Controllability of semilinear boundary problems with nonlocal initial conditions

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Abstract

The main purpose of this paper is the existence of solutions and controllability for semilinear boundary problems with nonlocal initial conditions. We show that the solutions are given by a variation of constants formula which allows us to study the exact controllability for this kind of problems with control and nonlinear terms at the boundary. The included application to a size structured population equation provides a motivation for abstract results.

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1. Introduction

In many cases, problems under consideration, principally arising from physics phenomena, suggest that the initial condition is an estimation through the solution of the problem
in some finite sequence of times, and then we say that the initial condition is nonlocal. Evolution problems with nonlocal initial conditions in Banach spaces are now well understood since their introduction for the first time by Byszewski [10,11], where the author has considered the following Cauchy problem with nonlocal initial conditions:

\[
\begin{align*}
\frac{d}{dt} x(t) &= A(t)x(t) + f(t, x(t)), & 0 \leq t \leq T, \\
x(0) + g(t_1, t_2, \ldots, t_p, x) &= x_0,
\end{align*}
\]

where \(0 \leq t_1 \leq t_2 \leq \cdots \leq t_p \leq T\), \(A\) is the generator of a \(C_0\)-semigroup, \(f : [0, T] \times X \to X\) and \(g : [0, T]^p \times C([0, T], X) \to X\). Subsequently, using different fixed point theorems, several authors have investigated the problem of nonlocal initial conditions for different classes of abstract differential equations in Banach spaces, among others, we refer to [1–6,18,20] and the references therein.

In this paper, we consider the following semilinear nonautonomous boundary problem with nonlocal initial conditions:

\[
\begin{align*}
\left(\text{BP}\right)_{f,g} \quad \begin{cases}
\frac{d}{dt} x(t) &= A_{\max}(t)x(t), & s \leq t \leq T, \\
L(t)x(t) &= f(t, x(t)), & s \leq t \leq T, \\
x(s) + g(t_1, t_2, \ldots, t_p, x) &= x_0,
\end{cases}
\end{align*}
\]

where \(0 \leq s \leq t_1 \leq \cdots \leq t_p \leq T\), \(x_0\) is in a Banach space \(X\), \(A_{\max}(t), L(t)\), for \(t \in [0, T]\), are defined on some subspace \(D\) in \(X\), \(f : [s, T] \times X \to \partial X\) and \(g : [s, T]^p \times C([s, T], X) \to X\). Recently in [7,8], the authors established the well-posedness and asymptotic behavior of this boundary problem but with local initial conditions. Using the same arguments and under appropriate assumptions (see Section 2), we prove the existence of the mild solutions of the boundary problem \((\text{BP})_{f,g}\), and that they are given by a variation of constants formula.

In Section 3, we present sufficient conditions for the exact controllability of the semilinear nonautonomous boundary problem with nonlocal initial conditions and boundary control,

\[
\begin{align*}
\left(\text{CBP}\right) \quad \begin{cases}
\frac{d}{dt} x(t) &= A_{\max}(t)x(t), & 0 \leq t \leq T, \\
L(t)x(t) &= f(t, x(t)) + B(t)u(t), & 0 \leq t \leq T, \\
x(0) + g(t_1, t_2, \ldots, t_p, x) &= x_0,
\end{cases}
\end{align*}
\]

where \(B(t), t \in [0, T]\), are the control operators which are bounded from a Banach space \(U\) to \(\partial X\).

The controllability of semilinear evolution equations has attracted in last years a big interest, see, for instance, [2–4,17] for the local initial conditions case and [6,13,21] for the nonlocal initial conditions case. Recently in [9], the authors have considered the controllability of the above problem \((\text{CBP})\) with local initial conditions.

Section 4 is devoted to an application of our abstract results to the size structured population equation of some kind of plants

\[
\begin{align*}
p_t(t, x) + (\partial_t(x)p(t, x)) & = -\mu(t, x)p(t, x), & x \in \mathbb{R}_+, & t \in [0, T], \\
\partial_t(0)p(0, 0) & = \int_0^\infty \beta(t, x)p(t, x)dx \\
& + h(t, p(t), \cdot) + b(t)u(t), & t \in [0, T], \\
p(0, x) + \sum_{i=1}^p c_i p(t_i, x) & = p_0(x), & x \geq 0.
\end{align*}
\]
where \((\cdot)_r\) is the derivation with respect to \(r\). The function \(p(t, x)\) stands for density of a population of size \(x\) at time \(t\). The function \(\vartheta\) is the growth rate depending on the size \(x\) and time \(t\) and the mappings \(\mu\) and \(\beta\) correspond to the aging and birth functions. Finally, the term of control represents the inflow of zero-size individuals from an external resource. This model, with local initial conditions, has been studied in [14, 15], where the authors proved the existence and uniqueness of the solution. Here we expect to study the exact controllability of (1.1).

2. Existence of mild solutions of semilinear boundary problems with nonlocal initial conditions

In this section, we consider the following semilinear nonautonomous boundary problem with nonlocal initial conditions:

\[
(BP)_{f,g} \begin{cases} 
\frac{d}{dt}x(t) = A_{\text{max}}(t)x(t), & s \leq t \leq T, \\
L(t)x(t) = f(t, x(t)), & s \leq t \leq T, \\
x(s) + g(t_1, t_2, \ldots, t_p, x) = x_0,
\end{cases}
\]

where \(0 \leq s \leq t_1 \leq t_2 \leq \cdots \leq t_p \leq T, x_0\) is in a Banach space \(X\), the operators \(A_{\text{max}}(t) \in \mathcal{L}(D, X), L(t) \in \mathcal{L}(D, \partial X), t \in [0, T]\), with \(D, \partial X\) are Banach spaces such that \(D\) is dense and continuously embedded in \(X\), and \(f: [s, T] \times X \to \partial X\) and \(g: [s, T]^p \times C([s, T], X) \to X\).

We assume the following hypotheses:

(H1) there are positive constants \(C_1, C_2\) such that \(C_1 \|x\|_D \leq \|x\| + \|A_{\text{max}}(t)x\| \leq C_2 \|x\|_D\) for all \(x \in D\) and \(t \geq 0\);

(H2) \([0, T] \ni t \mapsto A_{\text{max}}(t)x\) is continuously differentiable for all \(x \in D\);

(H3) the family \(A(t) := A_{\text{max}}(t)|_{\ker L(t)}, t \in [0, T]\), is stable, i.e., there exist \(M \geq 1\) and \(\omega \in \mathbb{R}\) such that \((\omega, \infty) \subset \rho(A(t))\) (the resolvent set of \(A(t)\)) for all \(t \in [0, T]\), and

\[
\left\| \prod_{i=1}^{k} R(\lambda, A(t_i)) \right\| \leq M(\lambda - \omega)^{-k}
\]

for all \(\lambda > \omega\) and for any finite sequence \(0 \leq t_1 \leq \cdots \leq t_k \leq T\);

(H4) for each \(t \in [0, T]\), the operator \(L(t): D \to \partial X\) is surjective;

(H5) \([0, T] \ni t \mapsto L(t)x\) is continuously differentiable for all \(x \in D\);

(H6) there exists a constant \(\gamma > 0\) such that

\[
\|L(t)x\|_{\partial X} \geq \frac{\lambda}{\gamma} \|x\|_X
\]

for all \(x \in \ker(\lambda - A_{\text{max}}(t)), \lambda > \omega\) and \(t \in [0, T]\).

The following assertions are consequence of these hypotheses. For the proof, we can see [12, Lemma 1.2]. For each \(t \in [0, T]\), we have

(i) \(D = D(A(t)) \oplus \ker(\lambda - A_{\text{max}}(t))\).
(ii) $L(t)|_{\ker(\lambda - A_{\max}(t))}$ is an isomorphism from $\ker(\lambda - A_{\max}(t))$ onto $\partial X$ and its inverse

$$L_{\lambda,t} := \left( L(t)|_{\ker(\lambda - A_{\max}(t))} \right)^{-1} : \partial X \to \ker(\lambda - A_{\max}(t))$$

satisfies the estimate

$$\|\lambda L_{\lambda,t}\| \leq \gamma.$$  \hfill (2.1)

Under hypotheses (H1)–(H6), it has been shown also (cf. [16,19]) that there is an evolution family $(V(t,s))(s,t) \in \Delta_T$, $\Delta_T := \{ (a,b): 0 \leq a \leq b \leq T \}$, generated by the family of operators $A(t) := A_{\max}(t)$ with $D(A(t)) := \{ x \in D: L(t)x = 0 \}, 0 \leq t \leq T$. That means, $V(t,s)x \in D(A(t))$ and $\frac{d}{dt} V(t,s)x = A(t)V(t,s)x$ for all $x \in D(A(s))$ and $(s,t) \in \Delta_T$, which is equivalent to the well-posedness of the nonautonomous linear system $(BP)_{0,0}$. We have to note that, for all $(s,t) \in \Delta_T$, we have the following estimate:

$$\|V(t,s)\| \leq M e^{\omega(t-s)},$$  \hfill (2.2)

where the stability constants $M$ and $\omega$ are given in (H3) (see, e.g., [22, Chapter 5, Theorem 3.1]).

Recently in [7], the authors have considered inhomogeneous nonautonomous boundary evolution problems with local initial conditions, i.e., $f(t,x) = f(t)$ and $g \equiv 0$. They have showed the existence of mild solutions which are given by the variation of constants formula

$$x(t) = V(t,s)x_0 + \lim_{\lambda \to +\infty} t \int_s^t V(t,\sigma)\lambda L_{\lambda,\sigma} f(\sigma)d\sigma$$  \hfill (2.3)

for every $t \in [s,T]$ and $f \in L^1((0,T),\partial X)$.

Now, to deal with the semilinear boundary system with nonlocal initial conditions $(BP)_{f,g}$, we adopt the following definition.

**Definition 1.** Let $f \in L^1([0,T] \times X, \partial X)$. A function $x \in C([s,T], X)$ is said to be a mild solution of the problem $(BP)_{f,g}$ if $f(\cdot, x(\cdot)) \in L^1((0,T), \partial X)$ and it satisfies the integral equation

$$x(t) = V(t,s)(x_0 - g(t_1, t_2, \ldots, t_p, x)) + \lim_{\lambda \to +\infty} \int_s^t V(t,\sigma)\lambda L_{\lambda,\sigma} f(\sigma, x(\sigma))d\sigma$$  \hfill (2.4)

for every $t \in [s,T]$.

To obtain the existence of mild solutions of $(BP)_{f,g}$, we should assume the following conditions:

(C1) For some constant $\alpha > 0$,

$$\|f(t, x) - f(t, y)\|_{\partial X} \leq \alpha \|x - y\| \quad \text{for a.e. } t \in [s,T] \text{ and } x, y \in X.$$
(C2) There exists a function $\beta$ from $C([s, T], \mathbb{R}_+)$ to $\mathbb{R}_+$, with $\beta(ah) \leq a\beta(h)$ for all $a > 0$, $h \in C([s, T], \mathbb{R}_+)$, such that
\[ \| g(t_1, t_2, \ldots, t_p, \varphi) - g(t_1, t_2, \ldots, t_p, \psi) \| \leq \beta(\| \varphi(\cdot) - \psi(\cdot) \|) \]
for all $\varphi, \psi \in C([s, T], X)$.

(C3) $M\beta(e^{(\omega + M\alpha\gamma)(t-s)}) < 1$.

Note that, if the function $g$ is uniformly lipschitzien with respect to $\varphi$ and the Lipschitz constant $l$ is such that $0 < l < \{M \sup_{s \leq \tau \leq T} e^{(\omega + M\alpha\gamma)(\tau - s)}\}^{-1}$ then the conditions (C2)–(C3) are satisfied with $\beta(\zeta) := l \sup_{s \leq \tau \leq T} \zeta(\tau)$, $\zeta \in C([s, T], \mathbb{R}_+)$.  

**Theorem 2.** Under the assumptions (H1)–(H6) and (C1)–(C3), the boundary problem $(BP)_{f,g}$ has a unique mild solution $x \in C([s, T], X)$.

**Proof.** For every $y \in X$, consider the mapping $\Gamma_y$ defined by
\[ (\Gamma_y u)(t) := V(t, s)y + \lim_{\lambda \to +\infty} \int_s^t V(t, \sigma)\lambda L_{\lambda, \sigma} f(\sigma, u(\sigma)) d\sigma, \]
where $s \leq t \leq T$ and $u \in C([s, T], X)$. According to [7], $\Gamma_y$ map from $C([s, T], X)$ to $C([s, T], X)$. In (H3) we can take $\omega > 0$ and estimate
\[ \| (\Gamma^n_y u - \Gamma^n_y v)(t) \| \leq e^{\omega T} \frac{(M\gamma A T)^n}{n!} \| u - v \| \]
for $n \in \mathbb{N}$, $u, v \in C([s, T], X)$ and $s \leq t \leq T$. Hence, for $n$ sufficiently large, $\Gamma^n_y$ becomes a contraction on $C([s, T], X)$. Thus, by the fixed point theorem, there exists a unique function $w_y \in C([s, T], X)$ satisfying $\Gamma_y w_y = w_y$. Also, it follows that
\[ w_y(t) = V(t, s)y + \lim_{\lambda \to +\infty} \int_s^t V(t, \sigma)\lambda L_{\lambda, \sigma} f(\sigma, w_y(\sigma)) d\sigma, \quad s \leq t \leq T. \]

Now, for $y_1, y_2 \in X$, we have
\[ \| w_{y_1}(t) - w_{y_2}(t) \| \leq M e^{\omega(t-s)}\| y_1 - y_2 \| + M\alpha\gamma \int_s^t e^{\omega(t-\sigma)} \| w_{y_1}(\sigma) - w_{y_2}(\sigma) \| d\sigma \]
and the Gronwall’s inequality leads to
\[ \| w_{y_1}(t) - w_{y_2}(t) \| \leq M e^{(\omega + M\alpha\gamma)(t-s)}\| y_1 - y_2 \|, \quad s \leq t \leq T. \] \hfill (2.5)

Let us now define $\Phi$ from $X$ to $X$ by $\Phi x := x_0 - g(t_1, t_2, \ldots, t_p, w_x)$. Then, we have for all $y_1, y_2 \in X$,
\[ \| \Phi y_1 - \Phi y_2 \| = \| g(t_1, t_2, \ldots, t_p, w_{y_1}) - g(t_1, t_2, \ldots, t_p, w_{y_2}) \| \leq \beta(\| w_{y_1}(\cdot) - w_{y_2}(\cdot) \|) \leq M\beta(e^{(\omega + M\alpha\gamma)(t-s)})\| y_1 - y_2 \|. \]
Hence, in view of assumption (C3), the operator $\Phi$ has a unique fixed point $x \in X$. The function $w_x$ associated to this point $x$ is the mild solution of our boundary problem $(BP)_{f,g}$ which we are looking for. □

We end up this section by a much considered special case of functions $g$ given by

$$g(t_1, t_2, \ldots, t_p, \varphi) := \sum_{i=1}^p c_i \varphi(t_i) \quad \text{for} \quad \varphi \in C([s, T], X),$$

(2.6)

where $c_1, \ldots, c_p \in \mathbb{R}$. If we set

$$G := I + \sum_{i=1}^p c_i V(t_i, s),$$

we obtain the following result.

**Theorem 3.** Assume that the hypotheses (H1)–(H6) and (C1) are satisfied. If $G^{-1}$ exists and

$$\|G^{-1}\| M \sum_{i=1}^p |c_i| e^{\omega(t_i-s)} (e^{M\alpha\gamma(t_i-s)} - 1) < 1,$$

(2.7)

then the system $(BP)_{f,g}$ has a unique mild solution.

**Proof.** Let $y \in X$. In the proof of Theorem 2 we have proved that there exists a unique function $w_y \in C([s, T], X)$ which satisfies (2.5) and

$$w_y(t_i) = V(t_i, s)y + \lim_{\lambda \to +\infty} \int_s^{t_i} V(t_i, \sigma) \lambda L_{\lambda, \sigma} f(\sigma, w_y(\sigma)) d\sigma \quad \text{for} \quad i \in \{1, \ldots, p\}.$$

We also introduce the operator $\Phi : X \to X$ by

$$\Phi y := G^{-1}x_0 - G^{-1} \sum_{i=1}^p c_i \lim_{\lambda \to +\infty} \int_s^{t_i} V(t_i, \sigma) \lambda L_{\lambda, \sigma} f(\sigma, w_y(\sigma)) d\sigma.$$

For $y_1, y_2 \in X$, the inequality (2.5) yields

$$\|\Phi y_1 - \Phi y_2\| \leq \|G^{-1}\| \sum_{i=1}^p |c_i| \int_s^{t_i} M e^{\omega(t_i-\sigma)} \gamma \alpha \|w_{y_1}(\sigma) - w_{y_2}(\sigma)\| d\sigma$$

$$\leq \|G^{-1}\| M \sum_{i=1}^p |c_i| e^{\omega(t_i-s)} (e^{M\alpha\gamma(t_i-s)} - 1) \|y_1 - y_2\|.$$

From (2.7), it follows that $\Phi$ has a unique fixed point $x$, which satisfies
\[ x = x_0 - \sum_{i=1}^{p} c_i V(t_i, s)x - \sum_{i=1}^{p} c_i \lim_{\lambda \to +\infty} \int_{s}^{t_i} V(t_i, \sigma)\lambda L_{\lambda, \sigma} f(\sigma, w_x(\sigma)) \, d\sigma \]

\[ = x_0 - \sum_{i=1}^{p} c_i w_x(t_i). \]

Consequently, \( w_x \) is the unique mild solution of the system \((BP)_{f,g}\).

**Remark 4.** In the setting of (2.6), if we assume moreover that the stability constant (given in (H3)) \( \omega < 0 \), i.e., \((V(t, r))_{(r,t) \in \Delta_T}\) is exponentially stable, then one can verify that the condition

\[ \sum_{i=1}^{p} |c_i| < \frac{e^{-\omega(t_1-s)}}{Me^{M\alpha\gamma(t_p-s)}} \]  

implies both invertibility of \( G \) and (2.7). In general case, one can take \( \omega \geq 0 \) and assume

\[ \sum_{i=1}^{p} |c_i| < \frac{1}{Me^{(M\alpha\gamma+\omega)(t_p-s)}} \]  

instead of (2.8).

**3. Boundary control problems with nonlocal initial conditions**

Consider the following controlled semilinear nonautonomous boundary system with nonlocal initial conditions:

\[
\begin{aligned}
\frac{dx(t)}{dt} &= A_{\text{max}}(t)x(t), \quad 0 \leq t \leq T, \\
L(t)x(t) &= f(t, x(t)) + B(t)u(t), \quad 0 \leq t \leq T, \\
x(0) + g(t_1, t_2, \ldots, t_p, x) &= x_0,
\end{aligned}
\]

where the control operators are such that \( B(\cdot) \in L^2(0, T; \mathcal{L}(U, \partial X)) \). Under the same hypotheses (H1)–(H6) and (C1)–(C3), the mild solution of \((CBP)\) exists and satisfies the variation of constants formula (cf. Theorem 2)

\[ x(t) = V(t, 0)(x_0 - g(t_1, \ldots, t_p, x)) + \lim_{\lambda \to +\infty} \int_{0}^{t} V(t, \sigma)\lambda L_{\lambda, \sigma} \left[ f(\sigma, x(\sigma)) + B(\sigma)u(\sigma) \right] d\sigma \]

for every \( t \in [0, T] \).

**Definition 5.** The system \((CBP)\) is said to be exactly controllable on \([0, T]\), for some \( T > 0 \), if for every \( x_0, v \in X \) there is a control \( u \in L^2(0, T; U) \) such that the solution \( x(\cdot) \) of \((CBP)\) satisfies \( x(T) = v \).
For the controllability of the boundary system (CBP) we shall need the following hypotheses:

(C4) The operator $\Gamma: L^2([0, T], U) \to X$ defined by

$$\Gamma u := \lim_{\lambda \to +\infty} \int_0^T V(T, \sigma) \lambda L_{\lambda, \sigma} B(\sigma) u(\sigma) \, d\sigma$$

(3.2)

has an induced inverse bounded operator $\Gamma^{-1}$ which takes values in $L^2([0, T], U)/\ker \Gamma$.

(C5) There exists constants $K_f, K_g > 0$ such that

\begin{align*}
(a) & \quad \|f(t, x) - f(t, y)\|_{\partial X} \leq K_f \|x - y\|, \quad t \in [0, T], \ x, y \in X, \\
(b) & \quad \|g(t_1, \ldots, t_p, \phi) - g(t_1, \ldots, t_p, \psi)\| \leq K_g \|\phi - \psi\|, \quad \phi, \psi \in C([0, T], X).
\end{align*}

(C6) We assume that

$$M \left(1 + \gamma M \|B(\cdot)\|_{L^2} \|\Gamma^{-1}\| (K_g + T\gamma K_f)\right) < 1,$$

where $M := \sup_{(s,t) \in \Delta_T} \|V(t, s)\|$.

**Theorem 6.** If the hypotheses (H1)–(H6) and (C4)–(C6) are satisfied then the system (CBP) is exactly controllable on $[0, T]$.

**Proof.** For any function $x \in C([0, T], X)$, we define the control function

$$u^*(t) := \Gamma^{-1} \left( v - V(T, 0)(x_0 - g(t_1, \ldots, t_p, x)) \right)$$

\[ - \lim_{\lambda \to +\infty} \int_0^T V(T, \sigma) \lambda L_{\lambda, \sigma} f(\sigma, x(\sigma)) \, d\sigma \right) (t) \]

for $t \in [0, T]$. Using this control, we have to show that the operator $\Phi: C([0, T], X) \to C([0, T], X)$ given by

$$\Phi(x)(t) := V(t, 0)(x_0 - g(t_1, \ldots, t_p, x))$$

\[ + \lim_{\lambda \to +\infty} \int_0^t V(t, \sigma) \lambda L_{\lambda, \sigma} \left[ f(\sigma, x(\sigma)) + B(\sigma)u^*(\sigma) \right] \, d\sigma \]

has a fixed point which is the solution of the system (CBP) satisfying $x(T) = v$.

Let $x_1, x_2 \in C([0, T], X)$ and let $u_i^*$ be the control associated to $x_i$, $i = 1, 2$. Then, one can write

$$\Phi(x_1)(t) - \Phi(x_2)(t) = V(t, 0)\left(g(t_1, \ldots, t_p, x_1) - g(t_1, \ldots, t_p, x_2)\right)$$

\[ + \lim_{\lambda \to +\infty} \int_0^t V(t, \sigma) \lambda L_{\lambda, \sigma} \left[ (f(\sigma, x_1(\sigma)) - f(\sigma, x_1(\sigma))) \right] \]


By using (C4)–(C5) together with Hölder’s inequality it follows
\[
\| \Phi(x_1)(t) - \Phi(x_2)(t) \| \\
\leq MK_g \| x_1 - x_2 \|_{\infty} \\
+ \lim_{\lambda \to +\infty} \left\| \int_0^t V(t,\sigma)\lambda L_{\lambda,\sigma} [f(\sigma, x_1(\sigma)) - f(\sigma, x_2(\sigma))] d\sigma \right\| \\
+ \lim_{\lambda \to +\infty} \left\| \int_0^t V(t,\sigma)\lambda L_{\lambda,\sigma} B(\sigma) \Gamma^{-1} \times \right. \\
\left. \left[ V(T,0) \left( g(t_1, \ldots, t_p, x_1) - g(t_1, \ldots, t_p, x_2) \right) \right. \\
+ \lim_{\lambda \to +\infty} \int_0^T V(T,\tau)\lambda L_{\lambda,\tau} [f(\tau, x_1(\tau)) - f(\tau, x_2(\tau))] d\tau \right] (\sigma) d\sigma \right\| \\
\leq MK_g \| x_1 - x_2 \|_{\infty} + \gamma M \int_0^t \| f(\sigma, x_1(\sigma)) - f(\sigma, x_2(\sigma)) \| d\sigma \\
+ \gamma \left[ \int_0^T \| V(t,\sigma) \|^2 \| B(\sigma) \|^2 d\sigma \right]^{1/2} \| \Gamma^{-1} \| \\
\times \left[ M \int_0^T \| f(\tau, x_1(\tau)) - f(\tau, x_2(\tau)) \| d\tau + MK_g \| x_1 - x_2 \|_{\infty} \right].
\]

That means
\[
\| \Phi x_1 - \Phi x_2 \|_{\infty} \leq MK_g \| x_1 - x_2 \|_{\infty} + M \gamma K_f \int_0^T \| x_1(s) - x_2(s) \| ds \\
+ M^2 \gamma \| B(\cdot) \|_{L^2} \| \Gamma^{-1} \| \\
\times \left( \gamma K_f \int_0^T \| x_1(s) - x_2(s) \| ds + K_g \| x_1 - x_2 \|_{\infty} \right) \\
\leq M \left( 1 + \gamma M \| B(\cdot) \|_{L^2} \| \Gamma^{-1} \| \right) (K_g + T \gamma K_f) \| x_1 - x_2 \|_{\infty}.
\]

Thus, the assumption (C6) and the Banach fixed point theorem allow to obtain the result. □
4. Exact controllability of a size structured population equation

To illustrate our abstract results, we consider the following controlled size structured population equation:

\[
\begin{aligned}
&\frac{p_t(t,x)}{} + \left(\vartheta(t,x)p(t,x) \right)_x = -\mu(t,x)p(t,x), \quad x \geq 0, \quad t \in [0,T], \\
&\vartheta(t,0)p(t,0) = \int_0^\infty \beta(t,x)p(t,x) \, dx + h(t,p(t,\cdot)) + b(t)u(t), \quad t \in [0,T], \\
&p(0,x) + \sum_{i=1}^p c_i p(t_i,x) = p_0(x), \quad x \geq 0,
\end{aligned}
\]

(4.1)

where the coefficients are such that:

(i) \(0 < \vartheta \leq \mu(t,x)\) and \(0 \leq \mu(t,x) + \frac{\partial}{\partial x} \vartheta(t,x)\) for a.e. \(x \geq 0\) and \(t \in [0,T]\),

(ii) \(0 < \varphi \leq \vartheta(t,x)\) for a.e. \(x \geq 0\) and \(t \in [0,T]\),

(iii) \(\mu \in C^1([0,T], L^\infty(0,\infty))\),

(iv) \(\vartheta \in C^1([0,T], W^{1,\infty}(0,\infty))\),

(v) \(\beta \in L^\infty([0,T] \times \mathbb{R}_+)\) and \(h: [0,T] \times L^1(\mathbb{R}_+) \to \mathbb{R}\) satisfies (C5a),

(vi) \(p_0 \in L^1(0,\infty)\) and \(b(\cdot) \in L^2(0,T)\).

The population equation (4.1) can be reformulated as a boundary problem with nonlocal initial conditions. In fact, we consider the spaces \(X := L^1(0,\infty)\), \(U = \partial X := \mathbb{R}\), \(D := W^{1,1}(0,\infty)\), the operators

\[
A_{\max}(t)\varphi := -\varrho(t,\cdot)\frac{\partial}{\partial x}\varphi - \mu(t,\cdot)\varphi - \frac{\partial}{\partial x}\vartheta(t,\cdot)\varphi
\]

and

\[
L(t)\varphi := \vartheta(t,0)\varphi(0)
\]

for \(\varphi \in L^1(0,\infty)\) and \(0 \leq t \leq T\).

We show first that the hypotheses (H1)–(H6) are satisfied.

To show (H1), set \(\rho(t) := \mu(t,\cdot) + \frac{\partial}{\partial x} \vartheta(t,\cdot), t \in [0,T]\), and let \(\varphi \in W^{1,1}(0,\infty)\). Then we have

\[
\|\varphi\|_D = \int_0^\infty |\varphi(a)| \, da + \int_0^\infty |\varphi'(a)| \, da
\]

\[
\leq \int_0^\infty |\varphi(a)| \, da + \int_0^\infty |\varphi'(a)| + \rho(t,a)\varphi(a) \, da + \int_0^\infty \rho(t,a)\varphi(a) \, da
\]

\[
\leq \max\left(1, \sup_{t \in [0,T]} \frac{\|\rho(t,\cdot)\|_\infty}{\varphi(t,a)}\right)\|\varphi\|_X + \frac{1}{\varphi} \|A_{\max}(t)\varphi\|_X
\]

\[
\leq \max\left(1, \sup_{t \in [0,T]} \frac{\|\rho(t,\cdot)\|_\infty}{\varphi(t,a)}\right)\left(\|\varphi\|_X + \|A_{\max}(t)\varphi\|_X\right).
\]

On the other hand,
\[\|\varphi\|_X + \| A_{\text{max}}(t)\varphi\|_X = \int_0^\infty |\varphi(a)| \, da + \int_0^\infty |\vartheta(t, a) \varphi'(a) + \rho(t, a) \varphi(a)| \, da\]
\[\leq \max \left(1 + \sup_{t \in [0, T]} \|\rho(t, \cdot)\|_\infty, \sup_{t \in [0, T]} \|\vartheta(t, \cdot)\|_\infty\right) \|\varphi\|_D.\]

Hypothesis (H2) follows from the conditions (iii)–(iv).

For (H3), one can show that the resolvent operator associated to
\[A(t) := A_{\text{max}}(t) \mid_{\ker L(t)}\]
is given by
\[R(\lambda, A(t))\varphi = \frac{1}{\vartheta(t, \cdot)} \int_0^\infty e^{-\int_0^\tau \frac{\lambda + \rho(t, \sigma)}{\vartheta(t, \sigma)} \, d\sigma} \varphi(\tau) \, d\tau, \quad \varphi \in X, \ t \in [0, T].\]

Now, let \(\lambda > 0\). By using Fubini’s theorem and (i), we obtain
\[\| R(\lambda, A(t))\varphi \| = \int_0^\infty \left| (R(\lambda, A(t))\varphi)(\xi) \right| \, d\xi\]
\[= \int_0^\infty \frac{1}{\vartheta(t, \xi)} \left| \int_0^\xi e^{-\int_0^\tau \frac{\lambda + \rho(t, \sigma)}{\vartheta(t, \sigma)} \, d\sigma} \varphi(\tau) \, d\tau \right| \, d\xi\]
\[\leq \int_0^\infty \frac{1}{\vartheta(t, \xi)} \int_0^\xi e^{-\int_0^\tau \frac{\lambda + \mu(t, \sigma)}{\vartheta(t, \sigma)} \, d\sigma} \varphi(\tau) \, d\tau \leq \frac{1}{\lambda + \mu} \|\varphi\|\]
for all \(t \in [0, T]\). Since the domain of \(A(t)\) is dense in \(X\), we conclude that \(A(t)\) generates a contraction \(C_0\)-semigroup on \(X\) for every \(t \in [0, T]\). This implies the stability of the family \((A(t), D(A(t)))_{t \in [0, T]}\), with stability constants \(M = 1\) and \(\omega = -\mu\).

The operator \(L(t)\) is bounded. In fact, we have \(\varphi(0) = -\int_0^\infty \frac{\partial}{\partial a} \varphi(a) \, da\) for all \(\varphi \in W^{1,1}(0, \infty)\), and
\[\| L(t)\varphi \| = \| \vartheta(t, 0)\varphi(0) \| \leq \| \vartheta(t, \cdot)\|_\infty \|\varphi\|_D, \quad t \in [0, T].\]
The surjectivity of \(L(t)\) follows from the fact that \(L(t)\varphi = 1\) for \(\varphi(\cdot) := \frac{e^{-\vartheta(\cdot, \cdot)}}{\vartheta(\cdot, \cdot)}, t \in [0, T]\). Hence, (H4) is satisfied.

(H5) Follows from (iv).

Now, let \(\lambda > 0\) and let \(\varphi \in \ker(\lambda - A_{\text{max}}(t))\). Since \(\vartheta(t, \cdot)\varphi(\cdot) \in W^{1,1}(0, \infty)\) we can write
\[\| L(t)\varphi \| = \| \vartheta(t, 0)\varphi(0) \| = \int_0^\infty \left| \frac{\partial}{\partial a} \vartheta(t, a)\varphi(a) \right| \, da\]
\[= \int_0^\infty \left| \frac{\partial}{\partial a} \vartheta(t, a)\varphi(a) + \vartheta(t, a) \frac{\partial}{\partial a} \varphi(a) \right| \, da = \int_0^\infty (\lambda + \mu(t, a)) \varphi(a) \, da\]
\[\geq \lambda \|\varphi\|_X.\]
The assumption (H6) is then satisfied with \( \gamma = 1 \). Therefore, the evolution family \((V(t,s))(s,t)\in\Delta_T\) associated to this problem exists and is exponentially stable (see (2.2)). Using the construction presented in [23], we can assume that the bounded invertible operator \( \Gamma^{-1} \) exists. One can also verify, via (v), that the function \( f(t,\varphi) := \int_0^\infty \beta(t,x)\varphi(x)dx + h(t,\varphi) \) satisfies (C5a) with \( K_f = \|\beta\|_{L^\infty} + K_h \). Since the functional \( g(1,\ldots,p,\varphi) := \sum_{i=1}^p c_i \varphi(t_i) \) is linear bounded with respect to \( \varphi \), the condition (C5b) is then satisfied with \( K_g = \sum_{i=1}^p |c_i| \). Thus, by choosing the constants \( c_i, i = 1,\ldots,p, K_h \) and the function \( \beta \) such that
\[
(1 + \|b(\cdot)\|_{L^2} \|\Gamma^{-1}\|) \left( \sum_{i=1}^p |c_i| + T(\|\beta\|_{L^\infty} + K_h) \right) < 1
\]
one can deduce, via Theorem 6, that the size structured population equation (4.1) is exactly controllable on \([0, T]\).

References