Spectrum and analytical indices of the C*-algebra of Wiener–Hopf operators

Alexander Alldridge *, Troels Roussau Johansen 1

Universität Paderborn, Institut für Mathematik, Warburger Strasse 100, D-33098 Paderborn, Germany

Received 28 November 2006; accepted 12 March 2007
Available online 3 May 2007
Communicated by Alain Connes

Abstract

We study multivariate generalisations of the classical Wiener–Hopf algebra, which is the C*-algebra generated by the Wiener–Hopf operators, given by convolutions restricted to convex cones. By the work of Muhly and Renault, this C*-algebra is known to be isomorphic to the reduced C*-algebra of a certain restricted action groupoid, given by the action of Euclidean space on a certain compactification. Using groupoid methods, we construct composition series for the Wiener–Hopf C*-algebra by a detailed study of this compactification. We compute the spectrum, and express homomorphisms in K-theory induced by the symbol maps which arise by the subquotients of the composition series in analytical terms. Namely, these symbols maps turn out to be given by an analytical family index of a continuous family of Fredholm operators. In a subsequent paper, we also obtain a topological expression of these indices.

© 2007 Elsevier Inc. All rights reserved.

Keywords: Wiener–Hopf operator; Solvable C*-algebra; Analytical index

* Corresponding author.
E-mail addresses: alldridg@math.upb.de (A. Alldridge), johansen@math.upb.de (T.R. Johansen).

1 Part of the work was carried out while the author was supported by IHP ‘Harmonic Analysis and Related Problems,’ HPRN-CT-2001-00273. He is currently supported by a DFG post.doc grant under the International Research and Training Group “Geometry and Analysis of Symmetries,” http://irtg.upb.de.

0022-1236/$ – see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.jfa.2007.03.009
1. Introduction

1.1. The classical Wiener–Hopf equation

The classical Wiener–Hopf equation is of the form \((1 + W_f)u = v\), where

\[ W_f u(x) = \int_0^\infty f(x - y)u(y)\,dy \quad \text{for all } f \in L^1(\mathbb{R}), \ u \in L^2(0, \infty), \ x \in [0, \infty[. \]

The bounded operator \(W_f\) is called the Wiener–Hopf operator of symbol \(f\). The operator \(W_f\) is conjugate, via the Euclidean Fourier transform, to the Toeplitz operator \(T_{\hat{f}}\) defined on the Hardy space of the upper half plane, and thus has connections to both complex and harmonic analysis. Moreover, its multi-variate generalisations (see below) play a role in applications to wave propagation, e.g. in the presence of diffraction by an impenetrable wedge-shaped obstruction.

The one-variable Wiener–Hopf equation is well understood, by the following classical theorem.

**Theorem 1.** (See Gohberg–Krein [19].) Let \(\mathcal{W}(0, \infty)\) be the \(C^*\)-algebra generated by the operators \(W_f, f \in L^1(\mathbb{R})\).

(i) The following sequence is exact:

\[ 0 \to L(L^2(0, \infty)) \to \mathcal{W}(0, \infty) \xrightarrow{\sigma} C_0(\mathbb{R}) \to 0, \]

where \(\sigma\) is the Wiener–Hopf representation, defined by \(\sigma(W_f) = \hat{f}\).

(ii) The operator \(1 + W_f\) is Fredholm if and only if \(1 + \hat{f}\) is everywhere non-vanishing on \(\mathbb{R}^+\).

(iii) In this case, \(\text{Index}(1 + W_f)\) is the negative winding number of \((1 + \hat{f})(e^{i\theta})\).

1.2. Multivariate generalisation

It is quite straightforward to generalise the above setting to several variables. Indeed, let \(X\) be a finite-dimensional real vector space endowed with some Euclidean inner product \(\langle \cdot, \cdot \rangle\), and let \(\Omega \subset X\) be a closed, pointed and solid convex cone. I.e., \(\Omega\) contains no line, and has non-void interior. Consider Lebesgue measure on \(X\) to define \(L^1(X)\) and its restriction to \(\Omega\) to define \(L^2(\Omega)\).

Then, \(W_f\) is defined by

\[ W_f u(x) = \int_{\Omega} f(x - y)u(y)\,dy \quad \text{for all } f \in L^1(X), \ u \in L^2(\Omega), \ x \in \Omega. \]

Moreover, let \(\mathcal{W}(\Omega)\) be the Wiener–Hopf algebra, the \(C^*\)-subalgebra of all bounded operators on \(L^2(\Omega)\) generated by the collection of the \(W_f, f \in L^1(X)\).

The programme we propose to study then is the following.

(1) Determine a composition series of \(\mathcal{W}(\Omega)\) and compute its subquotients.
(2) Find Fredholmness criteria for Wiener–Hopf operators. (This is just a reformulation of the computation of subquotients.)

(3) Give an index formula which expresses their Fredholm index in topological terms.

These problems have been addressed from different angles in a quite extensive literature. Pioneering work was done in the series of papers Coburn, Douglas [7,8], Douglas et al. [9]; Coburn et al. [10]. Together with the work of Douglas and Howe [12], this culminated in the solution of problems (1)–(3) for the example of the (discrete) quarter plane. Berger and Coburn [4] were the first to address the structure of the Hardy–Toeplitz algebra (equivalent to the Wiener–Hopf algebra for symmetric tube type domains) for a symmetric domains of rank 2, the $2 \times 2$ matrix ball (the rank 1 case having been essentially solved by Venugopalkrishna). This led to the paper of Berger et al. [6] which treats the case of all Lorentz cones (also corresponding to rank 2 symmetric domains, the Lie balls).

Major advances were made by Upmeier [34–36] who solved (1) for the Hardy–Toeplitz algebras of all bounded symmetric domains (which properly include the Wiener–Hopf algebras for symmetric cones). Moreover, he developed an index theory, proving index formulae for the all Wiener–Hopf operators associated to symmetric cones, and thus solving problem (3) for this class of cones. A basic tool in his approach is the Cayley transform, which allows for the transfer to the situation of bounded symmetric domains.

Another approach was taken by Dynin [14], who uses an inductive procedure, based on the local decomposition of the cone $\Omega$ into a product relative to a fixed exposed face, to construct a composition series as in (1). This presupposes a certain tameness of the cone $\Omega$, which he calls ‘complete tangibility.’ Due to the weakness of this assumption, a large class of cones, including polyhedral, almost smooth and homogeneous cones, are subsumed.

The point of view we will adopt in this note is due to Muhly, Renault [29]. They describe a general procedure to produce a (locally compact, measured) groupoid whose groupoid C*-algebra is the Wiener–Hopf algebra, and compute composition series (1) for the opposite extremes of polyhedral and symmetric cones. Their construction is based on the specification of a convenient compactification of $\Omega$ (in fact, of $X$). Nica [31] has given a uniform construction of this Wiener–Hopf compactification for all pointed and solid cones. The main problem is to prove that the corresponding groupoid always has a Haar system. From the more general perspective of ordered homogeneous spaces, in which $X$ is replaced by a locally compact group and $\Omega$ by a submonoid satisfying certain assumptions, Hilgert, Neeb [21] have extended Nica’s results, at the same time giving a convenient alternative description of the Wiener–Hopf compactification.

As yet, none of the problems (1)–(3) have been solved in full generality. In fact, there is not even an index theorem for the polyhedral case. We show how the groupoid perspective allows for a unified treatment, for a large class of cones satisfying some global regularity assumption which arises in a natural fashion.

In this paper, we obtain a composition series of the Wiener–Hopf algebra, in the following manner. The Wiener–Hopf algebra is isomorphic to the reduced groupoid C*-algebra $C^*_r(\mathcal{W}_\Omega)$ of the groupoid $\mathcal{W}_\Omega$, defined as the restriction $(\overline{X \times X})|\Omega$ of the action groupoid given by the action of the vector space $X$ on $\overline{X}$, the order compactification of $X$ (see below), to the closure $\Omega$ of $\Omega$ in $\overline{X}$.

Order the dimensions of convex faces of the dual cone $\Omega^*$ increasingly by

\[
\{0 = n_0 < n_1 < \cdots < n_d = n\} = \{\dim F \mid F \subset \Omega^* \text{ face}\}.
\]
Let $P_j$ be the set of $n_{d-j}$-dimensional faces of $\Omega^*$, and assume that it is compact for all $j$, in the space of all closed subsets of $X$, endowed with the Fell topology. (The class of cones for which this condition is satisfied properly contains the polyhedral and symmetric cases, where the $P_j$ are, respectively, finite sets and certain compact homogeneous spaces including, in particular, all spheres.) Then there is a surjection from $\overline{P} = \mathcal{W}_\Omega^{(0)}$ onto the set of all faces of $\Omega^*$ which is continuous when restricted to the inverse image $Y_j$ of $P_j$.

The sets $Y_j$ are closed and invariant, and $U_j = \bigcup_{i=0}^{j-1} Y_i$ are open and invariant. Thus, we obtain ideals $I_j = C^*_r(\mathcal{W}_\Omega|U_j)$ of the Wiener–Hopf $C^*$-algebra $C^*_r(\mathcal{W}_\Omega)$, and extensions

$$0 \rightarrow C^*_r(\mathcal{W}_\Omega|Y_{j-1}) \rightarrow I_{j+1}/I_{j-1} = C^*_r(\mathcal{W}_\Omega|(U_{j+1} \setminus U_{j-1})) \rightarrow C^*_r(\mathcal{W}_\Omega|Y_j) \rightarrow 0.$$

Moreover, we have Morita equivalences $\mathcal{W}_\Omega|Y_j \sim \Sigma_j$ where $\Sigma_j = \mathcal{W}_\Omega|P_j$ is the ‘co-tautological’ topological vector bundle over the space $P_j$ whose fibre at the face $F$ is the orthogonal complement $F^\perp$. We prove the following theorem.

**Theorem 2.** The Wiener–Hopf algebra admits an ascending filtration by ideals $(I_j)_{j=0}^d$ whose subquotients are stably isomorphic to $C_0(\Sigma_j)$, and in particular, is solvable of length $d$ in the sense of Dynin [13].

The spectrum can be computed in terms of a suitable gluing of the bundles $\Sigma_j$. As a particular case, we obtain the classical Wiener–Hopf extension (i.e., $X = \mathbb{R}$ and $\Omega = \mathbb{R}_{\geq 0}$).

Moreover, the above extensions give rise to index maps $\partial_j : K^1_c(\Sigma_j) \rightarrow K^0_c(\Sigma_{j-1})$. In this paper, we give an analytical expression of the $\partial_j$, as follows.

**Theorem 3.** The quotient $I_{j+1}/I_j$ is a field $K_j$ of elementary $C^*$-algebras over $\Sigma_j$. If a class $f \in K^1_c(\Sigma_j)$ is represented by a element invertible modulo matrices over $I_j$, then its image $\sigma_j$ in the matrices over $M(K_j)$ is a Fredholm family, and

$$\partial_j(f) = \text{Index}_{\Sigma_{j-1}} \sigma_j(f)$$

is the analytical family index.

In the second part of our work, Alldridge, Johansen [1], we also give a topological expression of $\partial_j$, which we now proceed to explain. Assume that the cone $\Omega$ has a facially compact and locally smooth dual cone (compare Section 6 of Alldridge, Johansen [1]). Consider the compact space $P_j$ consisting of all pairs $(E,F) \in P_{j-1} \times P_j$ such that $E \supset F$. It has (not necessarily surjective) projections

$$P_{j-1} \xrightarrow{\xi} P_j \xrightarrow{\eta} P_j.$$

The map $\xi : P_j \rightarrow P_{j-1}$ turns $P_j$ into a fibrewise $C^1$ manifold over the compact base $\xi(P_j)$. Moreover, $\eta^* \Sigma_j$ is the trivial line bundle over $\xi^* \Sigma_{j-1} \oplus T P_j$ if $TP_j$ denotes the fibrewise tangent bundle. Then we have the following theorem.

**Theorem 4.** The $KK$-theory element representing $\partial_j$ is given by

$$\partial_j \otimes \xi^* = \eta^* \otimes \tau_j \quad \text{in} \quad KK^1(C^*_r(\Sigma_j), C^*_r(\Sigma_{j-1}|\xi(P_j))).$$
where \( y \in KK^1(\mathbb{C}, S) \) represents the classical Wiener–Hopf extension, \( \eta^* \) is associated to the projection \( \eta^* \Sigma_j \rightarrow \Sigma_j \), and \( \xi^* \) is associated to the inclusion \( \Sigma_{j-1} \vert \xi(\mathcal{P}_j) \subset \Sigma_{j-1} \).

Here, \( \tau_j \in KK(C^*_r(\Sigma_{j-1} \oplus TP_j), C^*_r(\Sigma_{j-1} \vert \xi(\mathcal{P}_j))) \) represents the topological Atiyah–Singer family index for \( \xi^* \Sigma_{j-1} \oplus TP_j \), considered as a vector bundle over \( \Sigma_{j-1} \vert \xi(\mathcal{P}_j) \).

2. The Wiener–Hopf C*-algebra

2.1. The Wiener–Hopf groupoid

Let \( X \) be a finite-dimensional Euclidean vector space, \( \Omega \subset X \) a closed convex cone which we assume to be pointed \((-\Omega \cap \Omega = 0)\) and solid \((\Omega - \Omega = X)\).

In order to construct a groupoid which conveniently describes the C*-algebra of Wiener–Hopf operators, we recall the order compactification of the Euclidean space \( X \). Here, we follow Hilgert, Neeb [21]. (The compactification was first described in Nica [31], in a quite different manner.)

Consider the set \( \mathbb{F}(X) \) of closed subsets of \( X \). The topology of Painlevé–Kuratowski convergence is a complete, compact and separable metric topology for which the convergent sequences \( (A_k) \subset \mathbb{F}(X) \) are those for which \( \lim_k A_k = \lim_k A_k \). Here, the Painlevé–Kuratowski limes inferior, respectively superior, are

\[
\lim_k A_k = \bigcap_{\varepsilon > 0} \bigcup_{k \in \mathbb{N}} (A_\varepsilon)_k = \{ a = \lim_k a_k \in X \mid a_k \in A_k \}
\]

and

\[
\lim_k A_k = \bigcap_{\varepsilon > 0} \bigcup_{k \in \mathbb{N}} (A_\varepsilon)_k = \{ a = \lim_k a_{\alpha(k)} \in X \mid a_{\alpha(k)} \in A_{\alpha(k)} \},
\]

where \( A_\varepsilon = \{ x \in X \mid \inf a \in A \| x - a \| < \varepsilon \} \).

Since \( X \) is a locally compact, separable metric space, the Painlevé–Kuratowski convergence topology coincides with the Fell topology, and the Vietoris topology on the one-point compactification \( X^+ \) [24, Proposition I.1.54, Theorem I.1.55], i.e. the topology induced by the Hausdorff metric of \( X^+ \). This is the fashion in which the topology on \( \mathbb{F}(X) \) was introduced in Hilgert, Neeb [21].

We inject \( X \) into \( \mathbb{F}(X) \) by \( \eta : X \rightarrow \mathbb{F}(X) : x \mapsto x - \Omega \). The map \( \eta \) is a homeomorphism onto its image, and \( \eta(X) \) is open in its closure [21, Lemma II.8, Theorem II.11]. Take \( \overline{X} \) to be the closure of \( X \) in \( \mathbb{F}(X) \), and similarly, denote the closure of \( \Omega \) by \( \overline{\Omega} \).

The elements of \( \overline{\Omega} \) are non-void, and for any \( a \in \overline{X} \), \( a \neq \emptyset \), there exists \( x \in X \) such that \( a + x \in \overline{\Omega} \); i.e., \( \overline{\Omega} \) intersects every orbit of \( \overline{X} \), save one, under the action of \( X \). Consequently, \( \overline{\Omega} \) completely determines \( \overline{X} \).

Define a right action of \( X \) on \( \mathbb{F}(X) \) by \( A.x = A + x \). Clearly, this action leaves \( X \) invariant. Hence, it also leaves \( \overline{X} \) invariant, and we may form the transformation groupoid \( \overline{X} \rtimes X \). We define \( \mathcal{W}_{\Omega} = (\overline{X} \rtimes X) \vert \overline{\Omega} \), the Wiener–Hopf groupoid, as its restriction. Recall

\[
r(\omega, x) = \omega \quad \text{and} \quad s(\omega, x) = \omega + x \quad \text{for all} \ (\omega, x) \in \mathcal{W}_{\Omega}.
\]

The locally compact groupoid \( \mathcal{W}_{\Omega} \) is topologically amenable. Indeed, \( \overline{X} \) is an amenable \( X \)-space, since the group \( X \) is amenable [2, Corollary 2.2.10]. Moreover, \( \mathcal{W}_{\Omega} \) is topologically equivalent to
the restriction of $\overline{X} \times X$ to the non-void elements of $X$ [30, Example 2.7, p. 16], and amenability is preserved under topological equivalence [2, Theorem 2.2.17] and restriction to open invariant subsets.

A Haar system of $\overline{X} \times X$ is given by $\lambda^A = \delta_A \otimes \lambda$, $\lambda$ denoting the Lebesgue measure on $X$. That this Haar system restricts to $\mathcal{W}_\Omega$ is a non-trivial matter related to the regularity of the compactification $\overline{X}$. It was proved in [29] for polyhedral and symmetric cones, in [31, Proposition 1.3] for the present setup and subsequently in [21, Lemma III.4], by a different method, for the more general setting of ordered homogeneous spaces.

**Theorem 5** (Muhly–Renault, Nica, Hilgert–Neeb). The $C^*$-algebra $C^*(\mathcal{W}_\Omega)$ of the locally compact groupoid $\mathcal{W}_\Omega$ is isomorphic to the $C^*$-algebra of Wiener–Hopf operators.

Moreover, the above authors also established that the ideal of compact operators on $L^2(\Omega)$ is naturally contained in $C^*(\mathcal{W}_\Omega)$. In order to describe $C^*(\mathcal{W}_\Omega)$ in greater detail, we embark on a closer study of the compactification $\overline{\Omega}$.

### 2.2. The Wiener–Hopf compactification

As a motivating example, consider the quarter plane $\Omega = [0, 1]^2 \subset \mathbb{R}^2 = X$. This cone is self-dual and simplicial. Identifying a point $x \in \Omega$ with the set $x - \Omega$, we see that limits of sequences $x_k$ can contribute to $\overline{\Omega} \setminus \Omega$ in two distinct fashions. Either, one of the components of $x_k$ remains bounded; in this case, the limit point will be an affine half space not completely containing $\Omega$. Or, both components tend to infinity; in which case, the limit shall be the entire space $X$. This is illustrated in Fig. 1.

This example suggests that $\overline{\Omega}$ is a local fibre bundle over the spaces of faces. More precisely, denote by

$$P = \{\emptyset \neq F \subset \Omega^* \mid F \text{ convex face}\}$$

and for all $A \subset X$, let

$$(A) = A - A, \quad A^* = \{x \in X \mid x \in (x : A) \geq 0\} \quad \text{and} \quad A^\circ = (A) \cap A^*$$

denote the linear span, the dual cone, and the relative dual cone, respectively. Then $\overline{\Omega} = \{x - F^* \mid x \in F^\circ, \ F \in P\}$, at least in the example.

In general, it seems to be a non-trivial matter to give a complete description of $\overline{\Omega}$. Nica [31, Proposition 4.6.2] has proved that at least the inclusion $\supset$ in the above equality always holds.
For the reader’s convenience, we give a streamlined proof, whilst claiming no originality on our part.

**Theorem 6** (Nica). We have the inclusion \( \{ x - F^* \mid x \in F^\oplus, \ F \in P \} \subset \Omega^\oplus \), where \( P \) is the face lattice of \( \Omega^\ast \). Moreover, \( x \) and \( F \) are uniquely determined by \( x - F^* \).

We first note the following lemmata.

**Lemma 7.** Let \( F \subset \Omega \) be an exposed face, i.e. the intersection of \( \Omega \) with a supporting hyperplane. Denote by \( F^\perp \) the orthogonal complement, and by \( \hat{F} = F^\perp \cap \Omega^* \) the dual face. Then

\[
\hat{F}^* = \Omega - F = \langle F \rangle \oplus p_{F^\perp}(\Omega) \quad \text{and} \quad p_{F^\perp}(\Omega) \cap (\hat{F}) = \hat{F}^\oplus.
\]

In particular, \( p_{\hat{F}^\perp}(\Omega) = \hat{F}^\oplus \). Here, for \( A \subset X \), \( p_A : X \to \langle A \rangle \subset X \) denotes orthogonal projection onto the linear span of \( A \).

**Proof.** The equality \( \hat{F}^* = \Omega - F \) follows from \([22, \text{Proposition I.1.9}]\). Since the intersection of the closed convex cones \( \langle F \rangle \) and \( p_{F^\perp}(\Omega) \) is 0, their sum is closed \([22, \text{I.2.32}]\). For the first assertion, it remains to prove that \( \Omega - F = \langle F \rangle + p_{F^\perp}(\Omega) \). But this follows from

\[
p_{F^\perp}(\Omega) \subset \hat{F}^\oplus \subset \hat{F}^* \quad \text{and} \quad \hat{F}^\oplus \subset \hat{F}^* \subset \hat{F}.
\]

As to the second assertion, \( p_{\hat{F}^\perp}(\Omega) \subset \hat{F}^\oplus \), so \( p_{F^\perp}(\Omega) \cap (\hat{F}) \subset \hat{F}^\oplus \). Conversely, for any \( x \in \hat{F}^\oplus \subset \hat{F}^* \), by the first assertion, \( x + f \in p_{F^\perp}(\Omega) \subset F^\perp \) for some \( f \in \langle F \rangle \). But then \( f \in F^\perp - x = F^\perp \), since \( x \in (\hat{F}) \subset F^\perp \). This implies \( f = 0 \), so \( x \in p_{F^\perp}(\Omega) \).

Finally, from the second assertion, we have \( \hat{F}^\oplus \subset p_{\hat{F}^\perp}(p_{F^\perp}(\Omega)) \subset p_{\hat{F}^\perp}(\Omega) \), since of course \( p_{\hat{F}^\perp}p_{F^\perp} = p_{\hat{F}^\perp} \) and \( \langle \hat{F}^\oplus \rangle = (\hat{F}) \). Here, the latter follows from the fact that \( \hat{F}^\oplus \) is the dual of the pointed cone \( \hat{F} \), taken in the vector space \( \langle \hat{F} \rangle \). Conversely,

\[
(\hat{F}^\perp : \hat{F}) = (p_{F^\perp}(\Omega) : \hat{F}) \subset (\hat{F}^* : \hat{F}) \subset \mathbb{R}_{\geq 0},
\]

so \( p_{\hat{F}^\perp}(\Omega) \subset \hat{F}^\oplus \cap (\hat{F}) = \hat{F}^\oplus \). This proves the lemma. \( \square \)

**Lemma 8.** Let \( x \in \Omega \), and denote by \( F = \Omega \cap (\Omega^* \cap x^\perp)^\perp \) the exposed face of \( \Omega \) generated by \( x \). Then \( \lim_{\lambda \to \infty}(\lambda \cdot x - \Omega) = F - \Omega = -F^* \) in \( F(X) \).

**Proof.** Let \( \lambda > 0 \) and \( \omega \in \Omega \). Then

\[
\lambda \cdot x - \omega = \lim_{\mu \to \infty} ((\lambda + \mu) \cdot x - (\omega + \mu \cdot x)) \in \lim_{\mu \to \infty} (\mu \cdot x - \Omega),
\]

and consequently

\[
\lim_{\lambda \to \infty} (\lambda \cdot x - \Omega) \subset \bigcup_{\lambda > 0}(\lambda \cdot x - \Omega) \subset \lim_{\lambda \to \infty} (\lambda \cdot x - \Omega) .
\]
This shows that
\[
\lim_{\lambda \to \infty} (\lambda \cdot x - \Omega) = \mathbb{R}_{\geq 0} \cdot x - \Omega = F - \Omega = -\tilde{F}^*,
\]
where we have used Lemma 7. This gives our contention. □

**Proof of Theorem 6.** Let \( F \in P, x \in F^{\circ}. \) By [20, Proposition 1.3(iv)], there are faces \( F = F_0 \subset \cdots \subset F_m = \Omega^*, \) \( F_j \) exposed in \( F_{j+1}. \) Suffices to show that all \( u - F_j^*, u \in F_j^\circ, \) are contained in the closure of \( \{ v - F_j^* \mid v \in F_{j+1}^\circ \}. \) By induction, we may assume \( m = 1, \) i.e. that \( F \) is exposed.

Taking \( y \) in the relative interior of \( F^\circ, \) we obtain \( -F^* = \tilde{F} - \Omega = \lim_{\lambda \to \infty} (\lambda \cdot y - \Omega) \) by Lemma 8, \( F \) being exposed. Since \( x \in F^\circ = p_F(\Omega) \) by Lemma 7, there exists \( y_k \in \Omega, \) such that \( x = \lim_k p_F(y_k). \) But then \( x - F^* = \lim_k (y_k + \lambda_k \cdot y - \Omega) \) for any \( \lambda_k \to \infty. \) Thus, \( x - F^* \in \Omega. \)

As to uniqueness, recall that for \( \emptyset \neq A \subset X, \) the support functional of \( A \) is defined by
\[
\sigma_A(x) = \sigma_{\text{co}(A)}(x) = \sup_{y \in A} (x : y) \in [0, \infty].
\]
The equality of sets \( u - E^* = v - F^* \) entails the equality of support functionals, so
\[
E = \text{dom } \sigma_{u-E^*} = \text{dom } \sigma_{v-F^*} = F.
\]
In particular, \( u, v \in \langle E \rangle. \) Moreover,
\[
(u : e) = \sigma_{u-E^*}(e) = \sigma_{v-F^*}(e) = (v : e) \quad \text{for all } e \in E.
\]
This proves \( u = v, \) and hence, the assertion. □

The natural question to ask is when the above theorem gives a complete description of \( \Omega. \) Nica [31, Proposition 6.1] shows that this is the case for the rather restricted class of tame cones. (Polyhedral cones, Lorentz cones, and cones of dimension \( \leq 3 \) are tame, but symmetric cones coming from irreducible Jordan algebras of rank \( \geq 2 \) are not.) On the other hand, he gives an example of a four-dimensional cone where equality fails, cf. [31, Example 5.3.5].

Nica’s work suggests that the description of \( \Omega \) is related to the compactness of \( P, \) considered as a subset of \( F(X). \) More precisely, we have the following theorem, the proof of which the remainder of this section is devoted to.

**Theorem 9.** Order the face dimensions increasingly,
\[
\{n_0 < n_1 < \cdots < n_d\} = \{\dim F \mid F \in P\},
\]
where we set \( \dim A = \dim \langle A \rangle \) for \( A \subset X. \) Let \( P_j = \{\dim = n_{d-j}\}, \)
\[
Y_j = \{x - F^* \mid x \in F^\circ, \ F \in P_j\} \quad \text{and} \quad U_j = \bigcup_{i=0}^{j-1} Y_i.
\]
Moreover, define projections \( \pi : U_{d+1} \to P \) and \( \lambda : U_{d+1} \to X \) by

\[
\pi(x - F^*) = F \quad \text{and} \quad \lambda(x - F^*) = x.
\]

Then the following holds.

(i) If \( \Omega = U_{d+1} \), then \( P \) is compact.
(ii) If \( P \) is compact, then \( (\pi, \lambda)|_{Y_j} \) is continuous for all \( 0 \leq j \leq d \).
(iii) If \( P_j \) is compact for all \( 0 \leq j \leq d \), then \( \Omega = U_{d+1} \).

Remark 10. A least if \( P \) is modular, \( F \mapsto \dim F \) is the rank function of the lattice \( P \). Thus, the condition that \( P_j \) be compact simply means that \( P \) has a continuous rank function.

Moreover, if all \( P_j \) are compact, \( \Omega \) is locally a fibre bundle. We shall prove this more precise statement below.

The following observation is fundamental, albeit elementary.

Proposition 11. Let \( C_k, C \in \mathbb{F}(X) \).

(i) We have \( \dim \lim_k C_k \leq \lim \inf_k \dim C_k \).
(ii) Assume that \( C_k, C \) are convex cones, and that \( C_k \to C \). Then \( \langle C \rangle \subset \lim_k \langle C_k \rangle \). Moreover, \( \langle C_k \rangle \to \langle C \rangle \) if and only if \( \dim C = \lim_k \dim C_k \).

Proof. (i) Let \( q = \dim C \) and choose \( x_1, \ldots, x_q \in C \), linearly independent. For \( C_k \) close to \( C \), there exist \( x_{ik} \in C_k \), \( x_{ik} \) close to \( x_i \) for \( i = 1, \ldots, q \). But then the matrices \( (x_{1k}, \ldots, x_{qk}) \) and \( (x_1, \ldots, x_q) \) are close. Since the rank function on \( \text{Hom}(\mathbb{R}^q, X) \) is l.s.c., the rank of these matrices is \( q \) for \( k \) sufficiently large. Thus, \( q \) is eventually a lower bound for \( \dim C_k \), and therefore \( q \leq m \), proving the lemma.

(ii) Since \( \langle C \rangle = C - C \), \( \langle C_k \rangle = C_k - C_k \), the inclusion \( \langle C \rangle \subset \lim_k \langle C_k \rangle \) is trivial. Moreover, \( \lim_k \langle C_k \rangle \) is manifestly a linear subspace of \( X \). By the first part, its dimension is \( \leq \dim C = \lim_k \dim C_k \) whenever this limit exists, so in that case, \( \langle C \rangle = \lim_k \langle C_k \rangle \).

Now, let \( x_{\alpha(k)} \in C_{\alpha(k)} \) converge to \( x \in X \), where \( \alpha(\mathbb{N}) \) is cofinal in \( \mathbb{N} \). Since \( C_{\alpha(k)} \to C \), the above shows that

\[
x = \lim_k x_{\alpha(k)} \in \lim_k \langle C_{\alpha(k)} \rangle = \langle C \rangle.
\]

So we have proved that \( \lim_k \langle C_k \rangle \subset \langle C \rangle \).

As for the converse, let \( \langle C \rangle = \lim_k \langle C_k \rangle \). Since \( X \) is locally compact, the Attouch–Wets topology on \( \mathbb{F}(X) \setminus \{ \emptyset \} \) coincides with the topology of Painlevé–Kuratowski convergence, by [5, Example 5.1.10]. Now, the dual cones \( \langle C \rangle^* = C^\perp \), \( \langle C_k \rangle^* = C_k^\perp \), so by continuity of polarity, [5, Corollary 7.2.12], \( C^\perp = \lim_k C_k^\perp \). The first part implies \( \dim C \leq \lim \inf_k \dim C_k \) and

\[
\dim C = n - \dim C^\perp \geq n - \lim \inf_k \dim C_k^\perp = \lim \sup_k \dim C_k,
\]

so \( \dim C = \lim_k \dim C_k \). \( \square \)
Corollary 12. Let $C_k, C \subset X$ be convex cones, such that $C_k \to C$.

(i) If $\pi_{C_k}, \pi_C$ denote the metric projections, cf. [38], then

$$\pi_{C_k}(x_k) \to \pi_C(x) \quad \text{whenever } x_k \to x, \ x_k, x \in X.$$

(ii) Let $\dim C_k = \dim C$. Then $p_{C_k} \to p_C$ and $p_{C_k}^\perp \to p_C^\perp$.

Proof. (i) Let $y_k = \pi_{C_k}(x_k)$. Then $\|y_k\| \leq \|x_k\|$, so $y_k$ is bounded, and we may assume that it converges to some $y \in X$. Then $y \in \lim_k C_k = C$. Let $u \in C$. There exist $u_k \in C_k$, $u_k \to u$. Thus

$$\|x - u\| = \lim_k \|x_k - u_k\| \geq \|y_k - u_k\| = \|y - u\|,$$

and it follows that $y = \pi_C(x)$.

(ii) By Proposition 11, $\langle C_k \rangle \to \langle C \rangle$. Thus $p_{C_k} \to p_C$ follows from the first part, because $p_C = \pi_{\langle C \rangle}$, and we have already noted $C_k^\perp \to C^\perp$ above.

The following lemma constitutes the main step in the theorem’s proof. For its proof, recall the following notions. Given a proper l.s.c. function $\varphi : X \to ]-\infty, \infty]$, its epigraph

$$\text{epi}\ \varphi = \{(x, y) \in X \times \mathbb{R} \mid y \geq \varphi(x)\}$$

is a closed non-void subset of $X \times \mathbb{R}$. Given $(\varphi_k), \varphi$ proper l.s.c., $(\varphi_k)$ is said to epi-converge to $\varphi$ if $\text{epi}\ \varphi_k \to \text{epi}\ \varphi$ in the Painlevé–Kuratowski sense (with respect to the box metric on $X \times \mathbb{R}$).

Lemma 13. Let $F_k \in \mathcal{F}(\Omega^*)$, $x_k \in F_k^\circ$, $-\Omega \subset C \subset X$ closed and convex, and $E \subset \Omega^*$ be closed. Assume that $x_k - F_k^* \to C$ and $F_k \to F$.

(i) We have $\text{dom} \sigma_C \subset F$.

(ii) If $(x_k)$ is bounded, then $x = \lim_k x_k \in F^\circ$ exists, $\text{dom} \sigma_C = F$, and $C = x - F^*$.

(iii) If $(x_k)$ is unbounded and $m = \dim F = \lim_k \dim F_k$, then $\dim \text{dom} \sigma_C < m$. In fact, there exist $E_k, E \in P, E_k \subset F_k, E \subset F$, such that

$$\dim E_k, \dim E < m, \ E_k \to E \quad \text{and} \quad x_k - E_k^* \to C.$$

Proof. (i) For the support functionals, $\varphi_k = \sigma_{x_k - F_k^*} \to \sigma_C = \varphi$ in the sense of epi-convergence, cf. [31, Corollary 3.4.5]. Whenever we have $\varphi(y) < \infty$, by [31, Lemma 6.2], there exist $y_k \in \Omega^*$ such that $y_k \to y$ and $\varphi_k(y_k) \to \varphi(y)$. Since $\varphi(y) < \infty$, we may assume $y_k \in \text{dom} \varphi_k = F_k$. Therefore, $y = \lim_k y_k \in \lim_k F_k = F$.

(ii) Let $(x_k)$ be bounded and assume there is some $y \in F$ such that $\varphi(y) = \infty$. Then by [31, Lemma 6.2] $\liminf_k \varphi_k(y_k) \geq \varphi(y) = \infty$ for any $y_k \in F_k, y_k \to y$. In particular,

$$(y_{\alpha(k)} : x_{\alpha(k)}) = \varphi_{\alpha(k)}(y_{\alpha(k)}) \to \infty$$

for some subsequence $\alpha$. Seeing that $(x_k)$ and $(y_k)$ are bounded, this is a contradiction. Thus, $\text{dom} \varphi = F$. Then there exists a unique $x \in F^\circ$ such that $C = x - F^*$, by [31, Lemma 6.1]. Let
\( z = \lim_k x_{\alpha(k)} \) be any accumulation point of \((x_k)\), and take \( u \in X \). We may write \( u = v - w \) for \( v, w \in \Omega^* \). By [31, Lemma 6.2], there exist \( v_k, w_k \), such that \( v_k \to v, w_k \to w \), and

\[
(z : u) = \lim_k (x_{\alpha(k)} : v_{\alpha(k)} - w_{\alpha(k)}) = \lim_k \varphi_k(v_k) - \lim_k \varphi_k(w_k)
= \varphi(v) - \varphi(w) = (x : u).
\]

Hence, \( x = z \), and thus \( \lim_k x_k = x \).

(iii) Now, consider the case that \((x_k)\) is unbounded and that \( \lim \dim F_k = m \). Define \( y_k = \|x_k\|^{-1} \cdot x_k \). Passing to a subsequence, we may assume \( y = \lim_k y_k \) exists, and \( \dim F_k = m \) for all \( k \). We have \( y \in F^* \), by continuity of polarity [5, Corollary 7.2.12], and \( \langle F \rangle = \lim_k \langle F_k \rangle \) by Proposition 11(ii). Consequently, \( y \in F^* \).

The exposed face \( E \) of \( F^* \) generated by \( y \) satisfies

\[
-\hat{E}^* = \hat{E} - F = \mathbb{R}_{\geq 0} \cdot y - F^*,
\]

so in order to prove \( C - \hat{E}^* = C \), it suffices to prove \( C + \mathbb{R}_{\geq 0} \cdot y \subseteq C \). Let \( \lambda \geq 0 \) and set \( \lambda_k = \|x_k\|^{-1} \). For any \( f_k \in F_k \) such that \( x_k - f_k \) converges,

\[
x_k - (1 - \lambda \lambda_k) \cdot f_k = \lambda \cdot y_k + (1 - \lambda \lambda_k)(x_k - f_k) \to \lambda \cdot y + \lim_k (x_k - f_k),
\]

and hence \( \lambda \cdot y + \lim_k (x_k - f_k) \subseteq \lim_k (x_k - F_k) = C \). Thus, \( C - \hat{E}^* = C \).

There exist \( y_k \in F_k^\circ \), \( y_k \to y \). Let \( E_k = F_k \cap y_k^\perp \subseteq P \). Clearly, \( \lim_k E_k \subseteq F \cap y^\perp \). Let \( f \in F \cap y^\perp \). There exist \( f_k \in F_k \), \( f_k \to f \). We can write \( f_k = e_k + u_k \) with uniquely determined \( e_k \in E_k \) and \( u_k \in -E_k^\circ \cap \langle F_k \rangle \). By Corollary 12(i), \( e_k \) converges to the projection of \( f \) onto \( F \cap y^\perp \), which is \( f \). This implies \( F \cap y^\perp \subseteq \lim_k E_k \), so \( E_k \to F \cap y^\perp \). Moreover,

\[
x_k - E_k^* = x_k - F_k^* - \hat{E}^* \to C - \hat{E}^* = C,
\]

so \( \text{dom} \varphi \subseteq F \cap y^\perp \) by the first part. Hence, \( \dim \text{dom} \varphi \leq \liminf_k \dim E_k \).

We need to see that eventually, \( y \not\subseteq F_k \). If it were true that \( y \perp F_k \) frequently, then, passing to a subsequence, we could assume that \( y \perp F_k \) for all \( k \). Hence, \( y \perp F \). But this would imply \( y \in \langle F \rangle \cap F^\perp = 0 \), a contradiction. Thus, eventually, \( y \not\subseteq F_k \), so \( \dim E_k \leq m - 1 \), and this proves that \( \dim \text{dom} \varphi < m \). \( \Box \)

**Lemma 14.** The map

\[(\pi, \lambda) : Y_j \to P \times X^+\]

has closed graph, where \( X^+ \) is the one point compactification of \( X \).

**Proof.** Let \( x_k - F_k^* \to x - F^* \) where \( F_k, F \in P_j \) and \( x_k \in F_k^\circ \), \( x \in F^\circ \). Further, let \( F_k \to E \in P \). By Lemma 13(i), \( F = \text{dom} \sigma_{x - F^*} \subseteq E \). But \( \dim E \leq n_{d - j} \) by Proposition 11, which proves that \( E = F \). By Lemma 13(iii), \((x_k)\) is bounded, so by part (ii) of that lemma, \( x = \lim_k x_k \). \( \Box \)

Now we are ready to prove Theorem 9.
Proof of Theorem 9. (i) If \( \overline{G} = U_{d+1} \), then the latter is compact. Let \( F_k \in P \), such that \( F_k \to F \in F(X) \). Then \( F \) is a convex cone. Passing to subsequences, we may assume \(-F_k^* = y - E^* \in U_{d+1}\). On the other hand, continuity of polarity gives \(-F_k^* \to -F^*\). Thus \( E = \text{dom}\, \sigma_{y-E^*} = \text{dom}\, \sigma_{-F^*} = F \in P \).

(ii) The map \((\pi, \lambda): Y_j \to P \times X^+\) has compact range and closed graph, by Lemma 14. Hence, it is continuous.

(iii) Let \( x_k - F_k^* \to A \in F(X) \) where \( F_k \in P \) and \( x_k \in F_k^\circ \). Since \( n = \dim X \) is finite, there exists \( 0 \leq j \leq d \) such that \( F_k \in P_j \) frequently. Passing to subsequences, we may assume \( F_k \in P_j \) for all \( k \), and \( F_k \to F \in P_j \). Let \( C = \text{dom}\, \sigma_A \). If \( \dim C < n_{d-j} \), then \( C \neq F \), and by Lemma 13(ii), \((x_k)\) is unbounded. Lemma 13(iii) provides us with \( E_k, E \in P \), such that \( x_k - E_k^* \to A, E_k \to E \), and \( \dim E_k, \dim E < n_{d-j} \).

We may write \( x_k = u_k + v_k \) where \( u_k \in E_k^\circ \) and \( v_k \perp E_k \). We claim that \( u_k - E_k^* \to A \). Let \( w \in A \). Then there exist \( e_k \in E_k^\perp \) such that \( x_k - e_k \to w \). Then \( v_k \in E_k^\perp \subset -E_k^* \), and

\[
w = \lim_k (u_k + v_k - e_k) \in \lim_k (u_k - E_k^*)\]

Conversely, let \( \alpha(\mathbb{N}) \) be cofinal in \( \mathbb{N} \) and \( e_{\alpha(k)} \in E_{\alpha(k)}^\circ \) such that \( w = \lim_k (u_{\alpha(k)} - e_{\alpha(k)}) \) exists in \( X \). Then \( v_{\alpha(k)} \in E_{\alpha(k)}^\perp \subset E_k^* \), and

\[
w = \lim_k (x_{\alpha(k)} - v_{\alpha(k)} - e_{\alpha(k)}) \in \lim_k (x_k - E_k^*) = A.
\]

Thus, \( \overline{\lim}_k (u_k - E_k^*) \subset A \), and this proves our claim.

Proceeding inductively (replace \( x_k \) by \( u_k \) and \( F_k \) by \( E_k \)), we may assume that we have \( \dim C = n_{d-j} \), so that \((x_k)\) is bounded by Lemma 13(iii). Then \( C = F \) and \( A = x - F^* \) where \( x = \lim_k x_k \), by part (ii) of the lemma. Thus, we conclude \( \overline{G} = U_{d+1} \), which proves the theorem. \( \square \)

Corollary 15. If the \( P_j \) are compact for all \( j \), then

\[
\mathcal{W}_\overline{G} = \{ (x - F^*, y_1 + y_2 - x) \mid x, y_1 \in F^\circ, y_2 \in F^\perp, F \in P \},
\]

with range and source given by

\[
r(x - F^*, y_1 + y_2 - x) = x - F^* \quad \text{and} \quad s(x - F^*, y_1 + y_2 - x) = y_1 - F^*.
\]

Proof. The condition \( s(x - F^*, y) \in \overline{G} \) reads \( x + y \equiv y_1 \) (mod \( F^\perp \)) for some \( y_1 \in F^\circ \), so we may set \( y_2 = x + y - y_1 \). \( \square \)

2.3. Transversals and the spectrum of \( C^*(\mathcal{W}_\overline{G}) \)

As is suggested by our study of \( \overline{G} \), we shall now always assume that \( P_j \) be compact for all \( j \). Moreover, somewhat abusing notation, we shall identify \( \overline{G} \) with its image under \((\pi, \lambda)\), i.e. we let \( x - F^* \equiv (F, x) \). Of course, one should beware that the components of \((F, x) \in \overline{G} \) depend continuously on \( x - F^* \) only when the latter is restricted to \( Y_j \), and not globally.

From [30, Example 2.7] recall that an abstract transversal \( T \) of some locally compact groupoid \( G \) is a closed subset of the unit space \( G^{(0)} \) meeting each orbit of the right action of \( G \) on \( G^{(0)} \), such that \( r|G_T \) and \( s|G_T \) are open, where \( G_T = s^{-1}(T) \).
Proposition 16. The natural embedding $P \to \Omega : F \mapsto (F,0)$ is a homeomorphism onto its closed image. Thus identified with its image in $\Omega$, $P_j$ is an abstract transversal for $W_\Omega|Y_j$. Therefore, $M_j = s^{-1}(P_j) = W_\Omega\cdot P_j$ is a topological $(W_\Omega|Y_j, \Sigma_j)$-equivalence, where $\Sigma_j = W_\Omega|P_j$ is an Abelian group bundle with unit space $P_j$.

Proof. The continuity of the embedding is just continuity of polarity, cf. [5, Corollary 7.2.12]. Since $P$ is compact, the embedding is topological. If $(F,x) \in Y_j$, then the groupoid element $\gamma = (F,x,\cdot x) \in W_\Omega|Y_j$ satisfies $(F,x).\gamma = (F,0) \in P_j$, so $P_j$ meets every orbit in $Y_j$. To check the openness of $r$ and $s$ on $M_j$, we first determine $M_j$.

Indeed, $M_j = \{(F,x,y - x) \mid F \in P_j, x \in F^\oplus, y \in F^\perp\}$.

As to the openness of $r|M_j$, we can produce a section by $\sigma : Y_j \to M_j : (F,x) \mapsto (F,x,\cdot x) = (F,x,\cdot \lambda(F,x))$.

This section is continuous, since $\lambda$ is continuous on $Y_j$. Similarly, a section for $s|M_j$ is given by $\tau : P_j \to M_j : F \mapsto (F,0,0)$. This section actually extends continuously to a section of $s|W_\Omega\cdot P$ (which is not the case for $\sigma$). Thus $P_j$ is indeed an abstract transversal, and by [30, Example 2.7], $M_j$ is therefore an equivalence.

As to the last statement, it suffices to check that $r|\Sigma = s|\Sigma$ are trivial and that the groups $r^{-1}(F) = s^{-1}(F)$ are Abelian. To that end, note

$\Sigma_j = W_\Omega|P_j = \{(F,0,y) \mid F \in P_j, y \perp F\}$,

so that $r$ and $s$ coincide, and their fibre at $F$ is $F^\perp$, with the usual group structure induced from the ambient vector space $X$. In passing, note that $\Sigma_j$ carries the relative topology induced from $P_j \times X$. □

The existence of a Haar system for $\Sigma_j$ follows from the openness of its range projection, but can also be checked by hand as follows.

Lemma 17. The Abelian group bundle $\Sigma_j$ has a Haar system given by $\lambda^F = \delta_F \otimes \lambda_{F^\perp}$, where $\lambda_{F^\perp}$ denotes Lebesgue measure on the subspace $F^\perp \subset X$ endowed with the induced Euclidean structure.

Proof. We need to check the continuity. To that end, note that $F \mapsto F^\perp : P_j \to \mathbb{F}(X)$ is continuous by Proposition 11. Let $m = n - n_{d-j} = \dim F^\perp$ for $F \in P_j$. Fix $G \in P_j$. For some neighbourhood $U \subset P_j$ of $G$, $p_{F^\perp} : G^\perp \to F^\perp$ is an isomorphism for all $F \in U$. For any $\varphi \in C_c(X)$,

$$\int_{F^\perp} \varphi \, d\lambda_{F^\perp} = \int \varphi \, d\mathcal{H}^m = \sqrt{\det(p_{F^\perp}^*p_{F^\perp})} \cdot \int \varphi \circ p_{F^\perp}^{-1} \, d\lambda_{G^\perp},$$

by the area formula, cf. [16, Corollary 3.2.20]. (Here, $\mathcal{H}^m$ denotes $m$-dimensional Hausdorff measure.) The continuity follows from Lebesgue’s dominated convergence theorem. □
Corollary 18. There is a completion of $C_c(M_j)$ to a $(C^*(\mathcal{W}_\Omega | Y_j), C^*(\Sigma_j))$ equivalence bimodule $C^*(M_j)$, thus establishing a strong Morita equivalence $C^*(\mathcal{W}_\Omega | Y_j) \sim C^*(\Sigma_j)$. Moreover, $C^*(\Sigma_j) \cong C_0(\Sigma_j)$ by Fourier transform. In particular, $C^*(\mathcal{W}_\Omega | Y_j)$ is liminary, of spectrum $\Sigma_j$.

Proof. Strong Morita equivalence follows from Proposition 16 and [30, Theorem 2.8]. Define, for $\varphi \in C_c(\Sigma_j)$, the fibrewise Fourier transform

$$\mathcal{F}(\varphi)(F, y) = \int_{F^\perp} e^{-2\pi i (y \cdot \eta)} \varphi(F, \eta) \, d\eta.$$  

From Euclidean Fourier analysis, $\mathcal{F} : (C_c(\Sigma_j), *) \to C_0(\Sigma_j)$, is a continuous *-morphism. Thus, there exists an extension to a contractive *-morphism $C^*(\Sigma_j) \to C_0(\Sigma_j)$. The image of $\mathcal{F}$ is dense by the Stone–Weierstrass theorem. Indeed, two points $(F_1, y_1), (F_2, y_2) \in \Sigma_j$ with $F_1 \neq F_2$ are easily separated. If $F_1 = F_2 = F$ and $y_1 \neq y_2$, we can separate $y_1$ and $y_2$ by the Fourier transform on $F^\perp$ of some $\varphi \in C_c(F^\perp)$. Now consider

$$\psi(F', y) = \chi(F') \cdot \varphi(p_{F^\perp}(y)) \quad \text{for all } (F', y) \in \Sigma_j,$$

where $\chi \in C(P_j)$ is such that $\chi(F) = 1$. $\psi$ is continuous because the $P_j$ are compact, and by Corollary 12(ii). Moreover, $\mathcal{F}\psi$ separates $(F, y_j), \ j = 1, 2$.

Thus, clearly, the locally compact space $\Sigma_j$ injects into the spectrum of the commutative $C^*$-algebra $C^*(\Sigma_j)$. Conversely, let $\chi$ be a character of $C^*(\Sigma_j)$. Denote by $I_F$ the ideal of $C^*(\Sigma_j)$ generated by the functions vanishing on the fibre of $\Sigma_j$ over $F$. A partition of unity argument shows that for $F \neq F', F, F' \in P_j$, $I_F + I_{F'} = C^*(\Sigma_j)$. Thus, there exists a unique $F \in P$ for which $\chi(I_F) \neq 0$. Clearly, $C^*(F^\perp) = C^*(\Sigma_j)/I_F$, so $\chi$ is given by the Fourier transform with respect to $F^\perp$, evaluated at some $y \in F^\perp$.

Hence, $\Sigma_j$ exhausts the spectrum, and Gelfand’s theorem shows that $\mathcal{F}$ is injective, and therefore an isometric *-isomorphism.

Since strong Morita equivalence implies stable isomorphism for $\sigma$-unital $C^*$-algebras, we find that $C^*(\mathcal{W}_\Omega)$ is stably isomorphic to $C_0(\Sigma_j)$. Indeed, the separability of these $C^*$-algebras follows from the separability of their underlying groupoids.  

\[\square\]

Remark 19. (i) We have $\Sigma_d = \{0\} \times X = Y_d$, so $C^*(\mathcal{W}_\Omega | Y_d) \cong C_0(X)$. Similarly, $\Sigma_0 = \{\Omega^*\} \times 0$ and $Y_0 = \Omega$, so $C^*(\mathcal{W}_\Omega | \Omega) \cong \mathbb{K}$.

(ii) As follows by the theory of *-algebraic bundles, the latter statement of the above corollary is true for any Abelian group bundle endowed with a Haar system, cf. [32, Theorem 1.3.3].

(iii) There is a delicate point to the above equivalences. Namely, although $P$ itself is a compact subset of the unit space $\overline{\Omega}$, meeting each orbit, it is not an abstract transversal in the sense defined above. Indeed, $C^*(\mathcal{W}_\Omega)$ has a faithful irreducible representation, so its spectrum contains a dense point, and unless $X = 0$, the spectrum is non-Hausdorff. However, if $P$ were a transversal, then $C^*(\mathcal{W}_\Omega)$ would be Morita equivalent to a commutative $C^*$-algebra (see Fig. 2).

Which condition fails can be inspected for the example of the classical Wiener–Hopf algebra where $X = \mathbb{R}$ and $\Omega = \mathbb{R}_{\geq 0}$. Then $\overline{X} = [-\infty, \infty]$ under the identification $x \mapsto ]-\infty, x]$, and this gives the order topology for this interval. Similarly, $\overline{\Omega} = [0, \infty]$. In this realisation, the action
of \( \mathbb{R} \) is by translation on \( \mathbb{R} \) and trivial at \( \pm \infty \). Thus, \( \mathcal{W}_{\mathbb{R}_{>0}} \) and \( \mathcal{W}_{\mathbb{R}_{>0}(0, \infty)} \) (note \( P = \{0, \infty\} \)) work out as in the illustration. The range and source projection are given by

\[
\begin{align*}
 r(x, -x) &= x, & r(\infty, x) &= \infty \\
s(x, -x) &= 0, & s(\infty, x) &= \infty.
\end{align*}
\]

Although \( s \) is open (it always is), \( r \) is not, since an open neighbourhood of \( \infty \) projects to the non-open point \( \infty \in [0, \infty] \). The first named author wishes to thank George Skandalis for pointing out this observation.

**Theorem 20.** The sets \( U_j \subset \mathcal{D} \), \( j = 0, \ldots, d + 1 \), form an ascending chain of open invariant subsets. The ideals \( I_j = C^*(\mathcal{W}_{\Omega} | U_j) \) form a composition series with liminary quotients \( I_{j+1} / I_j = C^*(\mathcal{W}_{\Omega} | Y_j) \). Hence, the \( C^* \)-algebra \( C^*(\mathcal{W}_{\Omega}) \) is of type \( I \) (i.e. postliminary). Its spectrum is the set \( \Sigma = \bigcup_{j=0}^{d} \Sigma_j \), the topology of which is given by the sets \( U \cup \bigcup_{i=0}^{j-1} \Sigma_i \) for all \( 0 \leq j \leq d \) and all open \( U \subset \Sigma_j \).

**Proof.** The computation of the quotients is given by any of the following sources: [33, Chapter II, Proposition 4.5], [23, 2.4], [32, Proposition 2.4.2]. We already know \( C^*(\mathcal{W}_{\Omega} | Y_j) \sim C_0(\Sigma_j) \), so this algebra is liminary of spectrum \( \Sigma_j \).

Set \( \Sigma = \mathcal{C}^*(\mathcal{W}_{\Omega}) \). Let

\[
V_j = \{ \varrho \in \Sigma \mid \varrho(I_{j+1}) \neq 0 \} \quad \text{and} \quad W_j = \{ \varrho \in \Sigma \mid \varrho(I_j) = 0 \}.
\]

Then \( V_j \) is open, and \( W_j \) is closed, and we have \( V_j \cap W_j \approx I_{j+1} / I_j \approx \Sigma_j \) [11, Proposition 3.2.1]. If \( U \subset \Sigma_j \) is open, then \( \Sigma_j \setminus U \) is closed in \( V_j \), and hence \( V_j \setminus (\Sigma_j \setminus U) = U \cup \bigcup_{i=0}^{j-1} \Sigma_i \) is open in \( \Sigma \).

Conversely, since \( C^*(\mathcal{W}_{\Omega}) \) has a faithful unitary representation, \( V_0 = V_0 \cap W_0 \approx \Sigma_0 = * \) is dense in \( \Sigma \). Hence, any open \( \varnothing \neq V \subset \Sigma \) is dense. The assertion follows. \( \square \)

**Corollary 21.** The \( C^* \)-algebra \( C^*(\mathcal{W}_{\Omega}) \) is solvable of length \( d \), in the sense of Dynin [13].
Proof. The above composition series is uniquely determined by the requirement that $I_{j+1}/I_j$ be the largest liminary subalgebra of $I_{j+1}/I_j$, by [11, Proposition 4.3.3]. That this requirement obtains in turn follows from [11, Proposition 4.2.6]. Hence, the length is exactly $d$. □

3. Analytical indices

3.1. Continuous fields of elementary $C^*$-algebras

The above considerations show that we have short exact sequences

$$0 \to C^*(\mathcal{W}_\Omega|Y_{j-1}) \to I_{j+1}/I_{j-1} \to C^*(\mathcal{W}_\Omega|Y_j) \to 0.$$ \n
These may be considered as elements $\partial_j \in KK^1(C_0(\Sigma_j), C_0(\Sigma_{j-1}))$, and the corresponding homomorphisms of the $K$-groups are then given by the Kasparov product with $\partial_j$. In order to give an analytical description of the $\partial_j$, we have to compute the subquotients $C^*(\mathcal{W}_\Omega|Y_j)$ of the composition series more explicitly. In fact, we shall exhibit them as continuous fields of elementary $C^*$-algebras, thereby giving an independent proof of results from Section 2.

Fix $0 \leq j \leq d$ and $E \in P_j$. A pair $(\psi_U, U)$ where $U \subset P_j$ is an open neighbourhood of $E$ and $\psi_U : U \times X \to X$ shall be called a positive local trivialisation at $E$ if the following conditions are satisfied:

(i) $\psi_U$ is continuous, $\psi_F = \psi_U(F, \omega) : X \to X$ is bi-Lipschitz for all $F \in U$;
(ii) for a.e. $x \in F^*$, $\psi_F'(x)$ exists, and $\det \psi_F'(x) > 0$;
(iii) for all $F \in U$, $\psi_F$ is linear when restricted to $F^\perp$; and
(iv) for all $F \in U$, $\psi_F(F^\oplus) = E^\oplus$ and $\psi_F(F^\perp) = E^\perp$.

We point out that the derivative exists for almost every $x \in X$, by Rademacher’s theorem, cf. [16, Theorem 3.1.6]. If, moreover, $\det \psi_F'(x) = 1$ whenever $\psi_F'(x)$ exists, then the local trivialisation shall be termed normalised.

Proposition 22. Let $(\psi_U, U)$ be a normalised local trivialisation $E \in P_j$. Then there exists a $\ast$-isomorphism

$$\Psi_U : C^*(\mathcal{W}_\Omega|\pi^{-1}(U)) \to C_0(U \times E^\perp) \otimes C^*(\mathcal{W}_E^\oplus|E^\circledast)$$

given by

$$\Psi_U(\varphi)(F, y, u, v - u) = \int_{E^\perp} e^{-2\pi i(y:z)} \varphi(F, \psi_F^{-1}(u), \psi_F^{-1}(v) + \psi_F^{-1}(z) - \psi_F^{-1}(u)) \, dz$$

for all $\varphi \in C_c(\mathcal{W}_\Omega|\pi^{-1}(U))$, $F \in U$, $y \in E^\perp$, $u, v \in E^\circledast$. Here, recall that

$$\mathcal{W}_E^\oplus|E^\circledast = (\langle E \rangle \times \langle E \rangle)|E^\circledast.$$ \n
Proof. We have seen in Corollary 18 how fibrewise Fourier transform establishes an isomorphism $C^*(U \times E^\perp) \cong C_0(U \times E^\perp)$, where $U \times E^\perp$ is considered an Abelian group bundle.
Thus, it remains to see that
\[ \Phi_U : \big\{ \mathcal{W}_\Omega \big| \pi^{-1}(U) \to (U \times E^\perp) \times (\mathcal{W}_{E^\perp}|E^\otimes) \big\}, \]
\[ (F, x, y_1 + y_2 - x) \mapsto (F, \psi F(y_2), \psi F(x), \psi F(y_1) - \psi F(x)) \]
is a topological isomorphism of groupoids in the sense of Muhly, Renault [29]. That it is a groupoid isomorphism is clear from the linearity of \( \psi F \) on \( F^\perp \); moreover, it is immediate that it is a homeomorphism. Finally, the Haar system of \( \mathcal{W}_\Omega \) is \( \lambda_{F,x} = \delta(F,x) \otimes \lambda_{F,\perp} \otimes \lambda_{E,\otimes} \). We find, by the change of variables formula,
\[ \Phi_U(\lambda_{F,x}) = \Phi_U(\delta_{(F,x)} \otimes \lambda_{F,\perp} \otimes \lambda_{F,\otimes} - x) = \left| \det \psi' F \right| \cdot \left( \delta_F \otimes \lambda_{E,\perp} \otimes \delta_{\psi F(x)} \otimes \lambda_{E,\otimes} - \psi F(x) \right), \]
as required, since \( \det \psi' F = 1 \) a.e. \( \square \)

In order to make this proposition substantial, we need to construct normalised local trivialisations. It is clear that given a positive local trivialisation, it can be normalised. Moreover, Corollary 12 shows that for \( F \) close to \( E \), \( p_F^\perp \) is a linear isomorphism of \( F^\perp \) onto \( E^\perp \). Hence, \( \psi = \psi_U \) can be constructed as \( \psi(F, x) = p_{E} \psi_1(F, p_{F}(x)) + p_{E^\perp} y \) as soon as a map \( \psi_1 \) can be given which satisfies all the conditions of a local trivialisation, apart from linearity on \( F^\perp \) and \( \psi F(F^\otimes) = E^\otimes \).

**Proposition 23.** For \( E \in P_j \), there exists an open neighbourhood \( E \in U \subset P_j \) and a map \( \psi_U : U \times X \to X \) such that:

(i) \( \psi_U \) is continuous, for all \( F \in U \), \( \psi_F = \psi_U(F, \omega) \) is bi-Lipschitz,
(ii) for a.e. \( x \in F^* \), the derivative \( \psi' F(x) \) exists and \( \det \psi' F(x) > 0 \), and
(iii) \( \psi_F(F^\otimes) = E^\otimes \).

In particular, there exists a normalised local trivialisation at \( E \).

First, note the following definition and lemma. Any \( x \in \partial C \) where \( C \subset X \) is closed and convex with \( C^\circ \neq \emptyset \), is called a \( C^1 \)-point, if there is a unique supporting hyperplane at \( x \).

**Lemma 24.** Let \( C \subset X \) be a compact convex neighbourhood of \( 0 \), and \( \mu : X \to [0, \infty[ \) denote its Minkowski gauge functional, i.e.
\[ \mu(x) = \inf\{ \alpha > 0 \mid \alpha^{-1} \cdot x \in C \}. \]
For all \( x \in X \setminus \{ 0 \} \), \( v \in X \), the right directional derivative \( \nabla^+ v \mu(x) = \frac{d}{dt} \mu(x + tv)|_{t=0^+} \) exists, and
\[ \nabla^+ v \mu(x) = \sigma_{n_x(C)}(v) \quad \text{where} \quad n_x(C) = \left\{ y \in N_{\mu(x)}^{1-x}(C) \mid (x : y) = \mu(x) \right\} \]
and \( N_x(C) = \{ y \in X \mid (y : x) = \sigma_C(y) \} \) is the normal cone of \( C \) at \( x \). In particular, \( \mu \) is differentiable at \( x \in X \setminus \{ 0 \} \) if and only if \( \mu(x)^{-1} \cdot x \) is \( C^1 \), with gradient
\[ \nabla \mu(x) = \frac{\mu(x)}{(\pi_{N_{\mu(x)}^{1-x}(C)}(x) : x)} \cdot \pi_{N_{\mu(x)}^{1-x}(C)}(x). \]
Here, we remind the reader that $\pi_C$ denotes the metric projection onto the closed convex set $C$, cf. [38].

**Proof.** Since $\mu$ is convex, [18, §3.2(i), Theorem 1] the right directional derivative $\nabla_v^+ \mu(x)$ exists everywhere and defines a sublinear functional in $v$. By the Hahn–Banach theorem, $\nabla_v^+ \mu(x)$ is the upper envelope of the linear functionals it majorises. By [18, §3.2(i), Theorem 3], the subdifferential of $\mu$ at $x$ is

$$\partial \mu(x) = \{ y \in X \mid -\nabla_v^- \mu(x) \leq (y : v) \leq \nabla_v^+ \mu(x) \}.$$

Since indeed $-\nabla_v^- \mu(x) \leq \nabla_v^+ \mu(x)$, we find

$$\nabla_v^+ \mu(x) = \sup_{y \in \partial \mu(x)} \{ (y : v) \mid -\nabla_v^- \mu(x) \leq (y : v) \leq \nabla_v^+ \mu(x) \} = \sup_{y \in \partial \mu(x)} (y : v).$$

Since $\mu$ is positively 1-homogeneous, $\nabla_v^+ \mu(x) = \nabla_v^+ \mu(rx)$ for all $r > 0$. Hence, we may restrict attention to the case $x \in \partial C$, i.e. $\mu(x) = 1$. By [18, §3.2(i), Lemma to Theorem 4], we have

$$\partial \mu(x) = \{ y \in X \mid 1 = (y : x) \geq (y : z) \text{ for all } z \in C \} = \{ y \in X \mid 1 = (y : x) = \sigma_C(y) \}.$$

Thus,

$$\nabla_v^+ \mu(x) = \sup \{ (y : v) \mid 1 = (y : x) = \sigma_C(y) \} = \sigma_{\pi Nx(C)}(v).$$

This proves the first assertion.

As to the second, if $y = \mu(x)^{-1} \cdot x$ is a $C^1$-point, the normal cone is just the ray $\mathbb{R}_{\geq 0} \cdot \pi_{\pi_{N_y(C)}(x)}$. Note

$$(x : r \cdot \pi_{N_y(C)}(x)) = \mu(x) \iff r = \mu(x) \cdot (\pi_{N_y(C)}(x) : x)^{-1},$$

which implies

$$\nabla_v^+ \mu(x) = \frac{\mu(x)}{(\pi_{N_y(C)}(x) : x)} \cdot (\pi_{N_y(C)}(x) : v).$$

Since the $C^1$-points of $C$ are exactly the boundary points of $C$ at which $\mu$ is differentiable, by [18, §3.2(i), Theorem 5], the assertion follows. \( \square \)

**Remark 25.** We note that the above formula for $\nabla \mu(x)$ at $C^1$-points also follows from [17, Proposition 3.1], who prove

$$\nabla \mu(x) = \frac{x - \pi_C(x)}{(\pi_C(x) : x - \pi_C(x))} \text{ for all } x \in X \setminus C$$

at which $\mu$ is differentiable.

Indeed, let $x \in \partial C$, $y = \pi_{N_y(C)}(x)$ and $z = \mu(y)^{-1} \cdot y$. Then $N_z(C) = N_x(C)$, and moreover, $\pi_C^{-1}(z) = z + N_z(C)$, by [38, §2, Lemma 2.4]. We have $\pi_C(r \cdot y) = z$ for all $r \geq \mu(y)^{-1}$, if $x$ is a $C^1$-point. This implies the above formula.
Proof of Proposition 23. We are done once we have constructed a map satisfying (i) and (ii), and which maps $p_{E^+}(F^{\circ})$ to $E^{\circ}$, since $p_{E}$ sets up an isomorphism $\langle F \rangle \rightarrow \langle E \rangle$ for $F$ close to $E$, by Corollary 12. So we may as well assume that $F \subset \langle E \rangle$ for all $F \in U$. Since we may then let $\psi_{F}$ be the identity on $E^{\perp}$, for simplicity, we may assume the cones we are considering to be solid in $X$.

Let $\xi_{0} \in E^{\circ} \cap E^{\ast \circ}$, $\|\xi_{0}\| = 1$, and $H = \{ x \mid (x : \xi_{0}) = 1 \}$. For $F$ close to $E$, we have $\xi_{0} \in F^{\circ} \cap F^{\ast \circ}$, too. Take $X_{+} = \xi_{0}^{\ast \circ} \subset X \setminus \xi_{0}^{\perp}$ to be the half-space containing $E^{\ast} \setminus 0$. Then $F^{\ast} \subset X_{+}$ for $F$ close to $E$. Let

$$\mu_{F}(x) = \inf \{ \alpha > 0 \mid \alpha^{-1} x \in H \cap F^{\ast} - \xi_{0} \} \quad \text{for all } x \in \xi_{0}^{\perp} = H - \xi_{0},$$

the Minkowski functional of the compact convex set $C_{F} = H \cap F^{\ast} - \xi_{0}$, which is a neighbourhood of zero in $\xi_{0}^{\perp}$. Let

$$\varphi_{F}(x) = \frac{\mu_{F}(x)}{\mu_{E}(x)} \cdot x \quad \text{for all } x \in \xi_{0}^{\perp}.$$ 

Then $\varphi_{F} : \xi_{0}^{\perp} \rightarrow \xi_{0}^{\perp}$, mapping $C_{F}$ to $C_{E}$. We may now define

$$\psi_{F}(x + r \cdot \xi_{0}) = \varphi_{F}(x) + r \cdot \xi_{0} \quad \text{for all } x \perp \xi_{0}, \ r \in \mathbb{R}.$$ 

In particular, $\psi_{F} = \varphi_{F}$ on $\xi_{0}^{\perp}$, and

$$\psi_{F}(x) = (x : \xi_{0}) \cdot \left( \varphi_{F}\left( \frac{x}{(x : \xi_{0})} - \xi_{0} \right) + \xi_{0} \right) \quad \text{for all } x \in X_{+}.$$ 

Then condition (iii) is clearly verified.

As to condition (i), we may assume $B(r, \xi_{0}) \subset H \cap F^{\ast} \subset B(R, \xi_{0})$ for all $F$ and some $0 < r < R$. This implies $r \cdot \|\xi\| \leqslant \mu_{F} \leqslant R \cdot \|\xi\|$. Assume that $\mu_{E}(x) \geqslant \mu_{E}(y)$. Then

$$\mu_{E}(\varphi_{F}(x) - \varphi_{F}(y)) \leqslant \mu_{F}(x) \cdot \mu_{E}\left( \frac{x}{\mu_{E}(x)} - \frac{y}{\mu_{E}(y)} \right) + \mu_{E}\left( \left( \mu_{F}(x) - \mu_{F}(y) \right) \cdot \frac{y}{\mu_{E}(y)} \right).$$

Further,

$$\mu_{E}\left( \left( \mu_{F}(x) - \mu_{F}(y) \right) \cdot \frac{y}{\mu_{E}(y)} \right) \leqslant \mu_{F}(x - y) \leqslant R \cdot \|x - y\|,$$

and

$$\frac{x}{\mu_{E}(x)} - \frac{y}{\mu_{E}(y)} \leqslant \frac{1}{\mu_{E}(x)} \cdot \mu_{E}(x - y) + \mu_{E}\left( \left( \mu_{E}(x)^{-1} - \mu_{E}(y)^{-1} \right) \cdot y \right) \leqslant \frac{R}{\mu_{E}(x)} \cdot \|x - y\| + \left| \mu_{E}(x)^{-1} - \mu_{E}(y)^{-1} \right| \cdot \mu_{E}(y) \leqslant \frac{R}{\mu_{E}(x)} \cdot \|x - y\| + \frac{\mu_{E}(x) - \mu_{E}(y)}{\mu_{E}(x)} \leqslant \frac{1}{\mu_{E}(x)} \cdot (1 + R) \cdot \|x - y\|.$$
Exchanging the role of $x$ and $y$, and noting that $\frac{\mu_F}{\mu_E} \leq \frac{R}{r}$, we find that $\varphi$ is $L$-Lipschitz, where

$$L = \frac{R}{r} \cdot (1 + R) + R.$$  

It follows that $\psi_F$ is $L'$-Lipschitz with $L' = \sqrt{2} \cdot \max(L, \|\xi_0\|)$. Since $\psi_F^{-1}$ is given by exchanging the roles of $E$ and $F$ in the definition of $\varphi_F$, it follows that $\psi_F$ is bi-Lipschitz. As to the joint continuity of $\psi$, it suffices to note

$$\|\varphi_F(x) - \varphi_F(y)\| = |\mu_F(x) - \mu_F(y)| \cdot \|\mu_E(x)^{-1} \cdot x\| \leq \frac{1}{r} \cdot |\mu_F(x) - \mu_F(y)|,$$

and that $\mu_F$ depends continuously on $F$.

Suffices to compute derivatives on $X_+$. By Lemma 24, for $x \perp \xi_0$,

$$\nabla \mu_F(x) = \frac{\mu_F(x)}{\langle \pi_{x,F}(x) : x \rangle} \cdot \pi_{x,F}(x) \text{ whenever the derivative exists,}$$

$\pi_{x,F}$ denoting the metric projection onto the normal cone $N_{\mu(x)^{-1} \cdot x}(C_F)$.

A simple calculation gives, for all $x \perp \xi_0$ for which the derivative exists,

$$\varphi'_F(x)v = \lambda \cdot v + \lambda \cdot q_x(v) \cdot x$$

where

$$q_x(v) = \frac{\langle \pi_{x,F}(x) : v \rangle}{\langle \pi_{x,F}(x) : x \rangle} - \frac{\langle \pi_{x,E}(x) : v \rangle}{\langle \pi_{x,E}(x) : x \rangle},$$

and $\lambda = \frac{\mu_F(x)}{\mu_E(x)} > 0$.

Let $\xi_1 = \|x\|^{-1} \cdot x$, and complete this to an orthonormal basis $\xi_1, \ldots, \xi_{n-1}$ of $\xi_0^\perp$. Then since $q_x(x) = 0$, $\varphi'_F(x)$ has the matrix expression

$$\varphi'_F(x) = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ q_x(\xi_2) & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ q_x(\xi_{n-1}) & q_x(\xi_{n-2}) & \cdots & \lambda \end{pmatrix}.$$

We find $\det \varphi'_F(x) = \lambda^{n-1} > 0$.

Next, let $x \in X_+ = \mathbb{R}_{>0} \cdot \xi_0 \oplus \xi_0^\perp$ be arbitrary, and define $N(x) = \frac{x}{\langle x : \xi_0 \rangle}$. Then

$$\psi'_F(x)v = (v : \xi_0) \cdot (\varphi_F(N(x) - \xi_0) + \xi_0) + (x : \xi_0) \cdot \varphi'_F(N(x) - \xi_0)N'(x)v.$$

Let $\lambda = \frac{\mu_F}{\mu_E}(N(x) - \xi_0)$ and $\xi_1 = \|N(x) - \xi_0\|^{-1} \cdot (N(x) - \xi_0)$. Observe that

$$(x : \xi_0)N'(x)v = v - \frac{(v : \xi_0)}{(x : \xi_0)} \cdot x = \begin{cases} (v : \xi_0) \cdot (\xi_0 - N(x)) & v \in \mathbb{R} \cdot \xi_0, \\ v & v \perp \xi_0. \end{cases}$$

In particular, we note that $N'(x)v \perp \xi_0$ for every $x$, and since $\varphi_F$ is $1$-homogeneous,

$$\varphi'_F(N(x) - \xi_0)(\xi_0 - N(x)) = -\varphi_F(N(x) - \xi_0).$$
In terms of the orthonormal basis $\xi_0, \xi_1, \ldots, \xi_{n-1}$, $\psi'_F(x)$ has the matrix expression

$$
\psi'_F(x) = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & & & \\
\vdots & & \varphi'_F(N(x) - \xi_0) & \\
0 & & & 
\end{pmatrix}.
$$

In particular,

$$
\det \psi'_F(x) = \det \varphi'_F(x) = \lambda^n - 1 = \frac{\mu F}{\mu E} (N(x) - x_0)^{n-1} > 0.
$$

This proves the proposition. \hfill \Box

**Corollary 26.** For $0 \leq j \leq d$, $M_j$ is an oriented real vector bundle over $Y_j$ of rank $n - n_{d-j}$. Similarly, $\Sigma_j$ is an oriented real vector bundle over $P_j$ of rank $n - n_{d-j}$.

Let $E \in P_j$. Then, for the groupoid $G = \mathcal{W}_E \circ |E^\otimes|$, we have

$$
G_v = s^{-1}(v) = \{(u, v - u) \mid u \in E^\otimes\} \text{ for all } v \in E^\otimes,
$$

so we may identify $L^2(G_v)$ with $L^2(E^\otimes)$. The regular representation $\varrho_E$ of $C^*(\mathcal{W}_E \circ |E^\otimes|)$ on this space is given by

$$
\varphi \ast h(u) = \int_{E^\otimes} \varphi(u, w - u) h(w) \, dw \quad \text{for all } \varphi \in C_c(\mathcal{W}_E \circ |E^\otimes|), h \in L^2(E^\otimes).
$$

This is manifestly independent of $v$. In the notation of [29, 2.12.1–2], the representation $\varrho_E$ is just $J^{-1} \text{ind} \delta_0 J$.

On the other hand, for $(F, y) \in \Sigma_j$ define $L_{\Omega}^{F, y} = L^{F, y}$ by

$$
L_{\Omega}^{F, y}(\varphi) h(v) = \int_{F^\perp} \varphi(F, v, w) e^{-2\pi i(w; y)} h(v + p_F(w)) \, dw
$$

$$
= \int_{F^\perp} \int_{F^\otimes} \varphi(F, v, w_1 + w_2 - v) e^{-2\pi i(w_2; y)} h(w_1) \, dw_1 \, dw_2
$$

for all $\varphi \in C_c(\mathcal{W}_\Omega), h \in L^2(F^\otimes)$, and $v \in F^\otimes$. The following proposition is then straightforward.

**Proposition 27.** Let $E \in P_j$, $(\psi_U, U)$ a normalised local trivialisation, and fix some $(F, y) \in \Sigma_j U$. If $\chi_{F, y}$ denotes the character of $C_0(U \times \mathbb{T}^\perp)$ given by evaluation at $(F, z)$, where $(\psi_F(F\perp)^\perp)z = y$, then $(\chi_{F, y} \otimes \varrho_E) \circ \psi_U$ and $L^{F, y}$ are equivalent representations of $C^*(\mathcal{W}_\Omega |\pi^{-1}(U))$.

For measurable $E \subset X$ and functions $f, g$, define the following abbreviations whenever they make sense:
\[ \mathcal{F}_E(f)(x) = \int_E e^{-2\pi i (x:y)} f(y) d\lambda_{\langle E \rangle}(y), \quad \mathcal{F}_E^*(f) = \int_E e^{2\pi i (x:y)} f(y) d\lambda_{\langle E \rangle}(y), \]

\[ f \ast_E g(x) = \int_E f(y) g(x-y) d\lambda_{\langle E \rangle}(y), \]

\[ f^*(x) = \overline{f(-x)}, \quad f^y(x) = f(x+y), \quad e_y(x) = e^{2\pi i (x:y)}. \]

Note the following equations:

\[ (f \ast_E g) \ast_F h = f \ast_E (g \ast_F h) \quad \text{for} \quad F = E = F, \]

\[ f \ast_E g(x) = g \ast_{x-E} f(x) \quad \text{for} \quad x \in E, \]

\[ f \ast_E g^*(x) = (g \ast_{E-x} f)^*(x) \quad \text{for} \quad x \in \langle E \rangle, \]

\[ f^y \ast_E g = f \ast_{E+y} g^y \quad \text{for} \quad y \in \langle E \rangle, \]

\[ \mathcal{F}_F(f) \ast_E \mathcal{F}_G(g) = \mathcal{F}_{F \cap G}(f \cdot g) \quad \text{for} \quad E = \langle F \rangle = \langle G \rangle = \langle F \cap G \rangle, \]

\[ \mathcal{F}_E(f^y) = e_y \cdot \mathcal{F}_{E+y}(f) \quad \text{for} \quad y \in \langle E \rangle, \]

which are standard applications of Euclidean Fourier analysis.

**Proposition 28.** The family \( \mathcal{E}_j = (L^2(F^\oplus))_{(F,y) \in \Sigma_j} \) is a continuous field of Hilbert spaces with a dense subspace \( \Theta \) of sections given by the maps \( (F,y) \mapsto \varphi_{F,y} \), where

\[ \varphi_{F,y}(x) = \mathcal{F}_{F^\perp}(\varphi^x)(y) \quad \text{for all} \quad \varphi \in C_c(X), \quad (F,y) \in \Sigma_j, \quad x \in F^\oplus. \]

**Proof.** By [11, Proposition 10.2.3], it suffices to show that \( \Theta \) is dense in every fibre and that \( \|\varphi\| \) is continuous for all \( \varphi \in \Theta \). The density follows by considering the algebraic tensor product \( C_c(F^\oplus) \otimes C_c(F^\perp) \). Moreover,

\[ \|\varphi_{F,y}\|^2 = \int_{F^\oplus} \mathcal{F}_{F^\perp}(\varphi^x)(y) \cdot \mathcal{F}_{F^\perp}(\varphi^x)(y) \, dx = \int_{F^*} e^{-2\pi i (y:x)}(\varphi \ast_{F^\perp} \varphi^{p_F(x)^*})(x) \, dx. \]

Since the Fourier transform is continuous \( L^1(X) \to C_0(X) \), we need to see that

\[ 1_{F^*}(x) \cdot (\varphi \ast_{F^\perp} \varphi^{p_F(x)^*})(x) = 1_{F^*}(x) \cdot \int_{F^\perp} \varphi(x-w)\varphi(p_F(x)-w) \, dw, \]

viewed as an \( L^1 \) function in \( x \), depends continuously on \( (F,y) \). This follows from Lebesgue’s theorem once we have point-wise continuous dependence, which is ensured by Lemma 29 below. \( \square \)

**Lemma 29.** Let \( \varphi, \psi \in C_c(X) \). Define

\[ \chi(F,u,v) = [\varphi \ast_{F^\perp} \psi^u](u+v) \quad \text{for all} \quad (F,u,v) \in \mathcal{W}_\Omega|Y_j. \]

Then \( \chi \in C_c(\mathcal{W}_\Omega|Y_j) \).
Proof. Clearly, $\chi$ has compact support, and we need to prove continuity. Let $m = n - n_{d-j}$ denote the common dimension of $F_{k}^\perp$, $F \in P_{j}$. Let $(F_{k}, u_{k}, v_{k})$ tend to $(F, u, v)$, and set

$$
\phi_{k}(w) = 1_{F_{k}^\perp}(w) \cdot \varphi(u_{k} + v_{k} - w)\psi(u_{k} + w) \quad \text{for all } k \in \mathbb{N}, \ w \in X.
$$

Then $\chi(F_{k}, u_{k}, v_{k}) = \int \phi_{k} d\mathcal{H}^{m}$ and $\phi_{k}(w) \to 1_{F^\perp}(w) \cdot \varphi(u + v - w)\psi(u + w)$ for all $w \in X$. (Here, $\mathcal{H}^{m}$ denotes $m$-dimensional Hausdorff measure.) There exist $r > 0$, $C > 0$ such that $|\phi_{k}| \leq C \cdot 1_{F_{k}^\perp \cap B_{r}}$. Note that $\mathcal{H}^{m}(F_{k}^\perp \cap B_{r})$ is independent of $k$, since the intersections are just the $m$-dimensional balls of radius $r$ in $F_{k}^\perp$, centred at the origin. Hence, Pratt’s lemma [15, Theorem 1.3.4], implies that $\chi(F_{k}, u_{k}, v_{k}) \to \chi(F, u, v)$. □

Theorem 30. The representation $\sigma_{j} = (L^{F,y})_{(F,y) \in \Sigma_{j}}$ exhibits $C^{\ast}(\mathcal{W}_{ij}|Y_{j})$ as isomorphic to the field of elementary $C^{\ast}$-algebras $\mathbb{K}(\mathcal{E}_{j})$ associated to $\mathcal{E}_{j}$. Moreover, this field is trivial.

Proof. Let $\varphi \in \mathcal{C}_{c}(\mathcal{W}_{ij}|Y_{j})$, and $E \in P_{j}$. By Proposition 23, we may choose a normalised local trivialisation $(\psi_{U}, U)$ at $E$. Proposition 27 shows that $\psi_{U}(L^{F,y}(\varphi))$ depends continuously on $(F, y) \in \Sigma_{j}|U$. In particular, $(F, y) \mapsto \|L^{F,y}(\varphi)\|$ is continuous. By Proposition 22, the image of $L^{F,y}_{\mathcal{W}_{i}}$ on $C^{\ast}(\mathcal{W}_{i}|\pi^{-1}(U))$ is $C^{\ast}(\mathcal{W}_{E^{0}}|E^{0}) \cong \mathbb{K}(L^{2}(E^{0}))$.

By a partition of unity argument, $A = \sigma_{j}(I_{j+1}/I_{j})$ is a locally trivial, field of elementary $C^{\ast}$-algebras. It is clear that $\sigma_{j}$ is injective on $C^{\ast}(\mathcal{W}_{i}|Y_{j})$, so it sets up an isomorphism with $A$.

To see that the $C^{\ast}$-algebra of the continuous field $\mathcal{E}_{j}$ is contained in $A$, it suffices to see that $\vartheta_{\varphi, \psi} : (F, y) \mapsto \psi_{F,Y}^{\ast}(\varphi)$ lies in $A$ for all $\varphi, \psi \in \mathcal{C}_{c}(X)$. Let $s_{F} = 2p_{F} - 1$ and

$$
\chi(F, u, v) = [(\vartheta_{F,Y} \ast F^{\perp} \psi^{\ast})(u + v)](u + v) \quad \text{for all } (F, u, v) \in Y_{j}.
$$

Then $\chi \in \mathcal{C}_{c}(\mathcal{W}_{ij}|Y_{j})$ by Lemma 29. Now,

$$
L^{F,y}_{\mathcal{W}_{i}}(\chi)h(u) = \int_{H^{0}} \mathcal{F}_{F^{\perp}}\left(\left[(\vartheta_{F,Y} \ast F^{\perp} \psi^{\ast})^{v}\right]^{v}\right)(y)h(v) \, dv
$$

$$
= \mathcal{F}_{F^{\perp}}(\psi^{v})(y) \cdot \int_{H^{0}} \mathcal{F}_{F^{\perp}}(\vartheta_{F,Y}^{v})(y)h(v) \, dv = \vartheta_{\varphi, \psi}h(u).
$$

This implies $\mathbb{K}(\mathcal{E}_{j}) \subset A$, and since the former separates points, equality, by [11, Lemma 10.5.3].

The triviality of the field $\mathcal{E}_{j}$ for $j = d$ is clear, since $\Sigma_{d} \approx X$ is contractible. For $j < d$, it follows from [11, Lemma 10.8.7] since its fibre $L^{2}(F^{0})$ is separable, and its base $\Sigma_{j}$ is finite-dimensional by Lemma 31 below. □

Lemma 31. For $0 \leq j \leq d$, the map $P_{j} \to \text{Gr}_{n_{j-d}}(X) : F \mapsto \langle F \rangle$ is a topological embedding onto the Grassmannian of $n_{j-d}$-planes. Consequently, the spaces $P_{j}$ and $\Sigma_{j}$ are finite-dimensional.

Proof. The map is continuous by Proposition 11, and injective since $F = \Omega^{0} \cap \langle F \rangle$. Thus, it is topological, seeing that $P_{j}$ is compact. The image of $P_{j}$ has dimension $\leq n_{d-j} \cdot (n - n_{d-j})$, by [25, Chapter III, §1, Theorem III.1]. Moreover, dimension is invariant under homeomorphisms,

cf. [25, Chapter III, §1, Remark A)]. The finite-dimensionality of $P_j$ follows, and [25, Chapter III, §4, Theorem III.4] entails that of $\Sigma_j \subset P_j \times X$. □

**Remark 32.** Needless to say, our proof of Theorem 30 follows the proof of the corresponding results in [29, Theorems 4.7, 6.4] for polyhedral and symmetric cones quite closely; the main new ingredient being the application of some convex analysis to the construction of local trivialisations.

**Corollary 33.** For $0 \leq j < d$, the $C^*$-algebra $C^*(\mathcal{W}_\Omega|Y_j)$ is stable.

**Proof.** Indeed, the trivial field $\mathcal{E}_j$ has separable infinite-dimensional fibre for $j < d$. □

### 3.2. Analytical index formula

For any Hilbert $C^*$-module $E$, let $Q(\mathcal{E}) = \mathcal{L}(\mathcal{E})/\mathbb{K}(\mathcal{E})$ denote its Calkin algebra. Let $\tau_j : C^*(\mathcal{W}_\Omega|Y_j) \to Q(\mathcal{E}_j)$ be the Busby invariant of the extension from Section 3.1. We call this the $j$th Wiener–Hopf extension.

If $q_j : C^*(\mathcal{W}_\Omega) \to I_{j+1}/I_{j-1}$ is a completely positive contractive section of $\sigma_j$, then $\tau_j = q_j \circ \sigma_j \circ q_j$ where $q_{j-1} : \mathcal{L}(\mathcal{E}_{j-1}) \to Q(\mathcal{E}_{j-1})$ is the canonical projection onto the Calkin algebra of the Hilbert $C_0(\Sigma_j)$-module $\mathcal{E}_{j-1}$, and $\sigma_{j-1} : I_{j+1}/I_{j-1} \to \mathcal{L}(\mathcal{E}_{j-1})$ is the strict extension of $\sigma_{j-1} : C^*(\mathcal{W}_\Omega|Y_{j-1}) \to \mathbb{K}(\mathcal{E}_{j-1})$.

Moreover, by naturality of connecting homomorphisms, $\sigma_{j-1}^* \partial_j = \tau_j^* \partial$ where $\partial$ is the connecting homomorphism of

$$0 \to \mathbb{K}(\mathcal{E}_{j-1}) \to \mathcal{L}(\mathcal{E}_{j-1}) \xrightarrow{q_j} Q(\mathcal{E}_{j-1}) \to 0.$$  

We call an element $a \in C^*(\mathcal{W}_\Omega)$ $j$-Fredholm if it represents an invertible in the unitisation of the quotient $C^*(\mathcal{W}_\Omega)/I_j$. Equivalently, $ab \equiv ba \equiv 1 \pmod{I_j}$ for some $b \in C^*(\mathcal{W}_\Omega)$. More generally, any $a \in C^*(\mathcal{W}_\Omega) \otimes \mathbb{C}^{N \times N}$ which is invertible modulo $I_j \otimes \mathbb{C}^{N \times N}$ shall be called a $j$-Fredholm matrix.

**Proposition 34.** If $a \in C^*(\mathcal{W}_\Omega)$ is $j$-Fredholm, then $\sigma_{j-1}(a) = (L^{F,y}(a))_{(F,y) \in \Sigma_{j-1}}$ is a continuous family of Fredholm operators. The corresponding statement about matrices is also valid.

To that end, we observe the following naturality of the representations $L^{F,y}$.

**Lemma 35.** Let $F \in P$, and let $P_F$ be the set of faces of $F$. Then we may define a $*$-homomorphism $r_F : C^*(\mathcal{W}_\Omega) \to C^*(\mathcal{W}_{F^\perp})$ by

$$r_F(\varphi)(E, u, v) = \int_{F^\perp} \varphi(E, u, y + v) dy \quad \text{for all } (E, u, v) \in \mathcal{W}_{F^\perp}, \varphi \in C_c(\mathcal{W}_\Omega).$$

Moreover, we have

$$L^{E,v}_{F^\perp} \circ r_F = L^{E,v}_\Omega \quad \text{for all } E \in P_F, \quad v \in E^\perp \cap \langle F \rangle.$$
Proof. Observe $E^* = (F) \cap E^* \oplus F^\perp$, since $F^\perp \subset E^\perp$ for all $E \in P_F$. For $\varphi \in C_c(\mathcal{W}_\Omega)$, we compute

$$L_{E,v}^F r_F (\varphi) h(u) = \int_{E^* - u} \varphi(E, u, y + w) e^{-i(w : v)} h(u + p_E(w + y)) dy dw = \int_{E^* - u} \varphi(E, u, w) e^{-i(w : v)} h(u + p_E(w)) dw = L_{E,v}^F \varphi h(u)$$

for all $E \in P_F$, $v \in (F) \cap E^\perp$, $h \in L^2(E^\sideset{\oplus}{\ast}^\ast)$, and $u \in E^\sideset{\oplus}{\ast}^\ast$, since in the integral, $y$ is perpendicular to $v, w$. This proves the second equality. Choosing $E = F, v = 0$, $L_{F,v}^F$ is an isomorphism onto its image, so $r_F$ is bounded, and an involutory algebra homomorphism. This proves the lemma.

Proof of Proposition 34. The statement about Fredholm matrices follows along the same lines as the first assertion, so for the sake of simplicity, we restrict ourselves to $N = 1$. Then the continuous dependence is clear from Proposition 28.

Let $b \in C^*(\mathcal{W}_\Omega)$, $ab \equiv ba \equiv 1 \pmod{I_j}$. Take $(F, y) \in \Sigma_{j - 1}$, and let $E \in P_F$, $E \neq F$. Then $E \in P$, and hence $\dim E \leq n_{d - j}$. Thus,

$$I_j \subset \ker L_{E,v}^\Omega \quad \text{for all } v \in (F) \cap E^\perp.$$

This implies

$$1 = L_{E,v}^\Omega (ab) = L_{E,v}^\Omega r_F (ab) \quad \text{for all } v \in (F) \cap E^\perp.$$

Since $E$ was arbitrary,

$$r_F(ab) - 1 \in \bigcap_{E \in P_F \setminus (F), v \in (F) \cap E^\perp} \ker L_{E,v}^\Omega = (L_{F,v}^F)\,^{-1}\,((\mathbb{K}(L^2(F^\sideset{\oplus}{\ast}^\ast)))),$$

by the composition series for $C^*(\mathcal{W}_F)$. We conclude

$$L_{\Omega}^{F,0}(a)L_{\Omega}^{F,0}(b) - 1 = L_{\Omega}^{F,0}(ab) - 1 = L_{F,v}^F(r_F(ab) - 1) \in \mathbb{K}(L^2(F^\sideset{\oplus}{\ast}^\ast)).$$

If we denote by $e^{-iy^*}$ the bounded continuous function $\mathcal{W}_\Omega \to \mathbb{C}: (E, u, v) \mapsto e^{-i(y : v)}$, then $L_{\Omega}^{F,y}(\varphi) = L_{\Omega}^{F,0}(e^{-iy^*} \cdot \varphi)$. Thus, the above entails

$$L_{\Omega}^{F,y}(a)L_{\Omega}^{F,y}(b) - 1 = L_{\Omega}^{F,0}(e^{-iy^*} \cdot ab) - 1 = L_{F,v}^F(e^{-iy^*} \cdot ab) - 1 \in \mathbb{K}(L^2(F^\sideset{\oplus}{\ast}^\ast)).$$

Similarly, $L_{\Omega}^{F,y}(b)L_{\Omega}^{F,y}(a) - 1$ is compact. Hence, $L_{\Omega}^{F,y}(a)$ is Fredholm.

Recall that $\left[ f \right] \in K^1_c(\Sigma_j)$ is given by a continuous map $f : \Sigma_j \to U(N)$ for some $N \in \mathbb{N}$, such that $(f_{k\ell}) = (\delta_{k\ell})$ outside some compact set. Fix some completely positive cross-section $\varrho_j : \mathbb{K}(E_j) \to I_{j+1}$ of $\sigma_j$. We claim that

$$\varrho_j(f) = 1_N + (\varrho_j(f_{k\ell} - \delta_{k\ell}))_{1 \leq k, \ell \leq N}$$
is a $j$-Fredholm matrix. (Here, we identify $f - 1_N \in C_0(\Sigma_j) \otimes \mathbb{C}^{N \times N}$ with its preimage in $K(\mathcal{E}_j \otimes \mathbb{C}^N)$. Indeed, denoting the unital extension of $\sigma_j \otimes \text{id}_{\mathbb{C}^{N \times N}}$ to the unitisation of $I_{j+1} \otimes \mathbb{C}^{N \times N}$ by $\sigma_j$, 

$$\sigma_j(1 + \varrho_j(f - 1)) = 1 + \sigma_j \varrho_j(f - 1) = f,$$

which is invertible in the unitisation of $K(\mathcal{E}_j \otimes \mathbb{C}^N)$. Since $\ker \sigma_j = I_j$, this means that $\varrho_j(f)$ is a $j$-Fredholm matrix.

It is therefore natural to ask whether the map $\partial_j$ can be interpreted as the Atiyah–Jänich family index of the family $\sigma_{j-1} \varrho_j(f) = 1 + \sigma_{j-1} \varrho_j(f - 1)$ of Fredholm operators. First, we need to see that such a family index is well defined.

**Proposition 36.** Let $[f] \in K^1_c(\Sigma_j)$. Then $\sigma_{j-1} \varrho_j(f)$ is trivial at infinity, i.e. there exists a compact $L \subset \Sigma_{j-1}$ such that

$$L^{F \cdot \gamma} \varrho_j(f_{k\ell} - \delta_{k\ell}) \in K(L^2(F^\otimes)) \quad \text{for all } (F, y) \in \Sigma_{j-1} \setminus L.$$

We first make the following observation. Let $\mathcal{P}$ denote the graph of the order relation $\supset$ of the face lattice $P$, and

$$\mathcal{P}_j = \mathcal{P} \cap (P_{j-1} \times P_j) = \left\{ (E, F) \in P_{j-1} \times P_j \mid E \supset F \right\}.$$

Moreover, denote its projections by $P_{j-1} \xleftarrow{\xi} \mathcal{P}_j \xrightarrow{\eta} P_j$.

**Lemma 37.** The relation $\mathcal{P}$ is closed. Thus, $\mathcal{P}_j$ is a compact subspace of $P_{j-1} \times P_j$. The projections $\xi$ and $\eta$ are continuous, closed, and proper.

**Proof.** Let $(E_k, F_k) \in \mathcal{P}$, $(E_k, F_k) \to (E, F) \in P \times P$. If $e \in E$, then $e = \lim_k e_k$ for some $e_k \in E_k \subset F_k$. Hence, $e \in \lim_k F_k = F$. Therefore, $\mathcal{P}$ is closed. The continuity of $\xi$ and $\eta$ is clear. The closedness and properness follow from the compactness of $\mathcal{P}_j$. \qed

**Proof of Proposition 36.** Let $[f] \in K^1_c(\Sigma_j)$ where $f : \Sigma_j \to U(N)$ for some $N \in \mathbb{N}$ and $f = 1_N$ on $\Sigma_j \setminus K$ where $K$ is compact. Since $\Sigma_j$ is a vector bundle over $P_j$, we may consider $\eta^* K \subset \eta^* \Sigma_j$. Due to the properness of $\eta$, this set is compact. The projection

$$\eta^* \Sigma_j \to \xi^* \Sigma_{j-1} : (E, F, y) \mapsto (E, p_E^\perp(y), F)$$

is continuous, so we obtain a compact subset of $\xi^* \Sigma_{j-1}$ which is necessarily of the form $\xi^* L$ for some compact $L \subset \Sigma_{j-1}$. Explicitly, $L$ may be written down as follows,

$$L = \left\{ (E, v) \in \Sigma_{j-1} \mid \exists F \in \eta(\xi^{-1}(E)), u \in F^\perp \cap \langle E \rangle : (F, u + v) \in K \right\}.$$

Fix $(E, v) \in \Sigma_{j-1} \setminus L$. Let $H \in P_E$, $H \neq E$, and $w \in H^\perp \cap \langle E \rangle$. We have $F \not\subset E$ for every $F \in P_j$, $u \in F^\perp \cap \langle E \rangle$ such that $(F, u + v) \in K$. On the other hand, $H \subset E$, so that $(H, v + w) \not\in K$. Hence,
\[ L_{E}^{H,v}(L_{E}^{E,0})^{-1}L_{E}^{E,y}(f_{kt} - \delta_{kt}) = L_{E}^{H,v}r_{E}(e^{-iy^{\ast}} \cdot \varrho_{j}(f_{kt} - \delta_{kt})) \]
\[ = L_{E}^{H,v+y}(\varrho_{j}(f_{kt} - \delta_{kt})) = (f_{kt} - \delta_{kt})(H, v + y) = 0. \]

Thus,
\[ (L_{E}^{E,0})^{-1}L_{E}^{E,y}(f_{kt} - \delta_{kt}) \in \bigcap_{(H,v)} \ker L_{E}^{H,v} = (L_{E}^{E,0})^{-1}(\mathbb{K}(L^{2}(E^{\otimes}))) \]
for all \((E, y) \in \Sigma_{j-1} \setminus L\), which proves our assertion. \(\Box\)

Proposition 36 enables us to define the Atiyah–Jänich family index of the continuous family \(\sigma_{j-1}\varrho_{j}(f)\) of Fredholm operators, where \(f \in K_{i}^{1}(\Sigma_{j})\), by the following standard device. Consider a filtration \(X_{0} \subset X_{1}^{0} \subset X_{1} \cdots \subset \Sigma_{j-1}\) by compact sets whose interiors \(X_{k}^{0}\) are non-void and whose union is \(\Sigma_{j-1}\). For each \(k \in \mathbb{N}\), the index
\[ \text{Index} \sigma_{j-1}\varrho_{j}(f) \mid X_{k} \in K_{i}^{0}(X_{k}) = K_{c}^{0}(X_{k}) \]
is well defined, cf. [3, p. 158], [26, p. 138].

Let \(\xi_{k} = \text{Index} \sigma_{j-1}\varrho_{j} \mid X_{k}^{0} \subset K_{i}^{0}(X_{k}^{0})\), and denote by \(j_{k} : K_{i}^{0}(X_{k}^{0}) \to K_{c}^{0}(X_{k+1}^{0})\) the respective wrong way maps (i.e. extension by zero). Then \(j_{k}(\xi_{k}) = \xi_{k+1}\) for \(k\) large enough, since outside some \(X_{k}^{0}\), \(\sigma_{j-1}\varrho_{j}(f)\) is trivial. If we write \(T = \sigma_{j-1}\varrho_{j}(f)\), this means \(T_{F,y} = 1_{N}\) for \((F, y) \notin X_{k}\), possibly replacing \(T\) by a homotopic family (the set of compact operators is convex). But then \(T_{F,y}(V) = V\) for any \(V\) of finite codimension. By construction of the family index (loc. cit.), this shows that the restriction of \(\xi_{k}\) to \(\Sigma_{j-1} \setminus X_{k}^{0}\) vanishes for \(k > k\). By naturality of the index, [3, p. 159], [26, Lemma 6], the restriction of \(\xi_{k}\) to \(X_{k}^{0}\) is \(\xi_{k}\). Thus, we indeed have \(j_{k}(\xi_{k}) = \xi_{k+1}\). Since \(K_{c}^{0}(\Sigma_{j-1}) = \lim_{k} K_{i}^{0}(X_{k}^{0})\), by [27, Chapter II, Proposition 4.21], we find that there exists a uniquely determined \(\xi \in K_{c}^{0}(\Sigma_{j-1})\) such that its restriction to \(X_{k}^{0}\) is \(\xi_{k}\). We denote the class \(\xi\) by \(\text{Index}_{\Sigma_{j-1}} \sigma_{j-1}\varrho_{j}(f)\).

**Theorem 38.** For \([f] \in K_{i}^{1}(\Sigma_{j})\), we have
\[ \partial_{j}[f] = \text{Index}_{\Sigma_{j-1}} \sigma_{j-1}\varrho_{j}(f), \]
for any choice of completely positive contractive section \(\varrho_{j} : \mathbb{K}(E_{j}) \to C^{\ast}(W_{Uj}^{\otimes} | U_{j+1})\) for \(\sigma_{j}\).

**Proof.** By naturality of connecting maps, it suffices to establish the fact that the connecting map for the extension
\[ 0 \to A \otimes \mathbb{K} \to M(A \otimes \mathbb{K}) \xrightarrow{q} Q(A \otimes \mathbb{K}) \to 0, \]
where \(A = C(Z)\) for some compact space \(Z\), is given by the Atiyah–Jänich index. This follows exactly as for \(Z\) a point. Indeed, let \([u] \in K_{1}(Q(A \otimes \mathbb{K}))\), such that
\[ u^{\ast}u = uu^{\ast} = 1 \pmod{A \otimes \mathbb{K}}. \]
By [28, Propositions 1.5, 1.7], there exists a partial isometry \( v \in M(A \otimes K) \) such that we have \( u - v \in A \otimes K \), and \( 1 - vv^* \) and \( 1 - v^*v \) have finitely generated range. So the ranges are contained in the range of the standard projection \( p_N : A \otimes \ell^2 \rightarrow A \otimes \mathbb{C}^N \) for \( N \gg 0 \). Then

\[
\text{Index}[u] = [1 - v^*v] - [1 - vv^*] = [wp_Nw^{-1}] - [p_N] = \partial[u],
\]

where

\[
 w = \begin{pmatrix}
 v & 1 - vv^* \\
 1 - v^*v & v^*
\end{pmatrix},
\]

which proves the theorem. \( \square \)

**Remark 39.** The above deduction of the analytic expression of the index maps \( \partial_j \) owes much to the exposition of [37] of the index maps for Toeplitz operators; the main differences again being the reconstruction of the Jordan algebraic computations performed there in terms of the convex geometry of the cone, and of course the groupoid framework for the \( \mathcal{C}^* \)-algebras involved. Let us remark that our proof of the topological index formula in [1] uses methods completely different from Upmeier’s, and in particular, contains as a special case an independent proof of the index formula from [36] for symmetric cones.

**References**