), a contradiction. Hence \(y'_2\) is not incident with \(x_2z_1\).

We now consider two possibilities.

- \(\triangle xx_1y_1\) is special for \(y_1\). Looking in \(\Gamma_{x_1}\), we then see that \(z_1\) and \(z'_1\) are collinear, and that \(x_1\) is collinear with the “third point” on \(y_1y_2\). This implies that \(y_1y_2y'_2\) is a line, giving an extra line in \(\Gamma_{x_1}\), a contradiction.

- \(\triangle xx_1y_2\) is special for \(y_1\). The situation is now symmetric in \(x_1\) and \(y_1\), and so \(z_1\) is collinear with the “third point” \(y_3\) on the line \(y_1y_2\). This implies that \(y_1y_2y'_2\) is a line, giving an extra line in \(\Gamma_{x_1}\), a contradiction.

Now consider \(\Gamma_{x_2}\). This already contains the lines \(x_2z_1\) and \(x_1y'_2\). Since the only points collinear with both \(x\) and \(x_2\) are \(x_1\) and \(z_1\) (considering \(\Gamma_{x_2}\)), \(x\) is not contained in the special triangle for \(x_2\). Hence either \(x_1\) is, or \(z_1\) is. In the first case, \(x_1\) is collinear with the “third point” on \(x_2z_1\), which is then either \(y_1\) or \(z'_1\). Clearly only \(z'_1\) qualifies. But then \(x_2z_1\) must coincide with one of the lines \(z'_1y_2\) or \(z'_1z_2\), a contradiction since this would imply that \(\{y_2, z_2\} \cap \{x_2, z_1\} \neq \emptyset\).

Hence \(z_1\) is collinear with \(y'_2\). Since \(x_2z_1y'_2\) is not a line, this implies that \(z_1y'_2y_3\) is a line. Interchanging the roles of \(x_1\) and \(y_1\), we see that also \(x_2y'_2y_3\) is a line. This is the final contradiction.

Our claim is proved.

So a triangle is special either for all its vertices, or for none of its vertices. Thus it makes sense to talk about special triangles without referring to the vertices. Moreover, we now deduce that the lines \(y_1y_2\) and \(x_1z_2\) meet in \(\Gamma\), say in \(u_2\), and this implies that the triangle \(\triangle u_2y_2z_2\) is special. So every line is an edge of a (unique) special triangle. Moreover, we observe that the “third point” on \(x_1y_1\) is collinear with \(x_2\), and that now \(\Gamma_{x_1} \cong \Gamma_{x_2} \cong LS(24)\) implies that \(\Gamma_L\), with \(L = xx_1\), is isomorphic to \(LS(24)\) as well! Moreover, we also see that \(\triangle xx_1y_1\) is special for \(L\) in the dual setting. All this implies that, if we consider the geometry of points, lines and nonspecial triangles in \(\Gamma\), with natural incidence, then this is a thin rank 3 geometry of type \(A_2\), and we can repeat the arguments of [8] to deduce that this is a quotient of the honeycomb geometry, enriched with the hexagons as a third type of elements (and natural incidence). But in this quotient, the
incidences in the special triangles are given by the incidences in the nonspecial ones, hence the special triangles must also be induced by the equivalence relation corresponding to the group that defines the quotient. This implies that \( \Gamma \) is one of the geometries in (HC1), (HC2), (HC3a), (HC3b), (HC4a) or (HC4b), with the parameters chosen in such a way that the local structure in every point is \( \text{LS}(24) \). In fact, we have to consider all the cases where one identifies points of the honeycomb geometry that are at graph theoretic distance 6 from each other. For (HC1), we obtain \( \text{LS}(3,0) \); for (HC2), we obtain \( \mathcal{M}(3,0),(i,d) \), with \( d \geq 4 \) and \( i \in \{0, 1, 2\} \) (\( d = 3 \) gives different local structure). But by subtracting an appropriate multiple of \( x_1 \), \( x_2 \), \( x_3 \), and, similarly, that \( x_0 \) has a geometric property in \( \Gamma_{x_0} \), with \( x_0 \) meets three lines of \( \Gamma_{x_0} \) that are also contained in \( \Gamma_{x_0} \). We will call \( x_1 \) the successor of \( x_0 \). Remark that, in view of \( \Gamma_{x_2} \cong \text{LS}(51) \), \( x_2 \) should be collinear with the “third point” \( x_{-4} \) on the line \( x_{-1}x_{-3} \) (since the transversal of \( \Gamma_{x_2} \) cannot be incident with \( x_0 \) and \( x_{-3} \) should be collinear with the “third point” on the line \( x_{-2}x_{-4} \). It follows easily that \( x_{-1} \) is the successor of \( x_{-2} \), and similarly \( x_{-3} \) is the successor of \( x_{-4} \). Considering \( \Gamma_{x_{-3}} \), we see that \( x_{-2} \) is the successor of \( x_{-3} \) and, similarly, that \( x_0 \) is the successor of \( x_{-1} \). From the line \( x_0x_{-2}x_{-3} \) we deduce that, whenever a point \( x \) is the successor of the point \( y \), then the “third point” on the line \( xy \) is the second successor of \( x \) (meaning, the successor of the successor). This now implies that \( x_2 \) is the second successor of \( x_0 \), and hence the successor of \( x_1 \). Similarly \( x_3 \) is the successor of \( x_2 \). We now see that we have chosen the indices such that \( x_i \) is the successor of \( x_{i-1} \), for all pairs \( \{i, i - 1\} \) of indices yet introduced.

It follows that the subgeometry of \( \Gamma \) induced by all successors and predecessors (with obvious meaning) of \( x_0 \) is bislim, and hence coincides with \( \Gamma \) itself. Hence we can denote the point set of \( \Gamma \) by \( \{x_i : i \in \mathbb{Z}\} \). There are now two cases to consider. Either \( i = j \) whenever \( x_i = x_j \) (and we denote in this case \( \Gamma \) by \( \Gamma^{(\infty)} \)), or there exist two numbers \( i, j \) such that \( x_i = x_j \). Since successors and predecessors are unique, we then see that \( x_{i+n} = x_{j+n} \), for all \( n \in \mathbb{Z} \), and so we obtain a unique geometry \( \Gamma^{(k)} \) for every given finite cardinality \( k \) of the point set. It is easily seen that \( k \geq 10 \), as otherwise we do not have local structure \( \text{LS}(51) \) in each point. For \( k \in \{7, 8, 9\} \), we obtain \( \text{LS}(77), \text{LS}(73) \) and \( \text{LS}(58) \), respectively.
It is also clear that $\Gamma^{(\infty)}$ is a $\frac{1}{2}$-cover of $\Gamma^{(k)}$, for every $k \in \mathbb{N}, k \geq 10$. Hence it is $\frac{1}{2}$-connected and it is the universal $\frac{1}{2}$-cover of $\Gamma^{(k)}$, for every $k \geq 10$.

This completes the proof of our Main Result 1.

6. Unambiguity for quotients of the honeycomb geometry

In this section we show that the parameters of the geometries in the families (HC1), (HC3a), (HC3b), (HC4a) and (HC4b) are determined by the isomorphism class of the geometries itself (in particular, no geometry in one family can be isomorphic to a geometry in another family). Also, no geometry in these families is isomorphic to a member of the family (HC2). Regarding the latter family, it is not true that the geometry determines the parameters unambiguously, but the following proposition restricts the sets of parameters in such a way that it will become true.

**Proposition 6.1.** The parameters $a, c, d$ of the geometry $\mathcal{M}_{(a,0),(c,d)}$ can be chosen such that either

(A) $d > \gcd(a, c), d > \gcd(a, c + d)$ and the unique number $c + d - ka, k \in \mathbb{Z}$, for which $-a < c + d - ka \leq 0$ satisfies $c + d - ka \leq -c$,

or

(B) $d = \gcd(a, c) = \gcd(a, c + d)$ and then we also have that

- the unique number $c + d - ka, k \in \mathbb{Z}$, for which $-a < c + d - ka \leq 0$ satisfies $c + d - ka \leq -c$,

- there are unique $k, \ell \in \mathbb{Z}$ such that $kc + \ell a - d = 0$ and $a > kd \geq 0$, and for that $k$ holds that $kd \geq c$,

- there are unique $k, \ell \in \mathbb{Z}$ such that $kc + \ell a - d = 0$ and $-a < kd + d \leq 0$, and for that $k$ holds that $kd \leq -c$,

- there are unique $k, \ell \in \mathbb{Z}$ such that $kc + \ell a + kd - d = 0$ and $a < kd \leq 0$, and for that $k$ holds that $kd \leq -c$,

- there are unique $k, \ell \in \mathbb{Z}$ such that $kc + \ell a + kd - d = 0$ and $-a < -kd + d \leq 0$, and for that $k$ holds that $kd \geq c + d$.

**Proof.** In the hexagonal tiling of $\mathbb{E}$, the points $(a, 0)$ and $(c, d)$ generate an integer lattice. The problem is to find a canonical basis for that lattice such that no lattice is described in more than one way. We start by choosing an origin $o$, which is a point of the lattice, and in particular also a point of the hexagonal tiling. We can then consider six half E-lines $[o x]$ through $o$ given by the points $x$ of the tiling at graph theoretical distance 2 from $o$. The idea is that we pick as the first basis vector $\overrightarrow{oa}$ a vector of minimal length on one of these six half E-lines, and we declare that half E-line as the $X$-axis. Let $x_0$ be the vertex of the tiling at distance 2 from $o$ on the $X$-axis. Then we consider the vertex $y_0$ at distance 2 from both $o$ and $x_0$ and not lying in the same hexagon of the tiling as $o$ and $x_0$, and we declare the half E-line $[o y_0]$ to be the $Y$-axis. As second basis vector of the lattice, we then look for a vertex $(c, d)$ of the lattice with $c \geq 0, d > 0$ and such that $c$ is minimal. One must also consider coordinate transformations (fixing the $X$-axis) to come to the choice of $(c, d)$. For instance, on the E-line $Y = d$, there could be a vertex $(c', d)$ such that, after the coordinate transformation $(x, y) \mapsto (-x - y, y)$, the first coordinate $-c' - d$ is smaller than $c$. The parameters of the proposition are now exactly chosen like this, noting that, given a lattice generated by $(a, 0)$ and $(c, d)$, the vectors of minimal length on the half E-lines
The parameter sets \(a\), \(\ell\), and \(c\), \(d\) of the geometries \(\mathcal{M}(a,0),(c,d)\) and \(\mathcal{M}(a',0),(c',d')\), respectively, satisfy the conditions (A) or (B) of Proposition 6.1. Then \((a, c, d) = (a', c', d')\) if and only if \(\mathcal{M}(a,0),(c,d) \cong \mathcal{M}(a',0),(c',d')\).

**Proof.** From the proof of the previous proposition, we know that \(2a\) is the length of a shortest cycle of points of \(\mathcal{M}(a,0),(c,d)\) such that all these points have representatives lying on one E-line \(Z\) in \(\mathbb{E}\). If through some point, there are three such minimal cycles (hence of the same length), then we know we are in case (B); otherwise we are in case (A). Then we can consider cycles \(\gamma\gamma'\) that consist of two paths: one path \(\gamma\) has representatives on \(Z\) and the other path \(\gamma'\) on one other E-line \(Z'\), where the half E-lines determined by the two paths make an angle of \(120^\circ\) and where the line incident with the first two points of \(\gamma'\) is incident with the endpoint of \(\gamma\) and with the ‘next’ point of the minimal cycle on \(Z\) (where the walking-through-sense of the cycle is determined by \(\gamma\)). First we consider those for which the length \(\ell'\) of \(\gamma'\) is minimal; and amongst these the ones for which the length \(\ell\) of \(\gamma\) is minimal. It follows from the definition of \(c\) and \(d\) in the proof of Proposition 6.1 that \(\ell = c\) and \(\ell' = d\). Hence the lemma will be proved if we can define such cycles and paths intrinsically in \(\mathcal{M}(a,0),(c,d)\) without referring to \(\mathbb{E}\). This we can do as follows.

Let \(A_0\) and \(A_2\) be two collinear points of \(\Gamma\). We first define a cycle \(C(A_0, A_2)\) that corresponds to an interval on the E-line \(Z\) containing representatives of \(A_0\) and \(A_2\) at distance \(2\) in the hexagonal tiling of \(\mathbb{E}\). Let \(A_1\) be the line incident with \(A_0\) and \(A_2\). We define inductively the element \(A_i\), for each \(i \geq 3\), as follows: \(A_i\) is the unique element incident with \(A_{i-1}\) different from \(A_{i-2}\) and such that each element incident with \(A_i\) and different from \(A_{i-1}\), does not belong to \(\Gamma_2(A_{i-3})\). It is easy to see that for each \(i, j \geq 1\), \(A_i = A_j\) implies \(A_{i-1} = A_{j-1}\). Since we are dealing with finite geometries, this implies that there is a minimal positive integer \(k\) such that \(A_k \in \{A_0, A_1, \ldots, A_{k-1}\}\) and then \(A_k = A_0\). Hence we obtain a cycle \(C(A_0, A_2)\) of length \(k\). Also, one can check easily that the points of \(C(A_0, A_2)\) have representatives on \(Z\). So if we consider all points \(A_2\) collinear with \(A_0\), then we obtain 6 such cycles and \(a\) is half the minimum of the lengths of these.

It remains to be shown how one can intrinsically define to make a turn of \(120^\circ\) in an arbitrary point of a minimal cycle \(C(A_0, A_2)\). Let \(2k\) be the length of \(C(A_0, A_2)\), and use the notation above. Let \(A_{2p+1}, 0 \leq p \leq k - 1\) be an arbitrary line of \(C(A_0, A_2)\). We choose the point \(B_{2p+2}\) different from \(A_{2p}\) and \(A_{2p+2}\) (there is a unique choice left). The elements \(B_{2p+i}, i \geq 2\) are then the elements of the unique cycle \(C(A_{2p}, B_{2p+2})\). But we choose \(p\) in such a way that the cycle \(C(A_{2p}, B_{2p+2})\) contains the point \(A_0\) (this is always possible: choose \(p = c\) and then \(B_{2p+2} = A_0\) by definition of \((c, d)\)). The distance between \(A_{2p}\) and \(A_0\) along the cycle \(C(A_{2p}, B_{2p+2})\) will be called the width of the cycle \(C'(A_0, A_2; p)\) that we thus obtain (but note that not all \(p\) are a priori possible here). In Case (A) we have two choices for \(A_2\). In Case (B)
there are six choices. Amongst this second type of cycles we choose those of minimal width, and this width is then the double of $d$. Then amongst all possible choices for $p$, we take the minimal one, and this is equal to $c$.

Hence we have reconstructed the parameters $a$, $c$, $d$ from $\mathcal{M}_{(a,0),(c,d)}^*$ in an intrinsic way and the lemma is proved. □

We now come to distinguishing all isomorphism classes of geometries obtained under (iii) of the Main Result 1. With class of a geometry we mean (HC1) up to (HC4b).

**Proposition 6.3.** The parameters of all geometries in (iii) of the Main Result 1 determine the corresponding geometry and its class.

**Proof.** The idea of the proof comes from the previous proof. For two collinear points $A_0$, $A_2$ in any geometry with uniform local structure $LS(13)$ and any positive integer $k$, we can define a path $P_{k,0}(A_0, A_2)$ of length $2k$ as follows. Let $A_1$ be the line incident with $A_0$ and $A_2$. We define inductively the element $A_i$, for each $3 \leq i \leq 2k$, as follows: $A_1$ is the unique element incident with $A_{i-1}$ different from $A_{i-2}$ and such that each element incident with $A_i$ and different from $A_{i-1}$, does not belong to $I_2(A_{i-3})$. Also, for any pair of positive integers $p$, $q$, we may define the path $P_{p,q}(A_0, A_2)$ as follows. Consider the elements of the path $P_{p+1,0}(A_0, A_2)$ as above, and let the points incident with $A_{2p+1}$ be $A_{2p}$, $A_{2p+2}$ and $B_{2p+2}$. Then the path $P_{p,q}(A_0, A_2)$ is the juxtaposition of $P_{p,0}(A_0, A_2)$ and $P_{q,0}(A_{2p}, B_{2p+2})$. For $p = 0$ and $q > 0$, we define $P_{0,q}(A_0, A_2)$ as $P_{q,0}(A_0, B_2)$, where $B_2$ is the unique point on $A_1$ distinct from $A_0$ and $A_2$. Finally $P_{0,0}(A_0, A_2)$ is the path $(A_0)$ of length 0.

In the geometry $S_{(r,s)}$, $0 < s \leq r$, it is clear that, for any given point $A_0$, there exist two points $A_2$ and $A_2'$ such that $P_{r,s}(A_0, A_2)$ and $P_{s,r}(A_0, A_2')$ are cycles of length $2(r + s)$, and this is the minimum we can obtain as length of a cycle of the form $P_{p,q}(A_0, A_2')$. Moreover, $s$ is the smallest among $r$, $s$ (which are determined by the above paths) and so $S_{(r,s)}$ is completely and uniquely determined by $r$, $s$, if $s > 0$. Note that in this case no path of the form $P_{p,0}(A_0, B_2)$ can be a cycle. If $s = 0$, then every cycle $P_{p,q}(A_0, A_2)$ has either $p = 0$ or $q = 0$ and $2r$ is the minimal length of such a cycle. Hence, in all cases, $S_{(r,s)}$ is completely and uniquely determined by $r$, $s$.

We consider $\mathcal{M}_{(r,s),0}^*$, with $r \geq 2$ and $s \geq 4$. Let $I'$ be such a geometry. Consider any cycle $P_{p,0}(A_0, A_2)$. We know that such a cycle $C$ exists for $p = s$, and it corresponds in $\mathcal{E}$ with a path parallel to the translation vector $(s, 0)$. But there might exist smaller cycles; the minimal possible length if $s$ is very large compared to $r$ is $4r$ (and there always exists such a cycle $C'$). In $\mathcal{E}$, this corresponds to a path that crosses the axis of a glide reflection. But $C' = P_{2r,0}(A_0, A_2)$ has the property that the second half of the path $P_{4r,0}(A_0, A_2)$ is never a copy (a “translate”) of $P_{2r,0}(A_0, A_2)$ (in other words, the second element is different from the $(4r + 1)$st element and we say that the cycle $P_{2r,0}(A_0, A_2)$ does not reproduce itself). While, if $C = P_{s,0}(A_0, A_2')$, then $P_{2s,0}(A_0, A_2')$ consists of two identical cycles $C$ (and we say that $P_{s,0}(A_0, A_2')$ reproduces itself). Hence $2s$ is the length of a minimal cycle of the form $P_{p,0}(A, B)$ that reproduces itself, while $4r$ is the length of a minimal cycle of the form $P_{q,0}(A', B')$ that does not reproduce itself.

We can also distinguish between cycles that reproduce themselves and those that do not by noting that the former exist with arbitrary starting point, while the latter cannot have every point as starting point.
A similar argument holds for \( \mathcal{M}^{**}_{(r),(s,0)} \), \( r \geq 2 \) and \( s \geq 4 \). Here, \( 2s \) is again the length of a minimal cycle of the form \( P_{p,0}(A,B) \) that reproduces itself, while \( 4r + 2 \) is the length of a minimal cycle of the form \( P_{q,0}(A',B') \) that does not reproduce itself. Since this length is not a multiple of 4, unlike the previous case, we can easily distinguish the geometry \( \mathcal{M}^{*}_{(r),(s,0)} \) from any geometry \( \mathcal{M}^{**}_{(r'),(s',0)} \).

**Remark.** We can also distinguish these geometries by their number of points, which is always a multiple of \( s \), but for \( \mathcal{M}^{*}_{(r),(s,0)} \) it is an even multiple (namely, \( 2rs \)) and for \( \mathcal{M}^{**}_{(r'),(s',0)} \) it is an odd multiple (namely, \( (2r' + 1)s' \)) of \( s' \).

Note that none of \( \mathcal{M}^{*}_{(r),(s,0)} \) or \( \mathcal{M}^{**}_{(r'),(s',0)} \), for \( s \geq 4 \) and \( s' \geq 4 \), has a transitive group acting on the points. Hence this distinguishes these geometries from the other finite examples in class (HC2).

So we conclude that we can distinguish all finite quotients of the honeycomb with local structure \( \text{LS}(13) \) and their class by their parameters as given in the Main Result 1, or in Proposition 6.1 above.

There remain only the infinite examples \( S^{*}_{(r)} \) and \( S^{**}_{(r)} \), which can be distinguished already from the other infinite examples \( S_{(r,s)} \) by the fact that they are never point transitive.

But similarly to above for \( \mathcal{M}^{*}_{(r),(s,0)} \) and \( \mathcal{M}^{**}_{(r'),(s',0)} \), one sees that for \( S^{*}_{(r)} \), the number \( 4r \) is the length of a minimal cycle of the form \( P_{p,0}(A,B) \) (and no such cycle reproduces itself) and in \( S^{**}_{(r)} \), the number \( 4r + 2 \) is characterized in this way. This also distinguishes \( S^{*}_{(r)} \) from \( S^{**}_{(r')} \), for all \( r \) and \( r' \).

This completes the proof of the proposition.  

**Appendix. A list of local structures**

![List of local structures](image_url)
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References