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Topological invariance of integral Pontrjagin classes mod p

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Abstract

It is proved that integral Pontrjagin classes p_k mod p are topological invariant if p is odd, $k < n(p)$ and they are not topological invariant if $k \geqslant n(p)$, where $n(p)$ is the smallest value of k for which p divides e_k and e_k is the smallest positive integer such that $e_k p_k$ is topological invariant. For example, p_k mod p is topological invariant for $p = 3, 5, 11$ etc. for every $k \geqslant 1$ but not topological invariant for $p = 7$ and $k \geqslant 2$.

Keywords: Integral Pontrjagin classes; Pontrjagin classes modulo p; Topological invariance; Eilenberg–Moore spectral sequence

AMS (MOS) Subj. Class.: 55R40, 55T20, 57R20

1. Introduction

This paper settles the problem of topological invariance of integral Pontrjagin classes p_k modulo a prime p. In [4] we reproved a classical result of Wu [7] that for every $k \geqslant 1$, p_k mod p is topological invariant (or briefly TOP-invariant) for $p = 3$ and extended it for $p = 5$. In fact, this problem turns out to be connected with the problem of finding the smallest multiples of integral Pontrjagin classes p_k which are TOP-invariant. The latter was solved in [3].

Let e_k be the smallest positive integer such that $e_k p_k$ is TOP-invariant. The following is an extension of the TOP-invariance of p_k mod 3 and p_k mod 5.

Theorem A. *For any odd prime p, let $n(p)$ be the smallest value of k such that p divides e_k. Then p_k* mod *p is TOP-invariant for $k < n(p)$ and is not TOP-invariant for $k \geqslant n(p)$.*

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For example, for $p=7$, $n(p)=2$, Theorem A implies that p_1 mod 7 is the only TOP-invariant class and p_k mod 7, $k \geqslant 2$ are not TOP-invariant.

Corollary B. *If p is an odd prime which does not divide e_k, for every $k \geqslant 1$, then p_k mod p is TOP-invariant.*

For example, 3, 5, 11, 13, 17 etc. are odd primes that do not divide any e_k.

The above theorems are proved by using necessary and sufficient conditions for TOP-invariance of characteristic classes proved in [3]. We recall them in Section 2. In this section, we also recall the numbers e_k and associated numbers d_k and c_k and their properties which are needed for the proofs of the theorems. Section 3 contains the proofs of Theorem A and Corollary B.

2. Integral multiples related to TOP-invariance

Let BO and BTOP be the stable classifying spaces for vector bundles and topological bundles respectively. Let the canonical map [1]

$$\varphi : \mathrm{BO} \to \mathrm{BTOP}$$

be treated as a fibration and let TOP/O be its homotopy fibre with

$$\eta : \mathrm{TOP/O} \to \mathrm{BO}$$

the canonical inclusion. Let

$$\varphi^* : H^*(\mathrm{BTOP}; \Lambda) \to H^*(\mathrm{BO}; \Lambda),$$
$$\eta^* : H^*(\mathrm{BO}; \Lambda) \to H^*(\mathrm{TOP/O}; \Lambda)$$

be the induced maps in cohomology, where Λ is any commutative ring with identity.

We recall the following from [3].

Definition 2.1. Let $\xi_i = \{E_i \xrightarrow{\pi_i} X\}$, $i=0, 1$ be two vector bundles on the same base X. We say that ξ_0 and ξ_1 are *TOP-equivalent* if there exists a homeomorphism $h : E_0 \to E_1$ making the following diagram commutative

$$\begin{array}{ccc} E_0 & \xrightarrow{\;h\;} & E_1 \\ & {}_{\pi_0}\searrow \quad \swarrow_{\pi_1} & \\ & X & \end{array}$$

There is a similar definition for stable TOP-equivalence.

Definition 2.2. Let $x \in H^*(\mathrm{BO}; \Lambda)$ be a universal characteristic class of stable vector bundles. We say that x is *TOP-invariant* if for each pair (ξ_0, ξ_1) of stably TOP-equivalent vector bundles on X, we have $\xi_0^*(x) = \xi_1^*(x)$. [Here we use the same symbol ξ_i for the bundle and the homotopy class of its classifying map $X \to \mathrm{BO}$.]

The following lemma giving necessary and sufficient conditions for TOP-invariance plays a crucial role in this work.

Lemma 2.3. (a) (A necessary condition for TOP-invariance) *If a characteristic class* $x \in H^*(\mathrm{BO}; \Lambda)$ *is TOP-invariant, then* $x \in \operatorname{Ker} \eta^*$.

(b) (A sufficient condition for TOP-invariance) $x \in H^*(\mathrm{BO}; \Lambda)$ *is TOP-invariant if* $x \in \operatorname{Image} \varphi^*$.

(c) (A necessary and sufficient condition for TOP-invariance) *A characteristic class* $x \in H^*(\mathrm{BO}; \Lambda)$ *is TOP-invariant iff* $\mu^* x = x \times 1$ *where* μ *is the composite*

$$\mathrm{BO} \times \mathrm{TOP/O} \xrightarrow{1 \times \eta} \mathrm{BO} \times \mathrm{BO} \xrightarrow{\lambda} \mathrm{BO}$$

where λ *is the H-space multiplication.*

For a proof see [3].

Let $\Lambda = \mathbb{Z}[\frac{1}{2}]$ and let c_k be the smallest positive integer such that $c_k p_k \in \operatorname{Image} \varphi^*$. Let d_k be the smallest positive integer such that $d_k p_k \in \operatorname{Ker} \eta^*$. We consider here only the odd part of c_k. Let $\pi(k)$ denote the set of partitions of the positive integer k. Also let $c_\omega = c_{i_1} c_{i_2} \dots c_{i_r}$ for $\omega = (i_1, i_2, \dots, i_r) \in \pi(k)$ and $\tilde{\prod}_{\omega \in \pi(k)} c_\omega =$ the least common multiple of c_ω's when ω varies in $\pi(k)$. Let $\gamma_k = (2^{2k-1} - 1)\mathrm{Num}(B_{2k}/4k)$ where B_{2k} is the $(2k)$th Bernoulli number and $\mathrm{Num}(B_{2k}/4k)$ denotes the numerator of the fraction $B_{2k}/4k$ in the lowest terms. Let $v_p(m) =$ the p-valuation of an integer m for a prime p.

We require the following lemmas.

Lemma 2.4. *For every* $k \geqslant 1$, c_k *divides* $\gamma_k \cdot \tilde{\prod}_{\omega \in \pi(k), \omega \neq k} c_\omega$.

For a proof see [4, Lemma 2.1].

Lemma 2.5 [3, Theorem 1.4]. *If* p *is an odd prime which divides* γ_k *but does not divide* γ_i *for* $1 \leqslant i < k$, *then* $v_p(d_k) = v_p(\gamma_k)$.

Proposition 2.6 [3, Corollary 2.9]. $e_k =$ *l.c.m. of* $d_1, \dots, d_k$.

Proof. Applying μ^* on p_k we get

$$\begin{aligned} \mu^* p_k &= (1 \times \eta^*)\lambda^* p_k \\ &= (1 \times \eta^*)\Bigg(p_k \otimes 1 + \sum_{\substack{i+j=k \\ j \geqslant 1}} p_i \otimes p_j\Bigg) \\ &= p_k \otimes 1 + \sum_{\substack{i+j=k \\ j \geqslant 1}} p_i \otimes \eta^*(p_j). \end{aligned}$$

If $m_k =$ l.c.m. of $d_1, \dots, d_k$ then

$$\begin{aligned} \mu^*(m_k p_k) &= m_k p_k \otimes 1 + \sum p_i \otimes \eta^*(m_k p_j) \\ &= (m_k p_k) \otimes 1. \end{aligned}$$

Now Lemma 2.3(c) implies that $m_k p_k$ is TOP-invariant. Since e_k is the smallest such integer, e_k divides m_k. Also, the above calculation of $\mu^* p_k$ shows that m_k is the smallest integer such that the class $x = m_k p_k$ satisfies the necessary and sufficient condition $\mu^* x = x \times 1$. Since $e_k p_k$ is TOP-invariant, m_k divides e_k and hence $m_k = e_k$. □

Lemma 2.7. *If p is an odd prime and p divides c_k but not c_i for $1 \leqslant i < k$ then p divides e_k.*

Proof. Since p does not divide c_i for $1 \leqslant i < k$, p will not divide $\tilde{\prod}_{\omega \in \pi(k), \omega \neq k} c_\omega$. Lemma 2.4 implies that if p divides c_k then p divides γ_k, in fact $v_p(c_k) = v_p(\gamma_k)$. Combining with Lemma 2.5 we get $v_p(c_k) = v_p(\gamma_k) = v_p(d_k)$. But p divides d_k implies that p divides e_k since $e_k =$ l.c.m. of $d_1, \ldots, d_k$. □

Remark 2.8. Given below are some values of e_k for $k \leqslant 7$,

$e_1 = 1$, $e_2 = 7$,
$e_3 = 7.31$, $e_4 = 7.31.127$,
$e_5 = 7.31.127.73$, $e_6 = 7.31.127.73.23.89.691$,
$e_7 = 7.31.127.73.23.89.691.8191$.

3. Proofs of Theorem A and Corollary B

Theorem A can be divided into three cases:

Case 1: $k < n(p)$. If $n(p) = \infty$ then this case is Corollary B. In this case we apply the sufficient condition for TOP-invariance (Lemma 2.3(b)).

Case 2: $k = n(p)$. To prove this we apply the necessary condition for TOP-invariance (Lemma 2.3(a)), that is if $\hat{p}_k \notin \operatorname{Ker} \hat{\eta}^*$ then $\hat{p}_k$ is not TOP-invariant.

Case 3: $k > n(p)$. The necessary and sufficient condition for TOP-invariance (Lemma 2.3(c)) is used to prove this. We show that $\hat{\mu}^* \hat{p}_k \neq \hat{p}_k \times 1$ implying that $\hat{p}_k$ is not TOP-invariant for every $k > n(p)$.

For Case 1, denote by $\hat{p}_k$, the kth integral Pontrjagin class modulo p, p an odd prime. We know that $\hat{p}_k \in H^{4k}(\mathrm{BO}; \mathbb{Z}/p)$ is TOP-invariant if

$$\hat{p}_k \in \operatorname{Image}[\hat{\varphi}^* : H^*(\mathrm{BTOP}; \mathbb{Z}/p) \to H^*(\mathrm{BO}; \mathbb{Z}/p)].$$

Let us consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker}(\varphi^* \otimes 1) & \longrightarrow & \operatorname{Ker} \hat{\varphi}^* & \longrightarrow & \operatorname{Ker} f & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^{4k}(\mathrm{BTOP}; \mathbb{Z}) \otimes \mathbb{Z}/p & \longrightarrow & H^{4k}(\mathrm{BTOP}; \mathbb{Z}/p) & \longrightarrow & \operatorname{Tor}(H^{4k+1}(\mathrm{BTOP}; \mathbb{Z}); \mathbb{Z}/p) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi^* \otimes 1} & & \downarrow{\scriptstyle \hat{\varphi}^*} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & H^{4k}(\mathrm{BO}; \mathbb{Z}) \otimes \mathbb{Z}/p & \longrightarrow & H^{4k}(\mathrm{BO}; \mathbb{Z}/p) & \longrightarrow & \operatorname{Tor}(H^{4k+1}(\mathrm{BO}; \mathbb{Z}); \mathbb{Z}/p) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{Coker} \varphi^* \otimes \mathbb{Z}/p & \longrightarrow & \operatorname{Coker} \hat{\varphi}^* & \longrightarrow & \operatorname{Coker} f & &
\end{array}$$

Since $H^*(\mathrm{BO}; \mathbb{Z})$ does not contain odd prime torsion, $\mathrm{Tor}(H^{4k+1}(\mathrm{BO}; \mathbb{Z}); \mathbb{Z}/p) = 0$ implying that Coker $f = 0$. Also $\mathrm{Coker}(\varphi^* \otimes 1) \cong \mathrm{Coker}\ \varphi^* \otimes \mathbb{Z}/p$. By applying the Snake Lemma we obtain the exact sequence

$$0 \to \mathrm{Ker}(\varphi^* \otimes 1) \to \mathrm{Ker}\ \hat{\varphi}^* \to \mathrm{Ker}\ f \to \mathrm{Coker}\ \varphi^* \otimes \mathbb{Z}/p \to \mathrm{Coker}\ \hat{\varphi}^* \to 0.$$

Consider the part

$$\mathrm{Coker}\ \varphi^* \otimes \mathbb{Z}/p \to \mathrm{Coker}\ \hat{\varphi}^* \to 0$$

of the above sequence. Let $n_{\alpha_1 \ldots \alpha_k}$ be the smallest positive integer such that $n_{\alpha_1 \ldots \alpha_k} p_1^{\alpha_1} \ldots p_k^{\alpha_k} \in \mathrm{Image}\ \varphi^*$. Since $c_1^{\alpha_1} \ldots c_k^{\alpha_k} p_1^{\alpha_1} \ldots p_k^{\alpha_k} \in \mathrm{Image}\ \varphi^*$, $n_{\alpha_1 \ldots \alpha_k}$ divides $c_1^{\alpha_1} \ldots c_k^{\alpha_k}$. Coker $\varphi^* \otimes \mathbb{Z}/p$ will contain summands of the type $\mathbb{Z}/n_{\alpha_1 \ldots \alpha_k} \otimes \mathbb{Z}/p$. By hypothesis, $k < n(p)$, hence by definition, p does not divide e_k. Then by Lemma 2.7, if p is odd and if p does not divide c_i, $1 \leqslant i < k$, then p does not divide c_k. But for every i, $1 \leqslant i < k$, p does not divide e_i. Inductively we can show that p does not divide c_i, $1 \leqslant i < k$. Hence for $k < n(p)$, p does not divide c_i, $1 \leqslant i \leqslant k$, and therefore p does not divide $n_{\alpha_1 \ldots \alpha_k}$. This implies $\mathbb{Z}/n_{\alpha_1 \ldots \alpha_k} \otimes \mathbb{Z}/p = 0$, hence Coker $\hat{\varphi}^* = 0$. This in turn will imply that $\hat{p}_k \in \mathrm{Image}\ \hat{\varphi}^*$, thus proving that $\hat{p}_k$ is TOP-invariant by Lemma 2.3(b).

For the proof of Case 2 we require the decomposition of the fibration

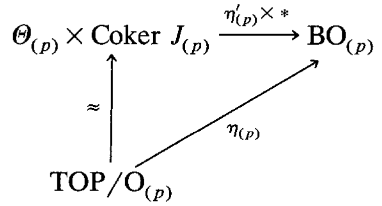
$$\mathrm{TOP/O} \xrightarrow{\ \eta\ } \mathrm{BO} \xrightarrow{\ \varphi\ } \mathrm{BTOP}$$

which when localised at an odd prime p is the product of two fibrations (see [3])

(i) $\Theta_{(p)} \xrightarrow{\eta'_{(p)}} \mathrm{BO}_{(p)} \xrightarrow{\ \theta\ } \mathrm{BO}^{\otimes}_{(p)}$,

(ii) $\mathrm{Coker}\ J_{(p)} \longrightarrow * \longrightarrow \mathrm{B\ Coker}\ J_{(p)}$

where $X_{(p)}$ and $g_{(p)}$ denote the space X and the map g localised at an odd prime p. θ is the Adams cannibalistic class, $\mathrm{BO}^{\otimes}$ is the space BO with H-space structure induced by the tensor product of vector bundles of virtual dimension 1, B Coker $J_{(p)}$ is the space whose homotopy groups are isomorphic to the cokernel of the J-homomorphism and Coker $J_{(p)} \cong \Omega$ B Coker $J_{(p)}$. Thus we obtain the following homotopy commutative diagram



which implies that Ker $\hat{\eta}^* =$ Ker $\hat{\eta}'^*$ where $\hat{}$ represents that cohomology is with $\mathbb{Z}/p$ coefficients. Lemma 3.1 below proves that Ker $\hat{\eta}'^* \cong (\mathrm{Image}\ \hat{\theta}^*)$ where $(\mathrm{Image}\ \hat{\theta}^*)$ denotes the ideal generated by the elements of Image $\hat{\theta}^*$ of degree > 0. Now to prove Case 2 of Theorem A we show that for $k = n(p)$, $\hat{p}_k \notin (\mathrm{Image}\ \hat{\theta}^*)$ which is isomorphic to Ker $\hat{\eta}'^*$. Thus the necessary condition for TOP-invariance establishes that $\hat{p}_k$ is not TOP-invariant.

The proof of Case 2 also requires the application of some results concerning the Eilenberg–Moore spectral sequence [5], which we recall below.

Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a fibration. Suppose that $\pi_1(B)$ operates trivially on the cohomology of the fibre F and the cohomology groups $H^*(E; \Lambda)$ and $H^*(B; \Lambda)$ are finitely generated, Λ being a commutative ring. Then there is a spectral sequence (Eilenberg–Moore spectral sequence) $\{E_r, d_r\}$ in the second quadrant, having the following properties:

(i) $E_2^{s,t} = \mathrm{Tor}^{s,t}_{H^*(B;\Lambda)}(H^*(E; \Lambda); \Lambda)$.

(ii) E_r converges to $H^*(F; \Lambda)$.

(iii) The edge homomorphism

$$e : \Lambda \otimes_{H^*(B;\Lambda)} H^*(E; \Lambda) = E_2^{0,*} \to E_\infty^{0,*} \hookrightarrow H^*(F; \Lambda),$$

coincides with the map

$$\Lambda \otimes_{H^*(B;\Lambda)} H^*(E; \Lambda) \to H^*(F; \Lambda),$$

which is obtained from the composite map

$$\Lambda \otimes H^*(E; \Lambda) \to H^*(F; \Lambda) \otimes H^*(F; \Lambda) \to H^*(F; \Lambda)$$

on passing to quotient.

A result of Munkholm [2] states that, if in addition to the above hypothesis we suppose that Λ is a principal ring with characteristic $\neq 2$ and $H^*(B; \Lambda)$ and $H^*(E; \Lambda)$ are polynomial algebras in at most countable variables, then the Eilenberg–Moore spectral sequence collapses.

Atiyah and Segal established an H-equivalence between $\mathrm{BO}_{(p)}$ and $\mathrm{BO}^{\otimes}_{(p)}$. Hence $H^*(\mathrm{BO}^{\otimes}; \mathbb{Z}/p) \cong H^*(\mathrm{BO}; \mathbb{Z}/p)$ which is a polynomial algebra in countable variables. Hence the Eilenberg–Moore spectral sequence of the fibration

$$\Theta_{(p)} \xrightarrow{\eta'_{(p)}} \mathrm{BO}_{(p)} \xrightarrow{\theta} \mathrm{BO}^{\otimes}_{(p)}$$

collapses.

The following lemma plays a crucial role in the proof.

Lemma 3.1. Ker $\hat{\eta}'^* \cong \overline{(\text{Image } \hat{\theta}^*)}$.

Proof. Let $\{E_r, d_r\}$ be the Eilenberg–Moore spectral sequence of the fibration

$$\Theta_{(p)} \xrightarrow{\eta'_{(p)}} \mathrm{BO}_{(p)} \xrightarrow{\theta} \mathrm{BO}^{\otimes}_{(p)}.$$

Consider the following commutative diagram where the edge homomorphism is surjective and $E_2 = E_\infty$ due to the collapsing of the spectral sequence. Let j be the quotient map.

$$\begin{array}{ccccc}
\mathbb{Z}/p \otimes_{H^*(\mathrm{BO}^{\otimes};\, \mathbb{Z}/p)} H^*(\mathrm{BO}; \mathbb{Z}/p) = E_2^{0,*} & \xrightarrow{e} & E_\infty^{0,*} & \hookrightarrow & H^*(\Theta_{(p)}; \mathbb{Z}/p) \\
\uparrow j & & \nearrow \hat{\eta}'^* & & \uparrow \cong \\
\mathbb{Z}/p \otimes H^*(\mathrm{BO}; \mathbb{Z}/p) \cong H^*(\mathrm{BO}; \mathbb{Z}/p) & & \longrightarrow & & \text{Image } \hat{\eta}'^*
\end{array}$$

Hence

$$\begin{aligned}\text{Image } \hat{\eta}'^{*} &\cong E_{\infty}^{0,*} \cong E_{2}^{0,*}\\ &\cong \mathbb{Z}/p \otimes_{H^{*}(\mathrm{BO}^{\otimes};\mathbb{Z}/p)} H^{*}(\mathrm{BO};\mathbb{Z}/p)\\ &\cong H^{*}(\mathrm{BO};\mathbb{Z}/p)/\big(\overline{\text{Image } \hat{\theta}^{*}}\big)\end{aligned}$$

(see [6, p. 67]) which implies that Ker $\hat{\eta}'^{*} \cong \overline{(\text{Image } \hat{\theta}^{*})}$. □

In the Universal Coefficient Theorem

$$0 \to H^{4k}(X;\mathbb{Z}) \otimes \Lambda \xrightarrow{\beta(X;\,\Lambda)} H^{4k}(X;\Lambda) \to \mathrm{Tor}\big(H^{4k+1}(X;\mathbb{Z});\Lambda\big) \to 0$$

the last term is zero for $X = \mathrm{BO}$ or $\mathrm{BO}^{\otimes}$ and $\Lambda = \mathbb{Z}_{(p)}$ or $\mathbb{Z}/p$ since $H^{4k+1}(X;\mathbb{Z})$ contains only 2-torsion and $p \neq 2$. In these cases $\beta(X;\Lambda)$ is an isomorphism. Consider the following diagrams

$$\begin{array}{ccccc}
H^{4k}(\mathrm{BO}^{\otimes};\mathbb{Z}/p) & \xleftarrow{\beta(\mathrm{BO}^{\otimes};\,\mathbb{Z}/p)} & H^{4k}(\mathrm{BO}^{\otimes};\mathbb{Z})\otimes\mathbb{Z}/p & \xleftarrow{B} & H^{4k}(\mathrm{BO}^{\otimes};\mathbb{Z})\otimes(\mathbb{Z}_{(p)}\otimes\mathbb{Z}/p)\\
\downarrow{\scriptstyle \hat{\theta}^{*}} & (4) & \downarrow{\scriptstyle \theta_{\mathbb{Z}}^{*}\otimes 1} & (3) & \downarrow{\scriptstyle \theta_{\mathbb{Z}}^{*}\otimes(1\otimes 1)}\\
H^{4k}(\mathrm{BO};\mathbb{Z}/p) & \xleftarrow{\beta(\mathrm{BO};\,\mathbb{Z}/p)} & H^{4k}(\mathrm{BO};\mathbb{Z})\otimes\mathbb{Z}/p & \xrightarrow{B} & H^{4k}(\mathrm{BO};\mathbb{Z})\otimes(\mathbb{Z}_{(p)}\otimes\mathbb{Z}/p)
\end{array}$$

and

$$\begin{array}{ccccc}
H^{4k}(\mathrm{BO}^{\otimes};\mathbb{Z})\otimes(\mathbb{Z}_{(p)}\otimes\mathbb{Z}/p) & \xrightarrow{A} & \big(H^{4k}(\mathrm{BO}^{\otimes};\mathbb{Z})\otimes\mathbb{Z}_{(p)}\big)\otimes\mathbb{Z}/p & \xrightarrow{\beta(\mathrm{BO}^{\otimes};\,\mathbb{Z}_{(p)})\otimes 1} & H^{4k}\big(\mathrm{BO}^{\otimes};\mathbb{Z}_{(p)}\big)\otimes\mathbb{Z}/p\\
\downarrow{\scriptstyle \theta_{\mathbb{Z}}^{*}\otimes(1\otimes 1)} & (2) & \downarrow{\scriptstyle (\theta_{\mathbb{Z}}^{*}\otimes 1)\otimes 1} & (1) & \downarrow{\scriptstyle \theta_{\mathbb{Z}_{(p)}}^{*}\otimes 1}\\
H^{4k}(\mathrm{BO};\mathbb{Z})\otimes(\mathbb{Z}_{(p)}\otimes\mathbb{Z}/p) & \xrightarrow{A} & \big(H^{4k}(\mathrm{BO};\mathbb{Z})\otimes\mathbb{Z}_{(p)}\big)\otimes\mathbb{Z}/p & \xrightarrow{\beta(\mathrm{BO};\,\mathbb{Z}_{(p)})\otimes 1} & H^{4k}\big(\mathrm{BO};\mathbb{Z}_{(p)}\big)\otimes\mathbb{Z}/p
\end{array}$$

The commutative square (1) is obtained by the naturality of the Universal Coefficient Theorem with respect to $\Lambda = \mathbb{Z}_{(p)}$ and the map $\theta : \mathrm{BO}^{\otimes} \to \mathrm{BO}$ and tensoring the square by $\mathbb{Z}/p$. Square (2) is commutative since the map A is the map of associativity. The commutative square (3) follows from the isomorphism $B : \mathbb{Z}_{(p)} \otimes \mathbb{Z}/p \cong \mathbb{Z}/p$. The commutativity of square (4) follows from the naturality of the Universal Coefficient Theorem for $\Lambda = \mathbb{Z}/p$. Now juxtaposing the above two diagrams we get the commutative diagram

$$\begin{array}{ccc}
H^{4k}(\mathrm{BO}^{\otimes};\mathbb{Z}/p) & \xrightarrow{[\beta(\mathrm{BO}^{\otimes};\,\mathbb{Z}_{(p)})\otimes 1]\circ A\circ B^{-1}\circ\beta(\mathrm{BO}^{\otimes};\,\mathbb{Z}/p)^{-1}} & H^{4k}(\mathrm{BO}^{\otimes};\mathbb{Z}_{(p)})\otimes\mathbb{Z}/p\\
\downarrow{\scriptstyle \hat{\theta}^{*}} & & \downarrow{\scriptstyle \theta_{\mathbb{Z}_{(p)}}^{*}\otimes 1}\\
H^{4k}(\mathrm{BO};\mathbb{Z}/p) & \xrightarrow{[\beta(\mathrm{BO};\,\mathbb{Z}_{(p)})\otimes 1]\circ A\circ B^{-1}\circ\beta(\mathrm{BO};\,\mathbb{Z}/p)^{-1}} & H^{4k}(\mathrm{BO};\mathbb{Z}_{(p)})\otimes\mathbb{Z}/p
\end{array}$$

Hence we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & \operatorname{Ker} \hat{\theta}^* & \longrightarrow & \operatorname{Ker}\left(\theta^*_{\mathbb{Z}_{(p)}} \otimes 1\right) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^{4k}(\mathrm{BO}^{\otimes}; \mathbb{Z}/p) & \longrightarrow & H^{4k}\left(\mathrm{BO}^{\otimes}; \mathbb{Z}_{(p)}\right) \otimes \mathbb{Z}/p & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \hat{\theta}^*} & & \downarrow{\scriptstyle \theta^*_{\mathbb{Z}_{(p)}} \otimes 1} & & \downarrow & & \\
0 & \longrightarrow & H^{4k}(\mathrm{BO}; \mathbb{Z}/p) & \longrightarrow & H^{4k}(\mathrm{BO}; \mathbb{Z}_{(p)}) \otimes \mathbb{Z}/p & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{Coker} \hat{\theta}^* & \longrightarrow & \operatorname{Coker}\left(\theta^*_{\mathbb{Z}_{(p)}} \otimes 1\right) & \longrightarrow & 0 & &
\end{array}$$

This implies that $\operatorname{Coker} \hat{\theta}^* \cong \operatorname{Coker}(\theta^*_{\mathbb{Z}_{(p)}} \otimes 1) \cong \operatorname{Coker} \theta^*_{\mathbb{Z}_{(p)}} \otimes \mathbb{Z}/p$. Consider the map $\overline{\varphi}^*$ localised at p,

$$\overline{\varphi}^*_{(p)} : \overline{H}^*(\mathrm{BTOP}; \mathbb{Z}_{(p)}) \to \overline{H}^*(\mathrm{BO}; \mathbb{Z}_{(p)}),$$

where $\overline{H}^*$ denotes the quotient of H^* by its torsion subgroup. By definition $\overline{\varphi}^*_{(p)} = \overline{\varphi}^* \otimes 1$ where

$$\overline{\varphi}^* \otimes 1 : \overline{H}^*(\mathrm{BTOP}; \mathbb{Z}) \otimes \mathbb{Z}_{(p)} \to \overline{H}^*(\mathrm{BO}; \mathbb{Z}) \otimes \mathbb{Z}_{(p)}.$$

Since $\varphi_{(p)}$ is the product of fibrations θ and $* \to \mathrm{B}\operatorname{Coker} J_{(p)}$,

$$\overline{\varphi}^*_{(p)} = \theta^*_{\mathbb{Z}_{(p)}}.$$

Hence $\operatorname{Coker} \theta^*_{\mathbb{Z}_{(p)}} = \operatorname{Coker} \overline{\varphi}^*_{(p)} = \operatorname{Coker}(\overline{\varphi}^* \otimes 1) = \operatorname{Coker} \overline{\varphi}^* \otimes \mathbb{Z}_{(p)}$. In $\operatorname{Coker} \overline{\varphi}^*$ we have summands of the type $\mathbb{Z}/n_{\alpha_1 \ldots \alpha_k}$ (refer to the proof of Case 1). Since $n(p)$ is the smallest value of k for which p divides c_k, the only nonzero summand on tensoring with $\mathbb{Z}/p$ will be $\mathbb{Z}/c_k$. Therefore

$$\begin{aligned}
\operatorname{Coker} \hat{\theta}^* \cong \operatorname{Coker} \theta^*_{\mathbb{Z}_{(p)}} \otimes \mathbb{Z}/p &\cong \Big(\mathbb{Z}/c_k \oplus \bigoplus_{n_{\alpha_1 \ldots \alpha_k} \neq c_k} \mathbb{Z}/n_{\alpha_1 \ldots \alpha_k}\Big) \otimes \mathbb{Z}_{(p)} \otimes \mathbb{Z}/p \\
&\cong \Big(\mathbb{Z}/c_k \oplus \bigoplus_{n_{\alpha_1 \ldots \alpha_k} \neq c_k} \mathbb{Z}/n_{\alpha_1 \ldots \alpha_k}\Big) \otimes \mathbb{Z}/p \\
&\cong \mathbb{Z}/p
\end{aligned}$$

which implies that $\hat{p}_k \notin \operatorname{Image} \hat{\theta}^*$. Also since $\overline{(\operatorname{Image} \hat{\theta}^*)}$ is the ideal consisting of elements of degree > 0, $\hat{p}_k \notin \overline{(\operatorname{Image} \hat{\theta}^*)}$.

To prove Case 3 we apply $\hat{\mu}^*$ on $\hat{p}_k$,

$$\begin{aligned}
\hat{\mu}^* \hat{p}_k &= (1 \times \hat{\eta}^*) \lambda^* \hat{p}_k \\
&= (1 \times \hat{\eta}^*) \Bigg[\hat{p}_k \otimes 1 + \hat{p}_{k-n(p)} \otimes \hat{p}_{n(p)} + \sum_{\substack{0 < j \neq n(p) \\ i+j=k}} \hat{p}_i \otimes \hat{p}_j \Bigg] \\
&= \hat{p}_k \otimes 1 + \hat{p}_{k-n(p)} \otimes \hat{\eta}^*\left(\hat{p}_{n(p)}\right) + \sum_{\substack{0 < j \neq n(p) \\ i+j=k}} \hat{p}_i \otimes \hat{\eta}^* \hat{p}_j .
\end{aligned}$$

Since $\hat{p}_{n(p)} \notin \operatorname{Ker} \hat{\eta}^*$, $\hat{\eta}^* \hat{p}_{n(p)} \neq 0$ implying that $\hat{\mu}^* \hat{p}_k \neq \hat{p}_k \otimes 1$ for $k > n(p)$. Applying the necessary and sufficient condition for TOP-invariance we get that $\hat{p}_k$ is not TOP-invariant for $k > n(p)$. □

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