Approximate extension property of mappings

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Abstract

The extension problem is to determine the extendability of a mapping defined on a closed subset of a space into a nice space such as a CW complex over the whole space. In this paper, we consider the extension problem when the codomains are general spaces. We take a shape theoretic approach to generalize the extension theory so that the codomains are allowed to be general spaces. We extend the notion of extension type which has been defined for the class of CW complexes and introduce the notion of approximate extension type which is defined for general spaces. We define approximate extension dimension analogously to extension dimension, replacing the class of CW complexes by the class of finitistic separable metrizable spaces. For every metrizable space \( X \), we show the existence of approximate extension dimension of \( X \).

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1. Introduction

For given spaces \( X \) and \( Y \), a closed subset \( A \) of \( X \), and a mapping \( f : A \to Y \), the extension problem is to determine whether the mapping \( f \) extends to a mapping \( \tilde{f} : X \to Y \). This property is denoted by \( Y \in \text{AE}(X) \) or \( X \tau Y \). In this problem the codomain \( Y \) is assumed to be a nice space such as a CW complex, a polyhedron, or an ANR so that the homotopy extension property holds for mappings from compact Hausdorff spaces or metrizable spaces into \( Y \). A. Dranishnikov [5] extensively studied this problem for mappings from locally compact metrizable spaces into \( Y \). A. Dranishnikov [6] introduced the notion of extension dimension for compact Hausdorff spaces. In the process of defining the extension dimension for a class \( C \) of spaces, one first defines a relation \( K \leq L \) for CW complexes \( K \) and \( L \) as the property that for every \( X \in C \), the condition \( K \in \text{AE}(X) \) implies the condition \( L \in \text{AE}(X) \). This relation induces an equivalence relation on the class of CW complexes, and the equivalence classes are the extension types. For any space \( X \in C \), the extension dimension of \( X \) is the minimal element of the class of all extension types of CW complexes \( K \) so that \( K \in \text{AE}(X) \) (if it exists).

A. Dranishnikov [6] proved that extension dimension exists for the class of compact Hausdorff spaces. The extension dimension was extended by A. Dranishnikov and J. Dydak [7] to a class including all separable metrizable spaces and by...
approximate extension types of finitistic separable metric spaces. We write \( St \) mapping \( f \), respectively, \( g \), for all \( x \in X \), there exists \( V \in \mathcal{V} \) such that \( f(x), g(x) \in V \).

For any simplicial complex \( K \) and for any simplex \( \sigma \) of \( K \), let \( St(\sigma, K) = \bigcup \{ \tau \in K : \sigma \cap \tau \neq \emptyset \} \). If \( \sigma = [v] \) is a 0-simplex, we write \( St(v, K) \) for \( St(\sigma, K) \).

Let \( X \) be a space. If \( A \) is a closed subset of \( X \), then for any space \( Y \), we denote by \( Y \in \mathcal{A}E(X, A) \) the property that every mapping \( f : A \to Y \) extends to a mapping \( \overline{f} : X \to Y \). A space \( Y \) is called an absolute extensor of \( X \), denote \( Y \in \mathcal{A}E(X) \), provided \( Y \in \mathcal{A}E(X, A) \) for any closed subset \( A \) of \( X \). We also write \( X \upharpoonright Y \) if \( Y \in \mathcal{A}E(X) \). Extending this notation, for any mapping \( g : Y \to Z \) between spaces, we write \( g \in \mathcal{A}E(X) \) provided any mapping \( f : A \to Y \) from a closed subset \( A \) of \( X \) admits a mapping \( \overline{f} : X \to Z \) such that \( \overline{f}|A = g \circ f \).

A space \( Y \) is called a neighborhood absolute extensor of \( X \), in notation \( Y \in \mathcal{A}N\mathcal{E}(X) \), provided every mapping \( f : A \to Y \) from a closed subset \( A \) of \( X \) extends over some neighborhood of \( A \).

If \( C \) is a class of spaces, let \( \mathcal{A}E(C) \) (respectively, \( \mathcal{A}N\mathcal{E}(C) \)) denote the class of all spaces \( Y \) such that \( Y \in \mathcal{A}E(X) \) (respectively, \( Y \in \mathcal{A}N\mathcal{E}(X) \)) for all \( X \in C \). Let \( M \) denote the class of metrizable spaces.

Let \( Top \) denote the category of spaces and mappings, and let \( H(\text{Top}) \) denote the homotopy category of \( \text{Top} \). Let \( \text{ANR} \), \( \text{CW} \), \( \text{Pol} \) denote the subcategories of \( \text{Top} \) whose objects are ANR’s, \( \text{CW} \) complexes, and polyhedra, respectively. If \( C \) is a class of spaces (or a subcategory of \( \text{Top} \), respectively), an inverse system \( X = (X_i, \varphi_{ij}, \lambda) \) in \( \text{Top} \) or in \( H(\text{Top}) \) is called a \( C \)-system provided \( X_i \in C \) for all \( \lambda \in A \) (or \( X_i \) are objects of \( C \) for all \( \lambda \in A \), respectively). Let \( \text{Sh} \) denote the shape category. For details in shape theory, the reader is referred to [15].

One of the key tools extending mappings is the homotopy extension theorem.

\textbf{Theorem 2.1.} If \( Y \) is a space such that \( Y \in \mathcal{A}N\mathcal{E}(X \times I) \) and if \( X \) is a normal space, then \( Y \in \mathcal{A}E(X \times I, X \times \{0\} \cup A \times I) \) for any closed subset \( A \) of \( X \).

Since \( Y \in \mathcal{A}N\mathcal{E}(M) \) for any \( \text{CW} \) complexes, \( \text{ANR} \)’s, or polyhedra \( Y \), we have

\textbf{Corollary 2.2.} If \( Y \) is a \( \text{CW} \) complex, an \( \text{ANR} \), or a polyhedron, and if \( X \) is metrizable, then \( Y \in \mathcal{A}E(X \times I, X \times \{0\} \cup A \times I) \) for any closed subset \( A \) of \( X \).
3. Absolute approximate extensors

In this section, we introduce the notion of absolute approximate extensor and obtain its fundamental properties.

For any metrizable space $X$, a system $(Y, q_{\mu}, q_{\mu}^\prime, M)$ in $\text{Top}$ is an absolute approximate extensor of $X$ provided for each $\mu \in M$ there exists $\mu^\prime > \mu$ such that $q_{\mu^\prime} \in \text{AE}(X)$. Similarly, the notion of absolute approximate extensor is defined for $\text{ANE}(M)$-systems $Y = (Y, (q_{\mu}, q_{\mu}^\prime), M)$ in $\text{H}(\text{Top})$. This is well defined. For, if $q_{\mu^\prime} \simeq q_{\mu^\prime}^\prime$ and if a mapping $f : A \to Y_{\mu}$ from a closed subset $A$ of $X$ admits a mapping $f : X \to Y_{\mu}$ such that $f|A = q_{\mu^\prime} \circ f$, then the condition $Y_{\mu} \in \text{ANE}(M)$ together with the homotopy extension theorem implies there exists a mapping $f : X \to Y_{\mu}$ such that $f|A = q_{\mu^\prime} \circ f$ and $f \simeq f$. Write $Y \in \text{AAEsys}(X)$ if $Y$ is an absolute approximate extensor of $X$.

If an ANR $Y$ is homotopy dominated by an ANR $Z$, then for any metrizable space $X$, $Z \in \text{AE}(X)$ implies $Y \in \text{AE}(X)$ by the homotopy extension theorem. We prove an analogous result for absolute approximate extensors.

**Theorem 3.1.** Let $X$ be any metrizable space, and let $Y$ and $Z$ be $\text{ANE}(M)$-systems in $\text{H}(\text{Top})$. If $Y$ is dominated by $Z$ in $\text{pro-H}(\text{Top})$, then $Z \in \text{AAEsys}(X)$ implies $Y \in \text{AAEsys}(X)$.

**Proof.** Let $g : Y \to Z$ and $h : Z \to Y$ be morphisms in $\text{pro-H}(\text{Top})$ such that $h \circ g = 1_Y$, where $1_Y : Y \to Y$ is the identity morphism. Write $Y = (Y, (q_{\mu}, q_{\mu}^\prime), M)$ and $Z = (Z, (r_{\nu}, r_{\nu}^\prime), N)$, and let $((g_\mu), g) : Y \to Z$ and $((h_\mu), h) : Z \to Y$ be system mappings which represent $g$ and $h$, respectively. Let $\mu \in M$. Since $Z \in \text{AAEsys}(X)$, there exists $\nu > h(\mu)$ such that $r_{(\mu)\nu} \in \text{AE}(X)$. Choose $\mu^\prime > \mu$, $g_{(\mu)}$, $h(\mu)$, $h(\nu)$ such that

\[ q_{\mu^\prime} \simeq h_{(\mu)} \circ g_{(\mu)} \circ q_{h(\mu)\nu}^\prime, \]

\[ g_{h(\mu)} \circ q_{h(\mu)\nu}^\prime \simeq r_{h(\mu)\nu} \circ g_{(\nu)} \circ q_{h(\nu)\nu}^\prime. \]

Let $f : A \to Y_{\mu^\prime}$ be any mapping from a closed subset $A$ of $X$. Then there exists a mapping $f : X \to Z_{h(\mu)}$ such that $f|A = r_{h(\mu)\nu} \circ g_{(\nu)} \circ q_{h(\nu)\nu}^\prime \circ f$. By (3.1), (3.2), (3.3),

\[ h_{(\mu)} \circ f|A \simeq q_{\mu^\prime} \circ f. \]

By the condition $Y_{\mu} \in \text{ANE}(M)$ and the homotopy extension theorem, there exists a mapping $f : X \to Y_{\mu}$ such that $f|A = q_{\mu^\prime} \circ f$ as required.

**Corollary 3.2.** Let $X$ be any metrizable space, and let $Y$ and $Z$ be systems in $\text{H}(\text{Top})$, each of which is an ANR-system, a CW-system, or a Pol-system. Suppose that $Y$ and $Z$ are isomorphic in $\text{H}(\text{Top})$. Then $Y \in \text{AAEsys}(X)$ if and only if $Z \in \text{AAEsys}(X)$.

**Proof.** This follows from Theorem 3.1 and the fact that whenever $Y$ is an ANR, a CW complex, or a polyhedron, then $Y \in \text{ANE}(M)$.

A space $Y$ is an absolute approximate extensor of a metrizable space $X$, in notation $Y \in \text{AE}(X)$, provided there exists an ANR-expansion $p = ([p_{\mu}]) : Y \to Y = (Y, (q_{\mu}, q_{\mu}^\prime), M)$ of $Y$ such that $Y \in \text{AAEsys}(X)$.

Corollary 3.2 easily implies the following corollary.

**Corollary 3.3.** Let $X$ be any metrizable space, and let $Y$ be any space. If $q_1 : Y \to Y_1$, $q_2 : Y \to Y_2$, $q_3 : Y \to Y_3$ are expansions of $Y$ such that $Y_1$, $Y_2$, $Y_3$ are an ANR-system, a CW-system, a Pol-system in $\text{H}(\text{Top})$, respectively, then the following are equivalent:

1. $Y \in \text{AE}(X)$.
2. $Y_1 \in \text{AAEsys}(X)$.
3. $Y_2 \in \text{AAEsys}(X)$.
4. $Y_3 \in \text{AAEsys}(X)$.
Theorem 3.1 and Corollary 3.3 easily imply the following corollary.

**Corollary 3.4.** Let \( X \) be any metrizable space, and let \( Y \) and \( Z \) be any spaces.

1. If \( Y \) is dominated by \( Z \) in \( SH \), then \( Z \in AAE(X) \) implies \( Y \in AAE(X) \).
2. If \( X \) and \( Y \) are shape equivalent, then \( Y \in AAE(X) \) if and only if \( Z \in AAE(X) \).

The following is obvious from the definition.

**Proposition 3.5.** Let \( X \) be any metrizable space, let \( Y \) be any space, and let \( q = ([q_\mu]) : Y \to Y = (Y_\mu, [q_{\mu,\mu'}], M) \) be an ANR-expansion of \( Y \).

1. If \( Y_\mu \in AEE(X) \) for \( \mu \in M \), then \( Y \in AAE(X) \).
2. Suppose that \( Y \) is an ANR, a CW complex, or a polyhedron. Then \( Y \in AAE(X) \) if and only if \( Y \in AAE(X) \).

**Example 3.6.** If \( W \) is a Warsaw circle, then \( W \in AAE(S^1) \) but \( W \notin AEE(S^1) \). Indeed, \( W \in AAE(S^1) \) follows from the facts that \( S^1 \in AEE(S^1) \) and that there exists an ANR-expansion \( p = ([p_i]) : W \to W = (W_i, [p_i,i+1]) \) of \( W \) such that \( W_i = S^1 \) for each \( i \in \mathbb{N} \). We can easily see \( W \notin AEE(S^1) \) by finding a closed subset \( A \) of \( S^1 \) and a mapping \( f : A \to W \) which is not extendable over \( S^1 \).

If \( X \) is a metrizable space with covering dimension \( \leq 1 \), then \( S^1 \in AEE(X) \). This together with Proposition 3.5(1) implies \( W \in AEE(X) \). It is not known whether there exists a metrizable space \( X \) (with covering dimension \( > 2 \)) such that \( W \in AEE(X) \) but \( W \notin AAE(X) \).

If an ANR \( Y \) is homotopy equivalent to a point, then, by the homotopy extension theorem, \( Y \in AEE(X) \) for any metrizable space \( X \). The converse also holds since \( Y \) is embedded in an AR \( Y_0 \) as a closed subset and every AR is contractible. An analogous result holds for absolute approximate extensors.

**Theorem 3.7.** Let \( Y \) be any metrizable space. Then \( Y \in AAE(X) \) for any metrizable space \( X \) if and only if \( Y \) has the shape type of a point.

**Proof.** If \( Y \) has the shape type of a point, then for any metrizable space \( X \), \( Y \in AAE(X) \) follows from \(* \in AAE(X) \) and Corollary 3.4(2).

To prove the converse, suppose that \( Y \in AAE(X) \) for any metrizable space \( X \). Consider \( Y \) as a closed subset of an AR \( Y_0 \). Then there is a neighborhood system \( \{Y_\mu : \mu \in M \} \) of \( Y \) in \( Y_0 \) such that \( Y_{\mu'} \subseteq Y_\mu \subseteq Y_{\mu'} \) for \( \mu < \mu' \). For \( \mu, \mu' \in M \) with \( \mu < \mu' \), let \( q_{\mu,\mu'} : Y_{\mu'} \to Y_\mu \) and \( q_\mu : Y_\mu \to Y_0 \) be the inclusion mappings. Then \( q = ([q_\mu]) : Y \to Y = (Y_\mu, [q_{\mu,\mu'}], M) \) is an ANR-expansion of \( Y \). Let \( \mu \in M \). Then, since \( Y \in AAE(Y_0) \) by assumption, there exists \( \mu' > \mu \) such that \( q_{\mu,\mu'} \in AEE(Y_0) \).

This implies that the mapping \( q_{\mu,\mu'} \) factors as \( Y_{\mu'} \to Y_0 \to Y_{\mu} \) for some mapping \( r_{\mu,\mu'} : Y_0 \to Y_{\mu} \). Since \( Y_0 \) is contractible, \( q_{\mu,\mu'} \) is inessential. This shows that \( Y \) is of trivial shape. \( \square \)

The following theorem shows that the notion of absolute approximate extensor coincides with that of absolute extensor for nice spaces.

**Theorem 3.8.** Let \( X \) be any metrizable space, and let \( Y \) be any space that is homotopy equivalent to an ANR, a CW complex, or a polyhedron. Suppose \( Y \in ANE(X \times I) \). Then \( Y \in AEE(X) \) if and only if \( Y \in AAE(X) \).

**Proof.** Suppose that there exists a homotopy equivalence \( \varphi : Y \to P \) for some ANR \( P \). The same argument applies to the case that \( P \) is a CW complex or a polyhedron. If \( Y \in AEE(X) \), then it follows from the fact that \( P \in ANE(X \times I) \) and the homotopy extension theorem that \( P \in AEE(X) \). Since \( \varphi : Y \to (P) \) is an ANR-expansion of \( Y \), then \( Y \in AAE(X) \). Conversely, if \( Y \in AAE(X) \), then \( (P) \in AAE(X) \), or equivalently \( P \in AEE(X) \). It follows from the hypothesis \( Y \in ANE(X \times I) \) and the homotopy extension theorem that \( Y \in AEE(X) \). \( \square \)

We can easily show the subset theorem for absolute approximate extensors.

**Proposition 3.9.** Let \( X \) be any metrizable space, and let \( X_0 \) be a closed subset of \( X \). Then for any mapping \( g : Y \to Z \) between spaces \( Y \) and \( Z \), \( g \in AEE(X) \) implies \( g \in AEE(X_0) \). Hence, for any space \( Y \), \( Y \in AAE(X) \) implies \( Y \in AAE(X_0) \).
4. Absolute approximate extensors and mappings

In this section, for each metrizable space $X$ and for each mapping $h : Y \to Z$ between metrizable spaces, we study the relations between the properties $Y \in \text{AAE}(X)$ and $Z \in \text{AAE}(X)$. We show that the two properties are equivalent if each fibre is an absolute approximate extensor of $X$. This is a generalization of the result [7, Theorem 1.9] (see also [9, Theorem 2.3] and [5, p. 272] for restricted cases). We will use Theorem 4.1 to define approximate extension dimension, which will be discussed in the following section.

A space $X$ is said to be finitistic provided every open covering of $X$ has an open refinement of finite order [18]. Characterizations of finitistic spaces were obtained in [10], and properties of finitistic spaces from dimension theoretic viewpoints were studied by various authors [1–3,10–12].

**Theorem 4.1.** Let $X$ be a metrizable space, and let $h : Y \to Z$ be a closed mapping from a metrizable space $Y$ to a finitistic metrizable space $Z$ such that $h^{-1}(z) \in \text{AAE}(X)$ for each $z \in Z$. Then the following two conditions are equivalent:

1. $Z \in \text{AAE}(X)$,
2. $Y \in \text{AAE}(X)$.

Let $h : Y \to Z$ be a mapping from a metrizable space $Y$ to a finitistic metrizable space $Z$. To prove the theorem, we use the idea from [17] to construct a system mapping between ANR-systems representing $h$.

Embed $Y$ into an ANR $Y_0$ as a closed subset. Then the collection $\{h^{-1}(z) : z \in Z\} \cup \{\{y \in Y_0 \setminus Y\}$ of subsets of $Y_0$ defines an upper semi-continuous decomposition of $Y_0$ and the decomposition space $Z_0$. The quotient mapping $h_0 : Y_0 \to Z_0$ is a surjective closed mapping, and $h = h_0|Y$ and $h_0^{-1}(Z) = Y$. Note that $Z_0$ is perfectly normal and paracompact.

Let $\Omega = \{\mathcal{W}_\lambda : \lambda \in \Lambda\}$ be the set of all locally finite families $\mathcal{W}_\lambda$ of open subsets of $Z_0$ such that

(i) $Z \subseteq H_\lambda = \bigcup \mathcal{W}_\lambda$,
(ii) $\mathcal{W}_\lambda$ has finite order, and
(iii) the function $N(\mathcal{W}_\lambda) \to N(\mathcal{W}_\lambda|Z) : W \mapsto W \cap Z$ is a bijection.

Define an order $\leq$ on the index set $\Lambda$ by saying $\lambda \leq \lambda'$ if and only if $\mathcal{W}_{\lambda'}$ is a refinement of $\mathcal{W}_\lambda$. Then we have the following lemma.

**Lemma 4.2.**

1. If $\mathcal{V}$ is a locally finite normal open covering of $Z$, then there exists $\lambda \in \Lambda$ such that $\mathcal{W}_\lambda|Z \leq \mathcal{V}$.
2. If $\mathcal{V}$ is a locally finite family of open subsets of $Z_0$ such that $Z \subseteq \bigcup \mathcal{V}$, then there exists $\lambda \in \Lambda$ such that $\mathcal{W}_\lambda \leq \mathcal{V}$.

**Proof.** To see (1), let $\mathcal{V}$ be a locally finite open covering of $Z$. Since $Z$ is finitistic, there exists a locally finite open covering $\mathcal{V}'$ of $Z$ with finite order such that $\mathcal{V}' \leq \mathcal{V}$. Since $Z$ is perfectly normal, each open set in $\mathcal{V}'$ is an $F_\sigma$-set. This fact together with [16, Lemma 1] implies that there exists $\mathcal{W}_\lambda \in \Omega$ such that $\mathcal{W}_\lambda|Z = \mathcal{V}'$. This proves (1).

To see (2), note that by (1), there exists $\lambda' \in \Lambda$ such that $\mathcal{W}_{\lambda'}|Z \leq \mathcal{V}|Z$. Write $\mathcal{W}_{\lambda'} = \{W_\gamma : \gamma \in \Gamma\}$. Then, for each $\gamma \in \Gamma$, there exists $V_\gamma \in \mathcal{V}$ such that $W_\gamma \cap Z \subseteq V_\gamma \cap Z$. Consider the locally finite family $\mathcal{W} = \{W_\gamma \cap V_\gamma : \gamma \in \Gamma\}$ with finite order. Obviously $Z \subseteq \bigcup \mathcal{W}$ and $\mathcal{W} \leq \mathcal{V}$. For any finite subset $I_0 \subseteq \Gamma$, the following four conditions are equivalent:

\[
\bigcap_{\gamma \in I_0} (W_\gamma \cap V_\gamma \cap Z) = \emptyset, \tag{4.1}
\]
\[
\bigcap_{\gamma \in I_0} (W_\gamma \cap Z) = \emptyset, \tag{4.2}
\]
\[
\bigcap_{\gamma \in I_0} W_\gamma = \emptyset, \tag{4.3}
\]
\[
\bigcap_{\gamma \in I_0} (W_\gamma \cap V_\gamma) = \emptyset. \tag{4.4}
\]

In particular, the equivalence between (4.1) and (4.4) shows that the assignment $W_\gamma \cap V_\gamma \mapsto W_\gamma \cap V_\gamma \cap Z$ for $\gamma \in \Gamma$ defines an isomorphism $N(\mathcal{W}) \to N(\mathcal{W}|Z)$. This shows that $\mathcal{W} \in \Omega$, verifying (2). □

For each $\lambda \in \Lambda$, let $G_\lambda = h_0^{-1}(H_\lambda)$ and $K_\lambda = N(\mathcal{W}_\lambda)$, and let $P_\lambda$ be the geometric realization of $N(\mathcal{W}_\lambda)$. For $\lambda < \lambda'$, let $i_{\lambda'} : G_{\lambda'} \to G_\lambda$ and $j_{\lambda'} : H_{\lambda'} \to H_\lambda$ be the inclusion mappings, and let $\psi_{\lambda'} : P_{\lambda'} \to P_\lambda$ be the mapping induced by a simplicial projection. For each $\lambda \in \Lambda$, let $i_\lambda : Y \to G_\lambda$ and $j_\lambda : Z \to H_\lambda$ be the inclusion mappings, and let $\varphi_\lambda : H_\lambda \to P_\lambda$. 


be a canonical mapping such that \( \varphi_{\lambda}^{-1}(St(W, K_{\lambda})) = W \) for each vertex \( W \) of \( K_{\lambda} \). Then \( (\{i_{\lambda}\} : Y \to G = (G_{\lambda}, [i_{\lambda, \lambda}], \Lambda) \) and \( ((\varphi_{\lambda} \circ j_{\lambda}) : Z \to P = (P_{\lambda}, [\varphi_{\lambda, \lambda}], \Lambda)) \) define ANR-expansions of \( Y \) and \( Z \), respectively.

For each \( \lambda \in \Lambda \), define the mapping \( h_{\lambda} : G_{\lambda} \to H_{\lambda} \) by \( h_{\lambda}(y) = h_{0}(y) \) for \( y \in G_{\lambda} \). Then we have a level mapping \((\varphi_{\lambda} \circ h_{\lambda}) : G \to P \) which represents a shape morphism \( H : Y \to Z \).

![Diagram](image)

This setting gives ANR-expansions of point inverses. For \( z \in Z \), let \( \Omega_{2} \) be the set of all open subsets \( W \) from some \( \mathcal{V}_{\lambda} \) such that \( z \in W \), which is ordered by reverse inclusion. For \( W, W' \in \Omega_{2} \) with \( W' \supseteq W \), let \( i_{W, W'}^{\circ} : h_{0}^{-1}(W') \to h_{0}^{-1}(W) \) be the inclusion mapping, and for each \( W \in \Omega_{2} \), let \( i_{W}^{\circ} : h_{0}^{-1}(z) \to h_{0}^{-1}(W) \) be the inclusion mapping. Using the fact that \( h_{0} \) is a closed mapping, we see that \((i_{W}^{\circ}) : h_{0}^{-1}(z) \to h_{0}^{-1}(W), [i_{W}^{\circ}], \Omega_{2}) \) is an ANR-expansion of \( h_{0}^{-1}(z) \).

The following lemma is useful in proving Theorem 4.1.

**Lemma 4.3.** For each \( \lambda \in \Lambda \), there exists \( \lambda' > \lambda \) with the following property: for every \( W' \in \mathcal{W}_{\lambda} \), there exists \( W \in \mathcal{W}_{\lambda} \) such that

\[
St(W', \mathcal{W}_{\lambda}) \subseteq W,
\]

and

\[
(h_{0}^{-1}(W), h_{0}^{-1}(St(W', \mathcal{W}_{\lambda}))) \in \mathcal{E}(X).
\]

**Proof.** Let \( \lambda \in \Lambda \). For each \( z \in Z \), there exists the ANR-expansion \(([i_{W_{z}}], \Omega_{2}) : h_{0}^{-1}(z) \to (h_{0}^{-1}(W), [i_{W_{z}}], \Omega_{2}) \) of \( h_{0}^{-1}(z) \). For each \( z \in Z \), take a \( W_{z} \in \mathcal{W}_{\lambda} \) such that \( z \in W_{z} \). By \( h_{0}^{-1}(z) \in \mathcal{A}(X), \) there exists \( W_{z}' \in \Omega_{2} \) such that \( W_{z}' \subseteq W_{z} \) and \((h_{0}^{-1}(W_{z}'), h_{0}^{-1}(W_{z}')) \in \mathcal{E}(X). \) By Lemma 4.2(2), there exists \( \lambda' > \lambda \) such that \( St(W_{z} \mathcal{V}_{\lambda' < W_{z}}^{-} : z \in Z) \). This implies that for each \( W' \in \mathcal{W}_{\lambda} \), there exists \( W_{z}' \in \Omega_{2} \) such that \( St(W', \mathcal{W}_{\lambda}') \subseteq W_{z}' \subseteq W_{z} \). Then \((h_{0}^{-1}(W_{z}), h_{0}^{-1}(St(W', \mathcal{W}_{\lambda}))) \in \mathcal{E}(X), \) completing the proof of the lemma.

Now, to show (1) \( \Rightarrow \) (2), suppose that \( Z \in \mathcal{A}(X) \). Fix \( \lambda \in \Lambda \). Let \( n \) be the order of \( \mathcal{W}_{\lambda} \). Applying Lemma 4.3, we have \( \lambda_{0}, \lambda_{1}, \ldots, \lambda_{n+1} \in \Lambda \) such that \( \lambda = \lambda_{0} < \cdots < \lambda_{n+1} \), and for each \( W' \in \mathcal{W}_{\lambda_{n+1}} \), there exists \( W \in \mathcal{W}_{\lambda} \) with the following properties

\[
St(W', \mathcal{W}_{\lambda_{n+1}}) \subseteq W, \tag{4.5}
\]

and

\[
(h_{0}^{-1}(W), h_{0}^{-1}(St(W', \mathcal{W}_{\lambda_{n+1}}))) \in \mathcal{E}(X). \tag{4.6}
\]

By \( Z \in \mathcal{A}(X) \), there exists \( \lambda' > \lambda_{n+1} \) such that \( \varphi_{\lambda_{n+1} \lambda'} \in \mathcal{E}(X) \). Let \( f : A \to G_{\lambda} \) be a mapping from a closed subset \( A \) of \( X \). We wish to obtain a mapping \( \overline{f} : X \to G_{\lambda} \) such that \( \overline{f}|A = i_{\lambda} \circ f \). By the choice of \( \lambda' \), there exists a mapping \( g : X \to P_{\lambda_{n+1}} \) such that

\[
g|A = \varphi_{\lambda_{n+1} \lambda'} \circ \varphi_{\lambda} \circ h_{\lambda'} \circ f. \tag{4.7}
\]
Here the upper squares commute, and the lower squares commute up to contiguity (and hence up to homotopy). For each \( k = 0, 1, \ldots, n \), let

\[
X_k = A \cup \{ g^{-1}(\sigma) : \sigma \text{ is a } k\text{-simplex of } K_{k+1} \}.
\]

We wish to define mappings \( f_k : X_k \to G_{k+1} \) for \( k = 0, 1, \ldots, n \) such that for each \( k\)-simplex \( \sigma = [W_0, \ldots, W_k] \) of \( K_{k+1} \),

\[
f_k(g^{-1}(\sigma)) \subseteq h_0^{-1}(W) \quad \text{for some } W \in \mathcal{W}_{k+1} \text{ such that } \bigcup_{i=0}^{k} W_i \subseteq W,
\]

and

\[
f_k|A = i_{k+1} \circ f.
\]

First, we define \( f_0 : X_0 \to G_{k+1} \) as follows. Let \( \sigma = [W_0] \) be a 0-simplex of \( K_{k+1} \). Since \( \varphi_{k+1} \circ \varphi_k \circ h_k \) and \( \varphi_{k+1} \circ h_{k+1} \circ i \) are contiguous, by (4.7),

\[
\varphi_{k+1} \circ h_{k+1} \circ i \circ f(A \cap g^{-1}(\sigma)) \subseteq St(\sigma, K_{k+1}) = St(W_0, K_{k+1}).
\]

Since \( \varphi_{k+1} \) is a canonical mapping, i.e., \( \varphi_{k+1}^{-1}(St(W_0, K_{k+1})) = W_0 \),

\[
i \circ f(A \cap g^{-1}(\sigma)) \subseteq h_0^{-1}(W_0).
\]

This together with (4.6) and Proposition 3.9 implies that there exist \( W \in \mathcal{W}_{k+1} \) and a mapping \( f_0|g^{-1}(\sigma) \to h_0^{-1}(W) \subseteq G_{k+1} \) such that

\[
St(W_0, \mathcal{W}_{k+1}) \subseteq W,
\]

and

\[
f_0|A \cap g^{-1}(\sigma) = i \circ f|A \cap g^{-1}(\sigma).
\]

Thus we have a mapping \( f_0 : X_0 \to G_{k+1} \) such that for each 0-simplex \( \sigma = [W_0] \),

\[
f_0(g^{-1}(\sigma)) \subseteq h_0^{-1}(W)
\]

for some \( W \in \mathcal{W}_{k+1} \) such that \( W_0 \subseteq W \), and

\[
f_0|A = i \circ f.
\]

Suppose that we have defined a mapping \( f_k : X_k \to G_{k+1} \) for \( 0 \leq k < n \) with properties (4.8) and (4.9). Let \( \sigma = [W_0, \ldots, W_{k+1}] \) be a \((k+1)\)-simplex of \( K_{k+1} \). Consider the mapping \( f_k'(A \cap g^{-1}(\sigma)) \cup g^{-1}(\partial\sigma) : (A \cap g^{-1}(\sigma)) \cup g^{-1}(\partial\sigma) \to G_{k+1} \). Since \( \varphi_{k+1} \circ \varphi_k \circ h_k \) and \( \varphi_{k+1} \circ h_{k+1} \circ i \) are contiguous, by (4.7),

\[
\varphi_{k+1} \circ h_{k+1} \circ i \circ f(A \cap g^{-1}(\sigma)) \subseteq St(\sigma, K_{k+1}) = \bigcup_{i=0}^{k+1} St(W_i, K_{k+1}).
\]

Since \( \varphi_{k+1} \) is a canonical mapping, i.e., \( \varphi_{k+1}^{-1}(St(W_i, K_{k+1})) = W_i \) for each \( i \),

\[
\varphi_{k+1}^{-1} \left( \bigcup_{i=0}^{k+1} St(W_i, K_{k+1}) \right) = \bigcup_{i=0}^{k+1} W_i.
\]

By (4.9) and (4.10),
Next, consider \( \partial \sigma \) as the union of \( k+2 \) faces \( \tau_i \) of dimension \( k \) for \( i = 0, 1, \ldots, k+1 \). Write \( \tau_i = [W_i(0), \ldots, W_i(k)] \), where \( 0 \leq i(0) < \cdots < i(k) \leq k+1 \). Since each \( \tau_i \) is a \( k \)-simplex, by (4.7),

\[
f_k(\tau_i) \subseteq h_0^{-1}(W'_i) \quad \text{for some } W'_i \in \mathcal{W}_{\lambda_{n-k}} \text{ such that } \bigcup_{j=0}^{k} W_{i(j)} \subseteq W'_i.
\]

This implies

\[
h_{\lambda_{n-k}} \circ f_k(\partial \sigma) = h_{\lambda_{n-k}} \circ f_k\left(g^{-1}\left(\bigcup_{i=0}^{k+1} \tau_i\right)\right) \subseteq \bigcup_{i=0}^{k+1} W'_i.
\]

On the other hand, since the vertices \( W_0, \ldots, W_{k+1} \) form a simplex of \( K_{\lambda_{n+1}} \), then

\[
\emptyset \neq \bigcap_{i=0}^{k+1} W_i \subseteq \bigcap_{j=0}^{k} \left( \bigcup_{i=0}^{k+1} W_{i(j)} \right) \subseteq \bigcap_{i=0}^{k+1} W'_i,
\]

which implies \( \bigcap_{i=0}^{k+1} W'_i \neq \emptyset \), and hence,

\[
\bigcup_{i=0}^{k+1} W_i \subseteq \bigcup_{i=0}^{k+1} W'_i \subseteq \text{St}(W'_0, \mathcal{W}_{\lambda_{n-k}}),
\]

By (4.11), (4.12), and (4.13),

\[
h_{\lambda_{n-k}} \circ f_k\left((A \cap g^{-1}(\sigma)) \cap g^{-1}(\partial \sigma)\right) \subseteq \bigcup_{i=0}^{k+1} W'_i \subseteq \text{St}(W'_0, \mathcal{W}_{\lambda_{n-k}}).
\]

This together with (4.6) and Proposition 3.9 implies that there exist \( W \in \mathcal{W}_{\lambda_{n-k}} \) such that \( \text{St}(W'_0, \mathcal{W}_{\lambda_{n-k}}) \subseteq W \) and a mapping \( f_{k+1}(g^{-1}(\sigma)) : g^{-1}(\sigma) \to h_{\lambda_{n-k}}^{-1}(W) \subseteq G_{\lambda_{n-k}} \) such that

\[
f_{k+1}\left((A \cap g^{-1}(\sigma)) \cup g^{-1}(\partial \sigma)\right) = f_k\left((A \cap g^{-1}(\sigma)) \cup g^{-1}(\partial \sigma)\right).
\]

Thus we obtain a mapping \( f_{k+1} : X_{k+1} \to G_{\lambda_{n-k}} \) such that for each \((k+1)\)-simplex \( \sigma = [W_0, \ldots, W_{k+1}] \) of \( K_{\lambda_{n+1}} \),

\[
f_{k+1}(g^{-1}(\sigma)) \subseteq h_0^{-1}\left( \bigcup_{i=0}^{k+1} W_i \right) \quad \text{for some } W \in \mathcal{W}_{\lambda_{n-k}} \text{ such that } \bigcup_{i=0}^{k+1} W_i \subseteq W,
\]

and

\[
f_{k+1}|A = i_{\lambda_{n-k}}, \lambda' \circ f.
\]

We show that \( f_{k+1} \) is continuous. Since \( X \) is a \( k \)-space, it suffices to show that if \( C \subseteq X_{k+1} \) is compact, then \( f_{k+1}|C : C \to G_{\lambda_{n-k}} \) is continuous. Note that \( g(C) \subseteq Q \) for some finite sub-polyhedron \( Q \) of \( P_{\lambda'} \). So, \( C \subseteq g^{-1}(Q) \). But since \( f_{k+1}|g^{-1}(\sigma) \) is continuous for each \((k+1)\)-simplex \( \sigma \) of \( K_{\lambda'} \), \( f_{k+1}|g^{-1}(Q) \) is continuous, and hence \( f_{k+1}|C = (f_{k+1}|g^{-1}(Q))|C \) is continuous.

The mapping \( f = f_n : X_n = X \to G_{\lambda'_+} \) satisfies \( |A = i_{\lambda'_+}, \lambda' \circ f \) as required. This shows that \( Y \in \text{AAE}(X) \).

To show (2) \( \Rightarrow \) (1), we need the following lemma.

**Lemma 4.4.** For each \( \lambda \in A \), there exist \( \lambda' > \lambda \) and a mapping \( \psi_{\lambda, \lambda'} : P_{\lambda'} \to H_{\lambda} \) with the following properties:

1. \( \left(f_{\lambda, \lambda'}, \psi_{\lambda, \lambda'} \circ \phi_{\lambda'}\right) \in \mathcal{W}_{\psi_{\lambda, \lambda'}} \), and
2. \( \psi_{\lambda, \lambda'} \) and \( \psi_{\lambda'} \circ \psi_{\lambda, \lambda'} \) are continuous.
Proof. Let \( \lambda \in A \). By [15, Lemma 2, p. 316], there exist \( \lambda' > \lambda \) and a mapping \( \psi_{\lambda', \lambda} : P_{\lambda'} \to H_{\lambda} \) with the following properties (1) and that each \( W' \in \mathcal{W}_{\lambda'} \) admits \( W \in \mathcal{W}_{\lambda} \) such that

\[
\psi_{\lambda', \lambda}(\text{St}(W', K_{\lambda'})) \subseteq W,
\]

and

\[
W' \subseteq W.
\]

It remains to verify property (2). Let \( x \in P_{\lambda'} \), and let \( \sigma = [W'_0, \ldots, W'_n] \) be the carrier of \( x \). Then \( \varphi_{\lambda', \lambda}(x) \in \bigcap_{i=0}^n \text{St}(\varphi_{\lambda', \lambda}(W'_i), K_{\lambda}) \). For each \( i \), there exists \( W_i \in \mathcal{W}_{\lambda} \) such that

\[
\psi_{\lambda', \lambda}(\text{St}(W'_i, K_{\lambda'})) \subseteq W_i,
\]

and

\[
W'_i \subseteq W_i.
\]

So,

\[
\varphi_{\lambda} \circ \psi_{\lambda', \lambda}(x) \in \psi_{\lambda} \circ \psi_{\lambda', \lambda}\left(\bigcap_{i=0}^n \text{St}(W'_i, K_{\lambda'})\right) = \varphi_{\lambda}\left(\bigcap_{i=0}^n \psi_{\lambda', \lambda}(\text{St}(W'_i, K_{\lambda'}))\right) \subseteq \varphi_{\lambda}\left(\bigcap_{i=0}^n W_i\right) \subseteq \bigcap_{i=0}^n \psi_{\lambda}(W_i) \subseteq \bigcap_{i=0}^n \text{St}(W_i, K_{\lambda}).
\]

But, since \( W'_i \subseteq \varphi_{\lambda', \lambda}(W'_i) \cap W_i \) for each \( i \), we have

\[
\emptyset \neq \bigcap_{i=0}^n W'_i \subseteq \left(\bigcap_{i=0}^n \varphi_{\lambda', \lambda}(W'_i)\right) \cap \left(\bigcap_{i=0}^n W_i\right).
\]

Thus \( 2(n + 1) \) vertices \( \varphi_{\lambda', \lambda}(W'_0), \ldots, \varphi_{\lambda', \lambda}(W'_n), W_0, \ldots, W_n \) span a simplex \( \tau \) of \( K_{\lambda} \). Since \( [\varphi_{\lambda', \lambda}(W'_0), \ldots, \varphi_{\lambda', \lambda}(W'_n)] \) and \( [W_0, \ldots, W_n] \) are faces of \( \tau \), then \( \varphi_{\lambda', \lambda}(x), \varphi_{\lambda} \circ \psi_{\lambda', \lambda}(x) \in \tau \), verifying property (2). \( \square \)

Now, to show (2) \( \Rightarrow \) (1), suppose that \( Y \in \text{AAE}(X) \). Let \( \lambda \in A \). Then there exists \( \mu > \lambda \) such that \( i_{\lambda, \mu} \in \text{AE}(X) \) and for any two \( \text{St} \mathcal{W}_{\mu} \)-near mappings \( \alpha, \beta : Q \to H_{\mu} \) from any space \( Q \), \( i_{\lambda, \mu} \circ \alpha \) and \( i_{\lambda, \mu} \circ \beta \) are \( \mathcal{W}_{\lambda} \)-homotopic. By Lemma 4.4, there exists \( v > \mu \) and a mapping \( \psi_{\mu, v} : P_v \to H_{\mu} \) such that

\[
(j_{\mu, v}, \psi_{\mu, v} \circ \varphi_{\lambda}) < \mathcal{W}_{\mu},
\]

and

\[
\varphi_{\mu, v} \text{ and } \psi_{\mu, v} \circ \varphi_{\lambda} \text{ are contiguous}.
\]

By Lemma 4.3, we can choose \( \lambda_0, \lambda_1, \ldots, \lambda_{n+1} \in A \) such that \( v = \lambda_0 < \cdots < \lambda_{n+1} = \eta \), and for each \( W' \in \mathcal{W}_{\lambda_{i+1}} \), there exists \( W \in \mathcal{W}_{\lambda_i} \) such that

\[
\text{St}(W', \mathcal{W}_{\lambda_{i+1}}) \subseteq W,
\]

and

\[
(h_0^{-1}(W), h_0^{-1}(\text{St}(W', \mathcal{W}_{\lambda_{i+1}}))) \in \text{AE}(X).
\]

Let \( g : A \to P_\eta \) be a mapping from a closed subset \( A \) of \( X \). We wish to find a mapping \( f : X \to P_\lambda \) such that

\[
f|A = \varphi_{\lambda, \eta} \circ g.
\]

For that, it suffices to obtain a mapping \( \tilde{f} : X \to G_\lambda \) such that
\[ \varphi_\lambda \circ h_\lambda \circ \tilde{f} | A \simeq \varphi_\lambda \eta \circ g \]

since this together with the homotopy extension theorem implies the existence of \( f \).

Choose points \( y_0 \in h_\eta^{-1} \circ \varphi_\eta^{-1}(g(A)) \) and \( x_0 \in A \) such that \( g(x_0) = \varphi_\eta \circ h_\eta(y_0) \). Consider the constant mapping \( \alpha_0 : B = [x_0] \rightarrow G_\eta ; \alpha_0(x_0) = y_0 \). By the same argument as above, there exists a mapping \( \tilde{f}' : A \rightarrow G_\nu \) such that for each \( n \)-simplex \( \sigma = [W_0, \ldots, W_n] \) of \( K_\eta \),

\[ \tilde{f}'(g^{-1}(\sigma)) \subseteq h_0^{-1}(W) \text{ for some } W \in \mathcal{W}_\nu \text{ such that } \bigcup_{i=0}^n W_i \subseteq W, \]

and

\[ \tilde{f}' | B = i_{v\eta} \circ \alpha_0. \quad (4.16) \]

We claim that

\[ \varphi_\lambda \circ h_\lambda \circ i_{\lambda \nu} \circ \tilde{f}' \simeq \varphi_{\lambda \eta} \circ g. \]

To show that, we first show that

\[ (h_\mu \circ i_{\mu \nu} \circ \tilde{f}', \psi_{\mu \nu} \circ \varphi_{\nu \eta} \circ g) < St \mathcal{W}_\mu. \quad (4.17) \]

Let \( x \in A \), and let \( \sigma = [W_0, \ldots, W_n] \) be the carrier of \( g(x) \). Let \( W \in \mathcal{W}_\nu \) be as in (4.16). Then

\[ \varphi_{\nu \eta} \circ g(x) \in St(W, K_\nu) \]

since \( \emptyset \neq W \cap (\bigcap_{i=0}^n W_i) \subseteq W \cap (\bigcap_{i=0}^n \varphi_{\nu \eta}(W_i)) \) and so \( \varphi_{\nu \eta} \circ g(x) \) belongs to the simplex spanned by \( W, \varphi_{\nu \eta}(W_0), \ldots, \varphi_{\nu \eta}(W_n) \). Since \( W = \varphi_{\nu \eta}^{-1}(St(W, K_\nu)) \), (4.18) implies that there exists \( z \in W \) such that \( \psi_{\nu \eta}(z) = \varphi_{\nu \eta} \circ g(x) \). Then, by (4.14), there exists \( W' \in \mathcal{W}_\mu \) such that

\[ z = j_{\mu \nu}(z) \in W', \]

\[ \psi_{\mu \nu} \circ \varphi_{\nu \eta} \circ g(x) = \psi_{\nu \eta} \circ \varphi_{\nu \eta}(z) \in W'. \quad (4.19) \]

Since \( z \in W' \cap W \), there exists \( W'' \in St \mathcal{W}_\mu \) such that \( W' \cup W \subseteq W'' \). Then by (4.16),

\[ h_\mu \circ i_{\mu \nu} \circ \tilde{f}'(x) = j_{\mu \nu} \circ h_\nu \circ \tilde{f}'(x) \in W''. \quad (4.20) \]

By (4.19) and (4.20), we have (4.17).

By the choice of \( \mu \),

\[ h_\lambda \circ i_{\lambda \nu} \circ \tilde{f}' = j_{\lambda \mu} \circ h_\mu \circ i_{\mu \nu} \circ \tilde{f}' \simeq j_{\lambda \mu} \circ \psi_{\mu \nu} \circ \varphi_{\eta \nu} \circ g. \quad (4.21) \]

Since, by (4.15), \( \psi_{\nu} \circ j_{\nu \rho} \circ \psi_{\rho \sigma} \simeq \varphi_{\lambda \eta} \), then

\[ \varphi_{\lambda \eta} \circ j_{\lambda \mu} \circ \psi_{\mu \nu} \circ \varphi_{\nu \eta} \circ g \simeq \varphi_{\lambda \eta} \circ g. \quad (4.22) \]

By (4.21) and (4.22),

\[ \psi_{\nu} \circ h_\lambda \circ i_{\lambda \nu} \circ \tilde{f}' \simeq \varphi_{\lambda \eta} \circ g. \]

By the choice of \( \mu \), there exists a mapping \( \tilde{f} : X \rightarrow G_\lambda \) such that \( \tilde{f}|A = i_{\lambda \mu} \circ \tilde{f}' \). Then \( \varphi_\lambda \circ h_\lambda \circ \tilde{f}|A \simeq \varphi_{\lambda \eta} \circ g \). This completes the proof of the theorem.
Theorem 4.5. Let $X$ be a metrizable space, and let $Y$ and $Z$ be any spaces. If there is a shape retract $R : Y \to Z$, then $Y \in \text{AAE}(X)$ implies $Z \in \text{AAE}(X)$.

Proof. Suppose that $Y \in \text{AAE}(X)$, and let $R$ be represented by a level morphism $((R_\mu)) : Y \to Z$, where $q = ((q_\mu)) : Y \to Y = (Y_\mu, q_\mu, M)$ and $r = (r_\mu) : Z \to Z = (Z_\mu, r_\mu, M)$ are HANR-extractions of $Y$ and $Z$, respectively, such that $Z_\mu \subseteq Y_\mu$ for all $\mu \in M$. Then there exists $\mu' > \mu$ such that $R_\mu \circ q_{\mu \mu'} \circ i_{\mu'} \preceq r_{\mu \mu'}$, and $q_{\mu \mu'} \in \text{AAE}(X)$. Here $i_{\mu'} : Z_\mu \to Y_\mu$ is the inclusion mapping. Now let $f : A \to Z_{\mu'}$ be a mapping. Then there exists a mapping $\gamma' : X \to Y_\mu$ such that $\gamma'|A = q_{\mu \mu'} \circ i_{\mu'} \circ f$. Note that $R_\mu \circ \gamma'|A = R_\mu \circ q_{\mu \mu'} \circ i_{\mu'} \circ f \preceq r_{\mu \mu'} \circ f$.

The homotopy extension theorem implies that there exists a mapping $\gamma : X \to Z_\mu$ such that $\gamma'|A = r_{\mu \mu'} \circ f$ as required.

5. Approximate extension dimension

Theorem 5.1. Let $\{Y_\gamma\}_{\gamma \in \Gamma}$ be a family of pointed finitistic metrizable spaces, and let $X$ be a metrizable space. Then the following are equivalent:

1. $Y_\gamma \in \text{AAE}(X)$ for each $\gamma \in \Gamma$.
2. $\bigvee_{\gamma \in \Gamma} Y_\gamma \in \text{AAE}(X)$.

Proof. (2) $\Rightarrow$ (1). There exists a retract $R_\gamma : \bigvee_{\gamma \in \Gamma} Y_\gamma \to Y_\gamma$ for each $\gamma \in I$. Then $R_\gamma$ is also a shape retract. This together with Theorem 4.5 implies (1).

(1) $\Rightarrow$ (2). Let $y_{0Y}$ be a base point of $Y_\gamma$ for each $\gamma \in \Gamma$. Let $Cone(\Gamma) = (\Gamma \times I) / \sim$, where $\sim$ is the equivalence relation $(\gamma, 1) \sim (\gamma', 1)$ for $\gamma, \gamma' \in \Gamma$, and consider $Z = Cone(\Gamma) \cup \bigcup_{\gamma \in \Gamma} Y_\gamma / \sim_{\text{h}}$, where $[(\gamma, 0)] \sim_{\text{h}} y_{0Y}$ for each $\gamma \in \Gamma$. Since $Z$ is shape equivalent to $\bigvee_{\gamma \in \Gamma} Y_\gamma$, by Corollary 3.4, it suffices to show that $Z \in \text{AAE}(X)$. Let $v$ be the vertex of $Cone(\Gamma)$. For each $\gamma \in \Gamma$, let $[(\gamma, t)] \in Cone(\Gamma) : 0 \leq t \leq 1$. For each $\gamma \in \Gamma$, define a mapping $h_\gamma : (v, \gamma) \cup Y_\gamma / \sim_{\text{h}} \to (v, \gamma)$ by $h_\gamma((\gamma, t)) = (\gamma, t)$ for each $[(\gamma, t)] \in (v, \gamma)$, and $h_\gamma(x) = [(\gamma, 1)]$ for $x \in Y_\gamma$. The mappings $h_\gamma$ define a mapping $h : Cone(\Gamma) \to Cone(\Gamma)$ satisfying the condition of Theorem 4.1 since for each $p \in Cone(\Gamma)$, $h^{-1}(p)$ is either a point of $Z$ or $Y_\gamma$ for some $\gamma$. By (1), $h^{-1}(p) \in \text{AAE}(X)$ for each $p \in Cone(\Gamma)$. This together with Theorem 4.1 implies $\bigvee_{\gamma \in \Gamma} Y_\gamma \in \text{AAE}(X)$, showing (2).}

Let $C$ be a class of metrizable spaces, and let $D$ be a class of metrizable spaces. We define a relation $\leq_D$ on $C$ as follows: for $Z_1, Z_2 \in C$, $Z_1 \leq_D Z_2$ if and only if for each $X \in D$, $Z_1 \in \text{AAE}(X)$ implies $Z_2 \in \text{AAE}(X)$. We then define an equivalence relation $\sim_D$ by saying $Z_1 \sim_D Z_2$ if and only if $Z_1 \leq_D Z_2$ and $Z_2 \leq_D Z_1$. Each equivalence class with respect to $\sim_D$ is called an approximate extension type. The approximate extension type of $Z \in C$ is denoted by $[Z]$.

Extending the notation in [7], let $\text{ApExtTypes}(D, C)$ denote the collection of all approximate extension types.

We say that the approximate extension dimension of a metrizable space $X$ is less than or equal to $Z$, denoted $\text{aext-dim}(X) \leq Z$, provided $Z \in \text{AAE}(X)$ holds.

Example 5.2. By Example 3.6, if $W$ is the Warsaw circle, then $\text{aext-dim}(S^1) \leq W$ although $W \notin \text{AE}(S^1)$.

The partial order $\sim_D$ on $C$ induces a partial order on $\text{ApExtTypes}(D, C)$, which we denote by the same notation $\sim_D$ whenever there is no confusion.

The following theorem shows that the approximate extension dimension of any metrizable space is well defined.

Theorem 5.3. Let $\mathcal{M}_0$ be the class of finitistic separable metrizable spaces, and let $\mathcal{M}$ be the class of metrizable spaces. Then for every metrizable space $X$, the class
\[(Z \in \text{ApExtTypes}(\mathcal{M}, \mathcal{M}_0) : \text{aext-dim}(X) \leq Z) \]

has the minimal element.

**Proof.** Let \(X\) be a metrizable space, and let \(Z\) be the set of all subspaces \(Z\) of the Hilbert cube \(I^{\aleph_0}\) such that \(\text{aext-dim}(X) \leq Z\). For each \(Z \in Z\), choose any point of \(Z\) as a base point. We claim that \(Z_0 = \bigvee_{Z \in Z} Z\) is the minimal element. Indeed, since \(\text{aext-dim}(X) \leq Z\) for all \(Z \in Z\), by Theorem 5.1, \(\text{aext-dim}(X) \leq Z_0\). Moreover, let \(Z\) be a finitistic separable metrizable space such that \(\text{aext-dim}(X) \leq Z\). We must show that for any metrizable space \(X'\), \(\text{aext-dim}(X') \leq Z_0\) implies \(\text{aext-dim}(X') \leq Z\). Since \(I^{\aleph_0}\) is a universal space for finitistic separable metrizable spaces, \(Z\) is homeomorphic to a separable metrizable space \(Z' \in I^{\aleph_0}\), so \(\text{aext-dim}(X) \leq Z'\), and so \(Z' \in Z\). By Theorem 5.1, if \(\text{aext-dim}(X') \leq Z_0\), then \(\text{aext-dim}(X') \leq Z'\), which implies \(\text{aext-dim}(X') \leq Z\) as required. \(\square\)

The minimal element in Theorem 5.3 is called the approximate extension dimension of \(X\) and denoted by \(\text{aext-dim}(X)\).

**References**