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Graph cohomology and Kontsevich cycles☆

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Abstract

We use the duality between compactly supported cohomology of the associative graph complex and the cohomology of the mapping class group to show that the duals of the Kontsevich cycles $[W_\lambda]$ correspond to polynomials in the Miller–Morita–Mumford classes. We also compute the coefficients of the first two terms of this polynomial. This extends the results of (Combinatorial Miller–Morita–Mumford classes and Witten cycles, math.GT/0207042, 2002), giving a more detailed answer to a question of Kontsevich (Commun. Math. Phys. 147(1) (1992) 1) and verifying more of the conjectured formulas of Arbarello and Cornalba (J. Algebraic Geom. 5 (1996) 705).

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0. Introduction

This paper explains the relationship between the Kontsevich cycles in associative graph homology and the Miller–Morita–Mumford (MMM) classes in the cohomology of the mapping class group. We use a version of the forested graph complex of Conant and Vogtmann [3] to go from the double dual of associative graph homology to the cohomology of the mapping class group and we use our cyclic set cocycle [8] to evaluate the MMM classes on the Kontsevich cycles.

Graph homology was introduced by Kontsevich [12,13]. He constructed three graph complexes which are called the "Lie", "associative" and "commutative" graph complexes. In this paper, we study the associative graph complex which is a chain complex $\mathscr{G}^*$ generated by isomorphism classes

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of ribbon graphs (graphs with cyclic orderings of the half-edges incident to each vertex). We use nonstandard notation, indexing the graph complex so that trivalent graphs have degree zero and the boundary map has degree $+1$. This avoids a shift in degree which occurs in the standard formulation of Kontsevich's theorem.

In [12,13], Kontsevich constructed homology and cohomology classes in the associative (ribbon graph) case. We use the proof of this theorem given in [3]. We also use the notation M_g^s to denote the group of isotopy classes of orientation preserving self-homeomorphisms of a connected Riemann surface of genus g with s punctures. This group maps onto the symmetric group on s letters with kernel equal to the mapping class group of genus g surfaces with s marked points. We always assume that $s \geqslant 1$.

Theorem 0.1 (Kontsevich). *The rational finitely supported cohomology of the associative graph complex $\mathscr{G}^*$ is isomorphic to the rational homology of the disjoint union of classifying spaces BM_g^s of mapping class groups M_g^s:*

$$H_c^n(\mathscr{G}^*;\mathbb{Q}) \cong H_n\left(\coprod_{g,s} BM_g^s;\mathbb{Q}\right).$$

In this paper, we use the category of ribbon graphs $\mathscr{F}at$ whose geometric realization $|\mathscr{F}at|$ is homotopy equivalent to the disjoint union of classifying spaces of mapping class groups over all $s \geqslant 1$ with $s \geqslant 3$ when $g=0$ (see [9] where this is shown to follow from Culler–Vogtmann [4]):

$$|\mathscr{F}at| \simeq \coprod_{g,s} BM_g^s.$$

We denote by $\mathscr{G}_*$ the integral finitely supported cochain complex of the associative graph complex $\mathscr{G}^*$. The notation reflects the fact that the boundary map in $\mathscr{G}_*$ has degree -1.

Kontsevich's theorem above can now be restated as saying that there is a rational chain homotopy equivalence between the cellular chain complex of the category $\mathscr{F}at$ and the (associative) graph cohomology complex $\mathscr{G}_*$. We give an explicit description of this chain isomorphism. We also show that the dual Kontsevich cycles in graph cohomology are (pull-backs of) polynomials in the "adjusted" MMM classes. (They are adjusted by subtracting certain boundary classes.)

The main results of this paper were announced in [8] with short proofs. This paper gives more detailed proofs, expresses them in the language of graph cohomology and also extends these results to the next case. The calculation at the end of the paper shows that for $n \neq 1$ we have

$$[W_{n,1}^*] = 3(-2)^{n+3}(2n+1)!!(\widetilde{\kappa}_n\widetilde{\kappa}_1 - \widetilde{\kappa}_{n+1}) - (-2)^{n+2}(2n+5)!!\widetilde{\kappa}_{n+1}.$$

For $n=1$ we divide the right-hand side by 2. Here $\widetilde{\kappa}_n$ is the adjusted MMM class. This is given by taking the tautological bundle over BM_g^s whose fibers are Riemann surfaces with genus g and s punctures and pushing down the $(n+1)$th power of the Euler class of the vertical tangent bundle. Equivalently, $\widetilde{\kappa}_n = \kappa_n - \gamma_n$ where κ_n is the same construction using the tautological bundle with closed fibers having s distinguished points and γ_n is the push-down of the nth power of Euler class along these points. The "dual Kontsevich cycle" W_λ^* is defined below. In the special case $\lambda = (n,1)$, $W_{n,1}^* : \mathscr{G}_{2n+2} \to \mathbb{Q}$ is nonzero on the generator $[\Gamma]^*$ if and only if Γ has exactly two nontrivalent vertices with valence 5 and $2n+3$. The value of $W^*_{n,1}$ on $[\Gamma]^*$ is $\pm 1/|\mathrm{Aut}(\Gamma)|$ where the sign depends on the orientation of Γ.

In more detail the contents of this paper are as follows. In Section 1, we review Kontsevich's definition of graph homology using Conant and Vogtmann's formula for the Kontsevich orientation of a graph. We define $\mathscr{G}_*$ to be the integral finitely supported cochain complex of the associative graph complex $\mathscr{G}^*$. Thus, e.g., $\mathscr{G}_0$ is the group of all integer valued functions f on the set of all isomorphism classes $[\Gamma]$ of oriented trivalent ribbon graphs so that $f[\Gamma]=0$ for all but a finite number of $[\Gamma]$ and $f[-\Gamma]=-f[\Gamma]$ where $-\Gamma$ is Γ with the opposite orientation. This complex has an augmentation map

$$\varepsilon : \mathscr{G}_0 \to \mathbb{Q},$$

given by sending each dual generator $[\Gamma]^*$ to $o(\Gamma)/|\mathrm{Aut}(\Gamma)|$ where $o(\Gamma)=\pm 1$ depending on the orientation of Γ.

We define the *integral subcomplex* $\mathscr{G}^{\mathbb{Z}}_*$ of $\mathscr{G}_*$ to be the subcomplex generated by

$$\langle \Gamma \rangle := |\mathrm{Aut}(\Gamma)|[\Gamma]^*.$$

In $\mathscr{G}^{\mathbb{Z}}_0$ these elements have augmentation ± 1.

For any partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_r)$ of $n=\sum \lambda_i$ we define the *dual Kontsevich cycle* W^*_λ to be the homomorphism

$$W^*_\lambda : \mathscr{G}_{2n} \to \mathbb{Q},$$

which sends each $[\Gamma]^*$ in the Kontsevich cycle W_λ to $o(\Gamma)/|\mathrm{Aut}(\Gamma)|$. Since

$$W^*_\lambda \langle \Gamma \rangle = o(\Gamma) = \pm 1,$$

these are integral cocycles on $\mathscr{G}^{\mathbb{Z}}_*$. These cocycles were considered by Kontsevich [11]. They are the Poincaré duals of certain strata of the moduli space of stable curves.

In graph cohomology, the dual Kontsevich cycles are linear combinations of cocycles given by partition functions associated with certain one-dimensional A_∞ algebras. We give a detailed account of this construction, essentially repeating what Kontsevich says in [13] using the Conant–Vogtmann definition of graph orientation.

For every ribbon graph Γ we next construct an acyclic $\mathbb{Z}$-augmented chain complex $F_*(\Gamma)$ over $\mathscr{G}_*$ so that F_* gives an acyclic carrier, i.e., a functor from the category $\mathscr{F}at$ of all ribbon graphs to the category of augmented chain complexes over $\mathscr{G}_*$. We call it the *forest carrier*. This determines a chain map from the cellular chain complex $C_*(\mathscr{F}at)$ to $\mathscr{G}^{\mathbb{Z}}_*$ which is unique up to homotopy. We show that this map is a rational homotopy equivalence by constructing a rational inverse

$$\psi : \mathscr{G}^{\mathbb{Z}}_* \otimes \mathbb{Q} = \mathscr{G}_* \otimes \mathbb{Q} \to C_*(\mathscr{F}at; \mathbb{Q}).$$

The chain map ψ is defined by *dual cells* $D(\Gamma)$ modelled on the Poincaré duals of the Kontsevich cycles.

In Section 2, we discuss the Stasheff polyhedron. We use the Conant–Vogtmann version of Kontsevich orientation to determine the intrinsic orientation of the Stasheff polyhedron which corresponds to the sign of the simplices in the dual cell $D(\Gamma)$. One of the main purposes of this is to justify the sign convention used in [8]. We also use this discussion to prove that the forest carrier F_* is acyclic as claimed in the previous section.

In Section 3, we discuss the relationship between the adjusted MMM classes $\widetilde{\kappa}_k$ and the dual Kontsevich cycles. We review the formula for the adjusted MMM classes given by the cyclic set cocycle and we show that the dual Kontsevich cycles are polynomials in the adjusted MMM classes. This is a detailed version of a one page argument in [8].

Finally, Section 4 contains the calculation of the coefficients of $[W^*_{n,1}]$ as a polynomial in the adjusted MMM classes. We use the figures from Section 2 which were drawn with this second purpose in mind.

This paper started with a conversation with Karen Vogtmann about graph homology. I should also thank Michael Kleber for some very helpful discussions. Finally, I would like to thank both James Stasheff and the referee for numerous helpful suggestions about this manuscript.

(1) *Kontsevich cycles*:
  (a) category of ribbon graphs $\mathscr{F}at$;
  (b) associative graph cohomology $\mathscr{G}_*$;
  (c) cocycles W^*_λ in graph cohomology;
  (d) partition functions;
  (e) the forest carrier F_*;
  (f) dual cells.
(2) *Stasheff associahedra*:
  (a) Stasheff polyhedron K^n;
  (b) the category $\mathscr{A}_{n+3}$;
  (c) orientation of K^n;
  (d) orientation of K^{odd};
  (e) proof of Proposition 1.21.
(3) *MMM classes*:
  (a) cyclic set cocycle;
  (b) adjusted MMM classes in $H^{2k}(\mathscr{G}_*;\mathbb{Q})$.;
  (c) cup products of adjusted MMM classes;
  (d) computing the numbers $b^{k_*}_{n_*}$;
  (e) Kontsevich cycles in terms of MMM classes;
  (f) computing a^μ_λ.
(4) *Some computations*:
  (a) the degenerate case $n=0$;
  (b) computation of $b^{n+1}_{n,1}$;
  (c) conjectures.

1. Kontsevich cycles

(1) Category of ribbon graphs $\mathscr{F}at$.
(2) Associative graph cohomology $\mathscr{G}_*$.
(3) Cocycles W^*_λ in graph cohomology.
(4) Partition functions.
(5) The forest carrier F_*.
(6) Dual cells.

We review the basic definitions and give an explicit rational homotopy equivalence between the finitely supported cohomology of the associative graph complex and the cellular chain complex of the category of ribbon graphs.

1.1. Category of ribbon graphs $\mathcal{F}at$

By a *ribbon graph* (also known as *fat graph*) we mean a finite connected graph together with a cyclic ordering on the half-edges incident to each vertex. We will use the following set theoretic model for the objects in the category of graphs.

Definition 1.1. Choose a fixed infinite set Ω which is disjoint from its power set. (This occurs, e.g., if every element of Ω is a set having greater cardinality than Ω.) Then by a *graph* we mean a finite subset of Ω (the set of *half-edges*) together with two partitions of the set:

(1) A partition into pairs of half-edges which we call *edges*.
(2) A partition into sets of cardinality ($=$*valence*) $\geqslant 3$ which we call *vertices*.

To avoid straying too far from conventional terminology we refer to the elements of a vertex as *incident half-edges*. Equivalently, we define *incident* to mean not disjoint.

If $e=\{e^-,e^+\}$ is an edge in Γ then the vertices v_1,v_2 incident to e^-,e^+ are the *endpoints* of e. If the endpoints are equal then e is a *loop*. If e is not a loop then we can collapse e to a point forming a new graph

$$\Gamma/e$$

with one fewer edge, one fewer vertex and two fewer half-edges than Γ. Set theoretically, Γ/e is given by merging v_1,v_2 and deleting e^-,e^+.

If Γ is a ribbon graph and e is an edge in Γ which is not a loop then Γ/e can be given the structure of a ribbon graph in the obvious way by letting the new vertex be cyclically ordered as

$$v_*=(h_1,\dots,h_n,k_1,\dots,k_m)$$

if $v_1=(e^-,h_1,\dots,h_n)$ and $v_2=(e^+,k_1,\dots,k_m)$.

Morphisms of graphs and ribbon graphs can be given by collapsing several edges to points and by isomorphisms. In other words, certain subgraphs will be collapsed to points.

By a *subgraph* of a graph Γ we mean a subset of the set of vertices together with all incident half-edges and a set of edges both endpoints of which lie in the chosen set of vertices. For example, we could take all of the vertices and none of the edges. A subgraph will usually not be a graph since it usually has unpaired half-edges. The unpaired half-edges of a subgraph will be called its *leaves*. If the graph Γ is connected, i.e., if it is not the disjoint union of two graphs, then every subgraph is determined by its set of leaves.

A subgraph is a *tree* if it is connected and has one more vertex than edge. A *forest* is a disjoint union of trees. A forest *spans* the graph if it contains all of the vertices. If F is a spanning forest in a graph Γ, let Γ/F be the graph obtained by collapsing each tree in F to a separate point.

By this we mean collapse the edges of each tree to a point. Thus:

(1) the edges of Γ/F are the edges of Γ which do not lie in F;
(2) the vertices of Γ/F are the sets of leaves of the component trees of F.

Definition 1.2. A *morphism* of graphs $\phi:\Gamma_0 \to \Gamma_1$ is defined to be an isomorphism

$$\Gamma_0/F \cong \Gamma_1 \tag{1}$$

for some spanning forest F in Γ_0. In other words, the inverse image of every edge in Γ_1 is an edge in Γ_0 and the inverse image of every vertex of Γ_1 is a tree in Γ_0.

One thing is obvious from this definition. A morphism $\phi:\Gamma_0 \to \Gamma_1$ is uniquely determined by the value of $\phi^{-1}(e)$ for every half edge e in Γ_1. The reason is that this information specifies the forest F and also gives an isomorphism $\Gamma_1 \cong \Gamma_0/F$.

Morphisms of graphs (and ribbon graphs) also have the following left cancellation property.

Proposition 1.3. *Any two morphisms $f,g:\Gamma_0 \to \Gamma_1$ which are equalized by a morphism $h:\Gamma_1 \to \Gamma_2$ are equal, i.e.,*

$$hf = hg \;\Rightarrow\; f = g.$$

Remark 1.4. In category theoretic terminology, this proposition says that morphisms of graphs are *monomorphisms*. They are also obviously *epimorphisms*, i.e., they satisfy both left and right cancellation.

Proof. In order for f,g to be different, there must be a half-edge e in Γ_1 so that $f^{-1}(e) \neq g^{-1}(e)$. But $hf = hg$ cannot send two different half-edges of Γ_0 to the same half-edge in Γ_2. So $h(e)$ must be a vertex v. The inverse image of v is a tree T_0 in Γ_0 and another tree T_1 in Γ_1 and the leaves of both trees map bijectively onto the half-edges incident to v. Consequently, f,g give the same bijection of the leaves of T_0 with the leaves of T_1. Any edge in T_0 is uniquely characterized by the partitioning of the set of leaves which would result if we cut the edge. And each interior half-edge of T_0 is determined by the corresponding subset of the set of leaves. Thus, $f^{-1}(e) = g^{-1}(e)$ which is a contradiction. □

If Γ is a ribbon graph then the set of leaves of every tree in Γ inherits a cyclic order. Consequently, Γ/F has an induced structure as a ribbon graph. A graph morphism $\phi:\Gamma_0 \to \Gamma_1$ will be called a *ribbon graph morphism* if it respects these cyclic orderings, i.e., if (1) is an isomorphism of ribbon graphs.

If Γ,Γ' are ribbon graphs, let $\mathrm{Hom}(\Gamma,\Gamma')$ denote the set of all ribbon graphs morphisms $\Gamma \to \Gamma'$. Since this is a subset of the set of all graph morphisms, left and right cancellation hold for these morphisms as well. Thus we get the following corollary where $\mathrm{Aut}(\Gamma) = \mathrm{Hom}(\Gamma,\Gamma)$ is the group of ribbon graph automorphisms of Γ.

Corollary 1.5. $\mathrm{Aut}(\Gamma)$ *acts freely on the right on* $\mathrm{Hom}(\Gamma,\Gamma')$ *and* $\mathrm{Aut}(\Gamma')$ *acts freely on the left.*

Let $\mathscr{F}at$ denote the category of all ribbon graphs and ribbon graph morphisms. Let $|\mathscr{F}at|$ denote its geometric realization

$$|\mathscr{F}at| = \coprod_n \coprod_{\Gamma_* \in \mathscr{N}_n \mathscr{F}at} \Delta^n / \sim .$$

Then we have the following theorem which I learned from Penner [18], which goes back to Strebel [20] but which I prove using Culler–Vogtmann [4]. For details, see [9].

Theorem 1.6.

$$|\mathscr{F}at| \simeq \coprod_{g,s} BM_g^s,$$

where M_g^s is the mapping class group *of a surface of genus g with s punctures, i.e., the group of isotopy classes of orientation preserving self-homeomorphisms of such a punctured surface.*

Many ribbon graphs have an intrinsic orientation in the sense of Kontsevich graph homology.

1.2. Associative graph cohomology $\mathscr{G}_*$

Graph homology (of ribbon graphs) is rationally dual to the homology of the category of ribbon graphs. More precisely, we have an isomorphism between rational compactly supported cohomology of the associative graph complex $\mathscr{G}^*$ and the rational homology of the mapping class group. We will construct an integral chain map which realizes this rational equivalence.

However, the main purpose of introducing graph homology in the present context is to fix our orientation conventions. We use the definitions given in [3]. Since our graphs are connected this agrees with Kontsevich's orientation convention.

Definition 1.7 (Conant–Vogtmann). An *orientation* of a graph is defined to be an orientation on the vector space spanned by the set of vertices and half-edges.

One way to specify an orientation of Γ is to take the vertices of Γ in some order followed by the half-edges in pairs $h, \bar{h}$ forming the edges of Γ with the order of each pair given by some orientation of each edge. We prefer however to use orientations of the form

$$o(\Gamma) = \mathrm{Sgn}(v_1 e_{11} e_{12} \cdots e_{1n_1} v_2 e_{21} \cdots e_{2n_2} v_3 \cdots),$$

where $v_1, \ldots, v_m$ are the vertices of Γ and $e_{i1}, \ldots, e_{in_i}$ are the half-edges incident to v_i in cyclic order. Here Sgn indicates the equivalence class of the permutation up to sign, i.e., if we permute the entries Sgn changes by the sign of the permutation. Note that $o(\Gamma)$ is only a *relative sign*. However, if Γ has a natural orientation, we can assign a value of ± 1 to $o(\Gamma)$ depending on whether it agrees or not with the natural orientation.

Remark 1.8 (Conant–Vogtmann). A ribbon graph has a *natural orientation* if all its vertices have odd valence. This is because the sign of the permutation $e_{i1} e_{i2} \cdots e_{in_i}$ depends only on the cyclic order if n_i is odd. Also the words $v_i e_{i1} \cdots e_{in_i}$ have even length making the order of the vertices irrelevant. Furthermore, such a graph necessarily has an even number of vertices. So, it does not matter if we put the vertex first and then the incident half-edges in cyclic order or the other way around in $o(\Gamma)$. (See Example 1.13.)

Definition 1.9 (Conant–Vogtmann). Given a graph Γ and an edge e in Γ which is not a loop, let Γ/e be the graph obtained from Γ by collapsing the edge e. If an orientation on Γ is given by orienting all edges and ordering the vertices so that the source of e is first and its target is second, the *induced orientation* on Γ/e is given by taking the coalesced vertex to be first and letting the remaining vertices and edges be ordered and oriented as before.

As we warned earlier, we index the Kontsevich graph complex in a nonstandard way using "codimension."

Definition 1.10. A graph has *codimension* n if it is obtained from a trivalent graph by collapsing n edges. The codimension of a graph is also equal to the sum of the codimension of its vertices where the *codimension* of a vertex is defined to be its valence minus 3.

The *associative graph homology complex* can now be defined. For all $n \geqslant 0$ let $\mathscr{G}^n$ be the free abelian group generated by all isomorphism classes $[\Gamma]$ of connected oriented ribbon graphs Γ of codimension n modulo the relation $-[\Gamma]=[-\Gamma]$ where $-\Gamma$ is Γ with the opposite orientation. If Γ has an orientation reversing automorphism this implies that $2[\Gamma]=0$. Define the boundary operator $\partial:\mathscr{G}^n \to \mathscr{G}^{n+1}$ by

$$\partial[\Gamma]=\sum_e [\Gamma/e],$$

where the sum is over all edges in Γ which are not loops.

The compactly supported dual of this complex is the (*associative*) *graph cohomology complex* given as follows.

Definition 1.11. For all $n \geqslant 0$ let $\mathscr{G}_n$ be the additive group of all homomorphisms $f:\mathscr{G}^n \to \mathbb{Z}$ so that $f[\Gamma]\neq 0$ for only finitely many $[\Gamma]$. (In particular, $f[\Gamma]=0$ if Γ has an orientation reversing automorphism.) Thus, $\mathscr{G}_n$ is generated by duals $[\Gamma]^*$ of generators of $\mathscr{G}^n$. The boundary map $d:\mathscr{G}_n \to \mathscr{G}_{n-1}$ is given in terms of these dual generators by

$$d[\Gamma]^*=\sum \ell_i[\Gamma_i]^*, \tag{2}$$

where ℓ_i is equal to the number of edges e in Γ_i so that $\Gamma_i/e\cong\Gamma$ minus the number of edges in Γ_i so that $\Gamma_i/e\cong-\Gamma$. The sum is over a basis for $\mathscr{G}_{n-1}$.

The coefficient ℓ_i in (2) can be written as

$$\ell_i=\frac{|\mathrm{Hom}^+(\Gamma_i,\Gamma)|-|\mathrm{Hom}^-(\Gamma_i,\Gamma)|}{|\mathrm{Aut}(\Gamma)|}\in\mathbb{Z},$$

where $\mathrm{Hom}^{\pm}(\Gamma_i,\Gamma)$ is the set of morphisms $f:\Gamma_i\to\Gamma$ so that the orientation of Γ agrees/disagrees with the orientation induced from Γ_i by f. In other words, ℓ_i is the number of left equivalence classes of morphisms $\Gamma_i\to\Gamma$ counted with sign.

Let r_i be the number of right equivalence classes of such maps counted with sign. Then

$$r_i = \frac{|\mathrm{Hom}^+(\Gamma_i, \Gamma)| - |\mathrm{Hom}^-(\Gamma_i, \Gamma)|}{|\mathrm{Aut}(\Gamma_i)|} \in \mathbb{Z}.$$

So (2) can be written as

$$d\langle \Gamma \rangle = \sum r_i \langle \Gamma_i \rangle,$$

where

$$\langle \Gamma \rangle := |\mathrm{Aut}(\Gamma)|[\Gamma]^*.$$

Definition 1.12. Let $\mathscr{G}_*^{\mathbb{Z}}$ denote the subcomplex of $\mathscr{G}_*$ generated by the elements $\langle \Gamma \rangle$. We call $\mathscr{G}_*^{\mathbb{Z}}$ the *integral subcomplex* of $\mathscr{G}_*$.

The boundary map in $\mathscr{G}_*^{\mathbb{Z}}$ can be described in terms of expanding vertices. If Γ is an oriented ribbon graph, each vertex of valence n can be expanded into two vertices connected by an edge in

$$\frac{n^2 - 3n}{2}$$

different ways. Each of these choices gives a ribbon graph Γ_i with a distinguished edge e and an isomorphism

$$\Gamma_i / e \cong \Gamma,$$

so that Γ_i is unique up to isomorphism over Γ. We give Γ_i the orientation induced from Γ by this isomorphism. The boundary map of the integral subcomplex is then given by

$$d\langle \Gamma \rangle = \sum \langle \Gamma_i \rangle.$$

Example 1.13. Consider the ribbon graphs $\Gamma_0, \Gamma_0', \Gamma_1$ shown in Fig. 1. We take the natural orientation on the trivalent graphs Γ_0, Γ_0' given by taking the vertices in any order with each vertex

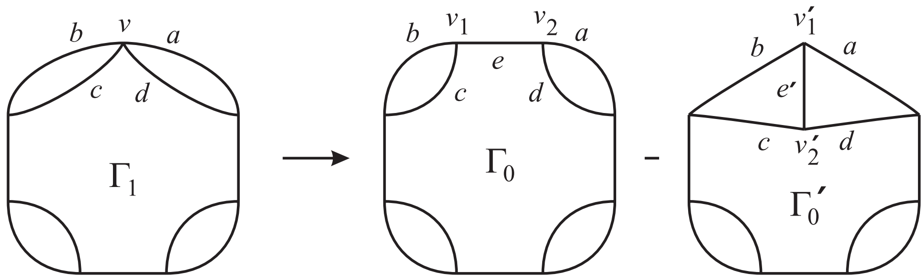

Fig. 1. $d\langle \Gamma_1 \rangle = \langle \Gamma_0 \rangle - \langle \Gamma_0' \rangle$.

followed by the incident half-edges in cyclic order:

$$o(\Gamma_0) = \mathrm{Sgn}(v_1 e_1 bc\, v_2 e_2 da \cdots),$$

$$o(\Gamma_0') = \mathrm{Sgn}(v_1' e_1' abv_2' e_2' cd \cdots).$$

We give Γ_1 the orientation induced by the isomorphism

$$\Gamma_1 \cong \Gamma_0/e.$$

This is given by bringing $v_1 v_2 e_1 e_2$ to the left in $o(\Gamma_0)$ and replacing it with the new vertex v. Thus

$$o(\Gamma_1) = -\mathrm{Sgn}(vbcda \cdots) = \mathrm{Sgn}(vabcd \cdots).$$

The automorphism group $\mathrm{Aut}(\Gamma_0) \cong D_8$ acts transitively on the set of four straight edges of Γ_0 and on the set of eight curved edges. Therefore, the 12 terms in the boundary $\partial[\Gamma_0]$ can be collected as

$$\partial[\Gamma_0] = 4[\Gamma_1] + 8[\Gamma_1'],$$

where Γ_1' is given by collapsing one of the curved edges of Γ_0.

The orientation of Γ_0'/e' is given by bringing $v_1' v_2' e_1' e_2'$ to the left in $o(\Gamma_0')$ and replacing it with v:

$$o(\Gamma_0'/e') = -\mathrm{Sgn}(vabcd \cdots) = -o(\Gamma_1).$$

In other words, $[\Gamma_0'/e'] = -[\Gamma_1]$. Since there are no other edges in Γ_0' equivalent to e', the term $[\Gamma_1]$ occurs only once with a minus sign in $\partial[\Gamma_0']$. Dualizing we get

$$d[\Gamma_1]^* = 4[\Gamma_0]^* - [\Gamma_0']^*.$$

Since the orders of the automorphism groups are $2, 8, 2$, respectively, we get

$$d\langle \Gamma_1 \rangle = \langle \Gamma_0 \rangle - \langle \Gamma_0' \rangle.$$

We will be looking at the rational cochain complex

$$\mathrm{Hom}(\mathscr{G}_*, \mathbb{Q}).$$

This is the rational double dual of the original graph homology complex $\mathscr{G}^*$. Thus cocycles in this complex, such as W_λ^* defined below, are "infinite cycles" in the graph homology complex.

Remark 1.14. Since $\mathscr{G}_n^{\mathbb{Z}}$ is a free abelian group whose generators $\langle \Gamma \rangle$ form a $\mathbb{Q}$-basis for $\mathscr{G}_n \otimes \mathbb{Q}$, its integral dual forms a lattice

$$\mathrm{Hom}(\mathscr{G}_*^{\mathbb{Z}}, \mathbb{Z}) \subseteq \mathrm{Hom}(\mathscr{G}_*, \mathbb{Q}),$$

which we call the *integral cochain complex*. Elements of this subcomplex will be called *integral cochains* on $\mathscr{G}_*$.

1.3. Cocycles W_λ^* in graph cohomology

Definition 1.15. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is a sequence of positive integers let W_λ be the set of all ribbon graphs Γ which are trivalent at all but r vertices $v_1, \ldots, v_r$ which have valence $2\lambda_i + 3$, resp. This set will be called the *Kontsevich cycle*. The *dual Kontsevich cycle* $W_\lambda^* \in \mathscr{G}^{2|\lambda|}$ (where $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_r$) is given by

$$W_\lambda^*[\Gamma]^* := \begin{cases} \dfrac{o(\Gamma)}{|\mathrm{Aut}(\Gamma)|} & \text{if } \Gamma \in W_\lambda, \\ 0 & \text{if } \Gamma \notin W_\lambda, \end{cases}$$

where $o(\Gamma) = \pm 1$ depending on whether the given orientation of Γ agrees with the natural orientation (Remark 1.8).

The dual Kontsevich cycle are integral cochains since they can be given by

$$W_\lambda^*\langle \Gamma \rangle = o(\Gamma)$$

if $\Gamma \in W_\lambda$. Also, note that W_λ, W_λ^* are independent of the order of the λ_i.

In the special case when $r = 1$, W_k is called the *Witten cycle* and W_k^* will be called the *dual Witten cycle*. In the case $r = 0$, $W_\emptyset$ is the set of all trivalent (connected) ribbon graphs and

$$W_\emptyset^* = \varepsilon : \mathscr{G}_0 \to \mathbb{Q}$$

is the map sending the dual $[\Gamma]^*$ of every trivalent graph Γ to $o(\Gamma)/|\mathrm{Aut}(\Gamma)|$. On the integral subcomplex this gives an epimorphism

$$\varepsilon : \mathscr{G}_0^{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}$$

sending each generator $\langle \Gamma \rangle$ to $o(\Gamma) = \pm 1$. We define these maps to be the *augmentation maps* for $\mathscr{G}_*$ and $\mathscr{G}_*^{\mathbb{Z}}$.

We will also consider *degenerate* cases where some of the indices are zero. We interpret these 0's as counting the number of distinct trivalent vertices:

$$W_{0^k}^*[\Gamma]^* := \begin{cases} \dbinom{n}{k} \dfrac{o(\Gamma)}{|\mathrm{Aut}(\Gamma)|} & \text{if } \Gamma \in W_\emptyset \text{ with } n \text{ vertices}, \\ 0 & \text{if } \Gamma \text{ is not trivalent}. \end{cases}$$

Proposition 1.16. *Each dual Kontsevich cycle W_λ^* is an integral cocycle.*

Proof. Let $n = 2|\lambda|$. Then we want to show that

$$W_\lambda^*(d\langle \Gamma \rangle) = 0$$

for all oriented ribbon graphs Γ of codimension $n+1$. However, the only case in question occurs when Γ has only one even valent vertex, call it v_0. The orientation on Γ can be given by first taking v_0, then the incident half-edges $e_1, \ldots, e_{2m}$, then all other vertices with their incident half-edges in cyclic order. The orientation depends on which of the half-edges at v_0 is first.

There are three cases.

Case 1: The graph Γ has $r-2$ odd valent vertices of codimension $\geqslant 2$. After re-indexing the λ_i we may assume that these codimensions are $2\lambda_3, 2\lambda_4, \ldots, 2\lambda_r$. The even valent vertex v_0 must have valence $2\lambda_1 + 2\lambda_2 + 4$ and it needs to split into two vertices of codimension $2\lambda_1$ and $2\lambda_2$. There is always an even number of ways to do this ($2\lambda_1+2$ ways if $\lambda_1=\lambda_2$ and $2\lambda_1+2\lambda_2+4$ ways if not) and half of them will give one sign and half the other. (The sign alternates as we rotate the half edges incident to v_0.) Consequently, the value of W_λ^* on $d[\Gamma]^*$ will be zero.

Case 2: The graph Γ has $r-1$ odd valent vertices of codimension $\geqslant 2$. We may assume that these codimensions are $2\lambda_2,\dots,2\lambda_r$ (after re-indexing the λ_i). The vertex v_0 must have valence $2\lambda_1+4$ and it needs to split into two vertices of valence $2\lambda_1+3$ and 3. There are $2\lambda_1+4$ ways to do this and half of them will give one sign and half the other.

Case 3: v_0 has valence 4. It can split into two trivalent vertices in two ways with opposite sign as we saw in Example 1.13. □

Proposition 1.16 also follows from an observation of Kontsevich that A_∞ superalgebras give partition functions on ribbon graphs which are cocycles on associative graph cohomology. We explain this sophisticated point of view in the following subsection which is not necessary to understand the rest of the paper.

1.4. Partition functions

In [12,13], Kontsevich explains how a finite-dimensional A_∞ superalgebra A gives a cocycle on the associative graph cohomology complex $\mathscr{G}_*$. Kontsevich assumed that A was an algebra over the real numbers. However, it is easy to see that the ground field can have any characteristic. In fact, we only need to assume that A is a finitely generated free module over a commutative ring R.

We will go over the definition of an A_∞ superalgebra following Getzler and Jones [5]. Then we revise Kontsevich's definition of the partition function using the Conant–Vogtmann definition of graph orientation. Finally, we examine the special case of one-dimensional algebras to verify Kontsevich's claim [13] that the cocycles coming from these examples linearly span the space of polynomials in the MMM classes. Translated into the present setting, these cocycles are easily seen to be linear combinations of the dual Kontsevich cycles (which come from Kontsevich's earlier paper [11]).

Definition 1.17. By an A_∞ *superalgebra* we mean a $\mathbb{Z}/2$-graded algebra $A=A_0\oplus A_1$ over a commutative ring R together with a sequence of R-linear mappings

$$m_k : A^{\otimes k}\to A,\quad k\geqslant 1,$$

which are homogeneous of degree $k \,(\mathrm{mod}\, 2)$ so that for homogeneous elements $x_1,\dots,x_k$ we have

$$\sum_{r+s+t=k}(-1)^u m_{r+1+t}(x_1,\dots,x_r,m_s(x_{r+1},\dots,x_{r+s}),x_{r+s+1},\dots,x_k)=0,$$

where $u=r+st+s|x_1|+\cdots+s|x_r|$.

Suppose that $A\cong R^n$ is finitely generated and free as an R-module. Suppose that $m_1=0$. And suppose that we have an *nondegenerate even scalar product*

$$\langle,\rangle : A\otimes A\to R.$$

This means the following.

(1) $\langle a,b\rangle = 0$ if $|a|+|b| = 1$.
(2) $\langle a,b\rangle = (-1)^{|a|}\langle b,a\rangle$. (This implies (1) if 2 is not a zero divisor in R.)
(3) There is a degree 0 R-linear isomorphism

$$D\colon \mathrm{Hom}_R(A,R) \xrightarrow{\approx} A,$$

so that $\langle a, Df\rangle = f(a)$.
(4) For all $x_0,\ldots,x_n \in A_0 \coprod A_1$ we have

$$\langle m_n(x_1,\ldots,x_n),x_0\rangle = (-1)^{n+|x_0|+n|x_0|}\langle m_n(x_0,\ldots,x_{n-1}),x_n\rangle.$$

Then we have a *partition function*

$$Z_A : \mathscr{G}_*^{\mathbb{Z}} \to R,$$

given on any generator $\langle \Gamma\rangle$ as follows.

First, choose an ordering for the vertices $v_1,v_2,\ldots$ of Γ. Next, label the half-edges incident to each v_i in *reverse* (*clockwise*) order $e_{i1},\ldots,e_{in_i},e_{i0}$ (if v_i has valence n_i+1). Let $\varepsilon_1 = \pm 1$ so that

$$o(\Gamma) = \varepsilon_1 \mathrm{Sgn}(v_1,e_{10},e_{1n_1},\ldots,e_{11},v_2,e_{20},e_{2n_2},\ldots,e_{21},v_3,\ldots).$$

Choose an R-basis $b_1,\ldots,b_n$ for A ($b_i \in A_0 \coprod A_1$) and a dual basis $b_1^*,\ldots,b_n^* \in \mathrm{Hom}_R(A,R)$ so that

$$\langle b_i, Db_j^*\rangle = b_j^*(b_i) = \delta_{ij}.$$

We note that if $c_i = \sum \phi_{ij}b_j$ is another basis for A then the corresponding dual basis is $c_i^* = \sum \psi_{ki}b_k^*$ where $(\psi_{ki}) = (\phi_{ij})^{-1} \in GL(n,R)$.

The partition function is given by the state sum

$$Z_A\langle\Gamma\rangle = \varepsilon_1 \sum_{\text{states}} \prod_i \langle m_{n_i}(x_{i1},\ldots,x_{in_i}),x_{i0}\rangle \varepsilon_2 \prod_j \langle D\bar{y}_j^*, Dy_j^*\rangle. \tag{3}$$

The sum is over all states where a *state* of Γ is given by assigning a basis element x_{ij} of A to each half-edge e_{ij} of Γ. The first product is over all vertices v_i. The second product is over all edges $(h_j,\bar{h}_j)$. Here y_j^* represents the dual basis element corresponding to the basis element y_j assigned to h_j and similarly for $\bar{y}_j^*$. The sign $\varepsilon_2 = \pm 1$ is the sign of the permutation of the odd half-edges (those assigned elements of A_1 as basis elements) as they appear in the sequence:

$$e_{11},e_{12},\ldots,e_{1n_1},e_{10},e_{21},\ldots,e_{2n_2},e_{20},e_{31},\ldots,$$

which places each next to its other half (placing $\bar{h}_j$ next to and on the right of h_j). If $h_j,\bar{h}_j$ are switched, the sign of ε_2 changes but so does the sign of $\langle D\bar{y}_j^*, Dy_j^*\rangle$. So, the sign of the expression $\varepsilon_2 \prod_j \langle D\bar{y}_j^*, Dy_j^*\rangle$ is independent of the choice of orientation of the edges.

Theorem 1.18 (Kontsevich). *Z_A is a cocycle on $\mathscr{G}_*^{\mathbb{Z}}$.*

Remark 1.19. The usual definition of the partition function has a factor of $1/|\mathrm{Aut}(\Gamma)|$:

$$Z_A[\Gamma]^* = \frac{\varepsilon_1}{|\mathrm{Aut}(\Gamma)|} \sum_{\text{states}} \prod_i \langle m_{n_i}(x_{i1},\ldots,x_{in_i}),x_{i0}\rangle \varepsilon_2 \prod_j \langle D\bar{y}_j^*, Dy_j^*\rangle.$$

This factor disappears on the integral subcomplex $\mathscr{G}_*^{\mathbb{Z}}$ since $\langle\Gamma\rangle = |\mathrm{Aut}(\Gamma)|[\Gamma]^*$.

Proof. It is easy to see that the partition function is well defined. For every ordered pair (i,j), the basis element x_{ij} and its dual appears (as one of the y_j's) exactly once as a variable to be summed over. So, formula (3) for $Z_A(\Gamma)$ is independent of the choice of basis. (If we sum over a different basis $c_i=\sum\phi_{ij}b_j$, the dual basis $c_i^*=\sum\psi_{ki}b_k^*$ also appears. But $\sum\psi_{ki}\phi_{ij}=\delta_{kj}$. So, this is the same as the original sum over b_j and b_j^*.) If we transpose two vertices v_1,v_2 then both ε_1 and ε_2 change by a factor of $(-1)^{n_1n_2}$. Finally, if we cyclically permute the half-edges around a vertex v of valence $n+1$ then the signs $\varepsilon_1,\varepsilon_2$ and the value change by factors of

$$\varepsilon_1'/\varepsilon_1=(-1)^n,$$

$$\varepsilon_2'/\varepsilon_2=(-1)^{|x_0|\sum|x_i|},$$

$$\text{value}'/\text{value}=(-1)^{n+|x_0|+|x_0|n}.$$

The product of these factors is 1 since the degrees of $x_0,\dots,x_n$ must add up to $n \bmod 2$. (Otherwise expression (3) is zero.) Thus Z_A is well defined. It remains to show that Z_A is a cocycle.

The boundary of any generator $\langle\Gamma\rangle$ in $\mathscr{G}_*^{\mathbb{Z}}$ is a sum over all vertices v of Γ of all ribbon graphs Γ' obtained from Γ by expanding v into two vertices. For each fixed v the sum of the values of $Z_A(\Gamma')$ add up to zero. To see this we label the half-edges clockwise around v. This means that the Conant–Vogtmann orientation starts as

$$o(\Gamma)=\mathrm{Sgn}(v\,e_0\,e_n\cdots e_1\,v'\cdots).$$

When we expand v we get Γ' with orientation

$$\begin{aligned}o(\Gamma')&=\mathrm{Sgn}(v_1v_2h\bar{h}e_0e_n\cdots e_1v'\cdots)\\&=(-1)^u\,\mathrm{Sgn}(v_1he_{r+s}\cdots e_{r+1}v_2e_0e_n\cdots e_{r+s+1}\bar{h}e_r\cdots e_1v'\cdots),\end{aligned}$$

where $n=r+s+t$ with

$$\varepsilon_1=(-1)^u=(-1)^{st+s+t+1}=(-1)^{r+st+n+1}.$$

The corresponding terms of the partition function are

$$\langle m_s(x_{r+1},\dots,x_{r+s}),y\rangle\langle m_{r+t+1}(x_1,\dots,x_r,\bar{y},x_{r+s+1},\dots,x_n),x_0\rangle\langle D\bar{y}^{\,*},Dy^*\rangle \tag{4}$$

with associated relative sign term

$$\varepsilon_2=(-1)^{s|x_1|+\cdots+s|x_r|},$$

since the degrees of $x_{r+1},\dots,x_{r+s},y$ must add up to r.

Using the identity

$$\sum_i\langle x,b_i\rangle\langle y,Db_i^*\rangle=\langle x,y\rangle,$$

we see that expression (4) contracts to

$$\langle m_{r+t+1}(x_1,\dots,x_r,m_s(x_{r+1},\dots,x_{r+s}),x_{r+s+1},\dots,x_n),x_0\rangle,$$

when summed over all allowed values of $y, \bar{y}, y^*, \bar{y}^*$. By definition of an A_∞ algebra, the product of this with $\varepsilon_1\varepsilon_2$ adds up to zero if we sum over all Γ' obtained from Γ by expanding v since n is constant. We need the assumption $m_1 = 0$ since Γ' has no bivalent vertices. □

Suppose that $x = (x_0, x_1, x_2, \ldots)$ is an infinite sequence of rational numbers. Then Kontsevich points out that there is a one-dimensional A_∞ algebra $A = A_0 = \mathbb{Q}$ with scalar product in which m_{2k} is multiplication by x_{k-1}, $m_{\mathrm{odd}} = 0$ and $\langle a, b\rangle = ab$. Since the states of Γ are given by assigning a basis vector to each half-edge, there is only one state and the partition function $Z_x = Z_A$, which is a sum over all states, has only one term.

Example 1.20. The *partition function*

$$Z_x : \mathscr{G}_* \to \mathbb{Q}$$

is the cocycle defined by the equation

$$Z_x[\Gamma]^* = \frac{o(\Gamma)}{|\mathrm{Aut}(\Gamma)|} x_0^{r_0} x_1^{r_1} \cdots$$

if Γ is a ribbon graph with r_i vertices of valence $2i+3$ for $i = 0, 1, 2, \ldots$ and no vertices of even valence.

Since the Euler characteristic of Γ is given by

$$\chi(\Gamma) = -\frac{1}{2} \sum r_i(2i+1),$$

the value of r_0 can be written as

$$r_0 = -2\chi - \sum_{i \geqslant 1} r_i(2i+1). \tag{5}$$

Thus, the partition function Z_x can be given in terms of the dual Kontsevich cycles by

$$Z_x = x_0^{-2\chi} \sum_{\lambda} y^\lambda W_\lambda^*, \tag{6}$$

where $y^\lambda = \prod_i (x_i/x_0^{2i+1})^{r_i}$ if $\lambda = (1^{r_1}, 2^{r_2}, \ldots)$ and the sum is over all λ so that r_0, as given by (5), is nonnegative. (So the right-hand side of (6) is well defined only when χ is fixed.)

Thus, if we restrict to the subcomplex of the graph cohomology complex $\mathscr{G}_*$ generated by $[\Gamma]^*$ where $\chi(\Gamma)$ is fixed, the linear span of these partition functions is the same as the linear span of the W_λ^* and, by Corollary 3.23, this is equal to the algebra generated by the adjusted MMM classes $\widetilde{\kappa}_k$ excluding $\widetilde{\kappa}_0 = \chi$. (However, we lose the linear independence of the $[W_\lambda^*]$ when we restrict to this finitely generated subcomplex of $\mathscr{G}_*$.)

Setting $x_0 = 1$ and taking partial derivatives of (6) with respect to y^λ using finite differences, we can conclude that, for each fixed χ, the dual Kontsevich cycles W_λ^* are linear combinations of partition functions Z_x for various multi-indices x. Consequently, Kontsevich's Theorem 1.18 that each Z_x is a cocycle implies that each W_λ^* is a cocycle.

To pull the cocycles W_λ^* back to the category of ribbon graphs we need to use an acyclic carrier related to the forested graph complex of [3].

1.5. The forest carrier F_*

Conant and Vogtmann use forested graph complexes to show that graph homology is rationally isomorphic to the cohomology of the mapping class group as claimed by Kontsevich. In our notation, the forested graph complex is the total complex of an integral acyclic carrier

$$F_* : \mathscr{F}at \to \mathscr{G}_*^{\mathbb{Z}} \subseteq \mathscr{G}_*,$$

which we call the "forest carrier."

Suppose that Γ_0 is a ribbon graph. Then we will construct a chain complex $F_*(\Gamma_0)$ generated by the isomorphism classes of all ribbon graphs Γ which map to Γ_0. Each such object is given by a "forested ribbon graph," i.e., a ribbon graph Γ with a spanning forest F (so that $\Gamma/F \cong \Gamma_0$). If Γ_0 has codimension n then $F_*(\Gamma_0)$ will be the augmented chain complex

$$0 \to F_n(\Gamma_0) \to F_{n-1}(\Gamma_0) \to \cdots \to F_0(\Gamma_0) \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

given as follows.

Let $F_k(\Gamma_0)$ be the free abelian group generated by all isomorphism classes of codimension k objects in $\mathscr{F}at$ over Γ_0 together with an orientation. In other words, generators of $F_k(\Gamma_0)$ are given by morphisms

$$f : \Gamma \to \Gamma_0,$$

where Γ is a ribbon graph of codimension k together with an orientation on Γ. Two such objects $f_i : \Gamma_i \to \Gamma_0$ for $i = 1, 2$ are *isomorphic* if there is an orientation preserving isomorphism $g : \Gamma_1 \to \Gamma_2$ so that $f_2 \circ g = f_1$. As usual, we equate reversal of orientation with reversal of sign. In particular, $F_n(\Gamma_0)$ has rank 1 with two generators corresponding to the two possible orientations of Γ_0.

The boundary map $d : F_k(\Gamma_0) \to F_{k-1}(\Gamma_0)$ is given by

$$d[f : \Gamma \to \Gamma_0] = \sum [f \circ g_i : \Gamma_i \to \Gamma_0],$$

where the sum is taken over all right equivalence classes of morphisms

$$g_i : \Gamma_i \to \Gamma,$$

which collapse only one edge. We take the unique orientation on each Γ_i which induces the given orientation on Γ.

The augmentation map $\varepsilon : F_0(\Gamma_0) \to \mathbb{Z}$ is given by

$$\varepsilon[\Gamma \to \Gamma_0] = o(\Gamma) = \pm 1.$$

Proposition 1.21. *Suppose that Γ_0 is trivalent except for r vertices $v_1, \ldots, v_r$ which have codimensions $n_1, \ldots, n_r$, resp. Then $F_*(\Gamma_0)$ is based chain isomorphic to the tensor product*

$$F_*(\Gamma_0) \cong C_*(K^{n_1}) \otimes C_*(K^{n_2}) \otimes \cdots \otimes C_*(K^{n_r}),$$

where $C_*(K^m)$ is the cellular chain complex of the m-dimensional Stasheff polyhedron K^m. In particular, $F_*(\Gamma_0)$ is acyclic.

Proposition 1.21 follow from the well-known properties of the Stasheff polyhedron which we will review shortly. Suppose for the moment that this is true.

For each Γ_0 there is a natural augmented chain map $p : F_*(\Gamma_0) \to \mathcal{G}_*^{\mathbb{Z}}$ given by

$$p[f : \Gamma \to \Gamma_0] = \langle \Gamma \rangle.$$

A morphism $g : \Gamma_0 \to \Gamma_1$ induces a chain map $g_* : F_*(\Gamma_0) \to F_*(\Gamma_1)$ by

$$g_*[f : \Gamma \to \Gamma_0] = [g \circ f : \Gamma \to \Gamma_1].$$

This is a chain map over $\mathcal{G}_*^{\mathbb{Z}}$ in the sense that $p \circ g_* = g_*$. Therefore, F_* is a functor from $\mathcal{F}at$ to the category of acyclic augmented chain complexes over $\mathcal{G}_*^{\mathbb{Z}}$. In other words, it is an acyclic carrier. We call F_* the *forest carrier*.

The acyclic carrier F_* carries a unique (up to homotopy) chain map

$$\phi_* : C_*(\mathcal{F}at) \to \mathcal{G}_*^{\mathbb{Z}}, \tag{7}$$

where $C_*(\mathcal{F}at)$ is the cellular chain complex of the category of ribbon graphs. (So $C_n(\mathcal{F}at)$ is the free abelian group generated by all elements

$$\Gamma_* = (\Gamma_0 \to \Gamma_1 \to \cdots \to \Gamma_n)$$

of the n-skeleton $\mathcal{N}_n\mathcal{F}at$ of the simplicial nerve $\mathcal{N}_\bullet\mathcal{F}at$ of $\mathcal{F}at$. We refer to such Γ_* as an *n-simplices* in $\mathcal{F}at$.)

Theorem 1.22. *Any chain map (7) carried by F_* is a rational homotopy equivalence.*

Remark 1.23. If we consider the forest carrier F_* as a diagram of chain complexes over $\mathcal{G}_*^{\mathbb{Z}}$ we see that there is an induced chain map from the homotopy pushout of this diagram into $\mathcal{G}_*^{\mathbb{Z}}$. This homotopy pushout is the *forested graph complex*

$$C_*(\mathcal{F}at; F_*) = \bigoplus_{n \geqslant 0,\, \Gamma_* \in \mathcal{N}_n\mathcal{F}at} \sigma^n \mathbb{Z}(\Gamma_*) \otimes F_*(\Gamma_0),$$

where $\mathbb{Z}(\Gamma_*)$ is the free abelian group of rank one generated by (Γ_*) and σ^n is the n-fold suspension operator. Since F_* is acyclic, $C_*(\mathcal{F}at; F_*) \simeq C_*(\mathcal{F}at; \mathbb{Z})$. This gives the following diagram:

$$C_*(\mathcal{F}at; \mathbb{Z}) \xleftarrow{\simeq} C_*(\mathcal{F}at; F_*) \xrightarrow{p} \mathcal{G}_*^{\mathbb{Z}}.$$

We are claiming that the right hand arrow, given by $p : F_* \to \mathcal{G}_*^{\mathbb{Z}}$ for $n = 0$ and zero for $n > 0$, is a rational homotopy equivalence.

We observe that the chain map (7), being well-defined up to homotopy, induces a well-defined map in cohomology. This gives the following observation.

Theorem 1.24. *The dual Kontsevich cycles pull back to well-defined integer cohomology classes*

$$\phi^*[W_\lambda^*] \in H^{2|\lambda|}(\mathcal{F}at; \mathbb{Z}) \cong \prod_{g,s} H^{2|\lambda|}(M_g^s; \mathbb{Z}).$$

The rest of this section is devoted to the proof of Theorem 1.22 assuming Proposition 1.21. Our strategy is to construct an explicit rational homotopy inverse for the chain map (7) using “dual cells.”

1.6. Dual cells

"Dual cells" are elements of $C_*(\mathscr{F}at)$ associated to every generator $\langle\Gamma\rangle \in \mathscr{G}_n^{\mathbb{Z}}$. Every oriented ribbon graph Γ has many dual cells but we will see that each of them is necessarily mapped to $(-1)^{\binom{n+1}{2}}\langle\Gamma\rangle$ by any chain map carried by the forest carrier.

A rational inverse

$$\psi : \mathscr{G}_*^{\mathbb{Z}} \otimes \mathbb{Q} \to C_*(\mathscr{F}at;\mathbb{Q})$$

is given by mapping each rational generator $\langle\Gamma\rangle$ to the average dual cell in a finite model for $C_*(\mathscr{F}at;\mathbb{Q})$ given by choosing one object from every isomorphism class of ribbon graphs and taking the average over all possible dual cells which lie in this finite model. We could have used this finite model all along if we were only interested in the rational instead of integral cohomology of the mapping class group.

The composition $\phi\psi$ will be the identity mapping on $\mathscr{G}_*^{\mathbb{Z}} \otimes \mathbb{Q} = \mathscr{G}_* \otimes \mathbb{Q}$ and the composition $\psi\phi$ will be homotopic to the identity on the finite model since it is carried by the "identity carrier". The identity carrier is a canonical acyclic carrier which carries the identity map on the cellular chain complex of any small category (see Lemma 1.29).

Suppose that Γ is an oriented ribbon graph of codimension n. Then a *dual cell* for Γ is given by choosing one representative from every isomorphism class of ribbon graphs over Γ (taking the identity map on Γ as one representative). Consider all n simplices

$$\Gamma_* = (\Gamma_0 \to \cdots \to \Gamma_n = \Gamma),$$

where Γ_i is a representative of codimension i. Then the dual cell is given by the signed sum of all of these n-simplices

$$D(\Gamma) = \sum o(\Gamma_*)(\Gamma_*) \in C_n(\mathscr{F}at),$$

where the sign $o(\Gamma_*) = \pm 1$ is positive iff the given orientation of Γ agrees with the one induced from the natural orientation of the trivalent graph Γ_0. Note that, if we reverse the orientation of Γ, this sign will change. So,

$$D(-\Gamma) = -D(\Gamma).$$

It is also trivial to see that, in the case $n = 0$, we have $D(\Gamma) = \Gamma$ assuming that we take the natural orientation on Γ.

Lemma 1.25. *The boundary of the dual cell is, up to sign, a sum of dual cells*

$$dD(\Gamma) = (-1)^n \sum D(\Gamma'),$$

where the sum is taken over all chosen representatives

$$\Gamma' \to \Gamma$$

of all isomorphism classes of ribbon graphs of codimension $n-1$ over Γ. We take the orientation on Γ' induced by the given map.

Proof. By Proposition 1.3, the set of isomorphism classes of objects over each Γ' maps monomorphically into the set of isomorphism classes of objects over Γ. Therefore, the given choices of representatives for Γ gives a complete set of representatives for the objects over Γ' and $D(\Gamma')$ is defined.

The boundary of $D(\Gamma)$ is given by

$$dD(\Gamma) = \sum_{\Gamma_*}\sum_{i=0}^{n-1}(-1)^i o(\Gamma_*)(\Gamma_0,\ldots,\widehat{\Gamma}_i,\ldots,\Gamma_n=\Gamma)$$
$$+(-1)^n\sum_{\Gamma_*} o(\Gamma_*)(\Gamma_0,\ldots,\Gamma_{n-1}).$$

However, the double sum is zero since, when Γ_i is deleted, there are exactly two ways to fill in the blank and these give opposite signs for $o(\Gamma_*)$. The second sum is equal to the sum of $D(\Gamma_{n-1})$ for all possible Γ_{n-1}. □

Lemma 1.26. *Given any oriented ribbon graph Γ of codimension n, any dual cell $D(\Gamma)\in C_*(\mathscr{F}at)$ and any augmented chain map $\phi: C_*(\mathscr{F}at)\to \mathscr{G}_*^{\mathbb{Z}}$ carried by the forest carrier F_* we will have*

$$\phi(D(\Gamma)) = (-1)^{n(n+1)/2}\langle\Gamma\rangle.$$

Proof. This will be by induction on n. Suppose that $n=0$. Then

$$\phi(D(\Gamma)) = \phi(\Gamma) = \langle\Gamma\rangle,$$

since the identity map $[\Gamma\to\Gamma]$ is the unique element of $F_0(\Gamma)$ with augmentation equal to 1.

Now suppose the statement holds for $n-1$. Then by Lemma 1.26 we have

$$\phi dD(\Gamma) = (-1)^n\phi\sum D(\Gamma') = (-1)^{n+n(n-1)/2}\sum\langle\Gamma'\rangle = (-1)^{n(n+1)/2}d\langle\Gamma\rangle.$$

However, $F_{n+1}(\Gamma)=0$. So, the value of $\phi D(\Gamma)$ in $F_n(\Gamma)$ is uniquely determined by the value of its boundary. This forces $\phi D(\Gamma)$ to be $(-1)^{n(n+1)/2}\langle\Gamma\rangle$. □

To construct a rational inverse for the chain map ϕ we choose a finite model for $\mathscr{F}at$. Let $\mathscr{F}in$ be a full subcategory of $\mathscr{F}at$ that contains exactly one object from every isomorphism class. Then $\mathscr{F}in$ is a deformation retract of $\mathscr{F}at$ and the cellular chain complex of $\mathscr{F}in$ is a deformation retract of $C_*(\mathscr{F}at)$. Although useful for computations, the category $\mathscr{F}in$ has certain defects that we need to watch out for. For example, if we take a graph $\Gamma\in\mathscr{F}in$ and collapse an edge e, the result Γ/e may not be an object of $\mathscr{F}in$. So, we would need to take the unique object $\Gamma'\in\mathscr{F}in$ isomorphic to Γ/e and choose an isomorphism $\Gamma'\cong\Gamma/e$. This construction become natural only if we "average" over all possible such isomorphisms.

If Γ is any oriented ribbon graph of codimension n, let $\bar{D}(\Gamma)\in C_n(\mathscr{F}in;\mathbb{Q})$ be the *average dual cell* of Γ given by

$$\bar{D}(\Gamma) = \sum\frac{o(\Gamma_*)}{|\mathrm{Aut}(\Gamma_0,\ldots,\Gamma_n)|}(\Gamma_0\to\cdots\to\Gamma_n\overset{\approx}{\to}\Gamma), \tag{8}$$

where

$$\mathrm{Aut}(\Gamma_0,\ldots,\Gamma_n) = \mathrm{Aut}(\Gamma_0)\times\cdots\times\mathrm{Aut}(\Gamma_n)$$

and the sum is taken over all possible sequences of morphism in $\mathscr{F}in$ so that Γ_i has codimension i for each i and all possible choices for the isomorphism $\Gamma_n \approx \Gamma$. We will view this sequence of $n+1$ morphisms in $\mathscr{F}at$ as a sequence of n morphisms in the category $\mathscr{F}in/\Gamma$ of objects of $\mathscr{F}in$ over Γ.

Lemma 1.27. *If $d\langle\Gamma\rangle = \sum\langle\Gamma'\rangle$ in $\mathscr{G}_*^{\mathbb{Z}}$ then $d\bar{D}(\Gamma) = (-1)^n \sum \bar{D}(\Gamma')$.*

Proof. The proof is analogous to the proof of Lemma 1.25. We just need to realize that when Γ_i is deleted, there are $|\mathrm{Aut}(\Gamma_i)|$ ways to put it back in as an isomorphic copy. Consequently,

$$d\bar{D}(\Gamma) = \sum_{i=0}^{n-1} (-1)^i \sum \frac{o(\Gamma_*)}{|\mathrm{Aut}(\Gamma_0,\ldots,\widehat{\Gamma}_i,\ldots,\Gamma_n)|} (\Gamma_0 \to \cdots \widehat{\Gamma}_i \cdots \to \Gamma_n \xrightarrow{\approx} \Gamma)$$

$$+ (-1)^n \sum \frac{o(\Gamma_*)}{|\mathrm{Aut}(\Gamma_0,\ldots,\Gamma_{n-1})|} (\Gamma_0 \to \cdots \to \Gamma_{n-1} \to \Gamma).$$

Then, in the second sum, the morphism $\Gamma_{n-1} \to \Gamma$ can be uniquely factored through some Γ' making it into a sum of terms of the form $(-1)^n \bar{D}(\Gamma')$. □

This lemma says that we have a chain map

$$\psi : \mathscr{G}_*^{\mathbb{Z}} \otimes \mathbb{Q} \to C_*(\mathscr{F}in; \mathbb{Q})$$

given by $\psi\langle\Gamma\rangle = (-1)^{n(n+1)/2}\bar{D}(\Gamma)$. Lemma 1.26 gives us:

Theorem 1.28. *The composition*

$$\mathscr{G}_* \otimes \mathbb{Q} \xrightarrow{\psi} C_*(\mathscr{F}in; \mathbb{Q}) \hookrightarrow C_*(\mathscr{F}at; \mathbb{Q}) \xrightarrow{\phi} \mathscr{G}_* \otimes \mathbb{Q}$$

is the identity map on $\mathscr{G}_* \otimes \mathbb{Q} = \mathscr{G}_*^{\mathbb{Z}} \otimes \mathbb{Q}$ for any chain map ϕ carried by the forest carrier.

Finally, Theorem 1.22 follows from the following.

Lemma 1.29. *The composition*

$$C_*(\mathscr{F}in; \mathbb{Q}) \hookrightarrow C_*(\mathscr{F}at; \mathbb{Q}) \xrightarrow{\phi} \mathscr{G}_* \otimes \mathbb{Q} \xrightarrow{\psi} C_*(\mathscr{F}in; \mathbb{Q})$$

is homotopic to the identity map.

Proof. To show that two chain maps are homotopic it suffices to construct an acyclic carrier that carries both of them. In this case it will be the "identity carrier."

Let X be any object of any small category $\mathscr{A}$. Then the category $\mathscr{A}/X$ of objects over X is contractible since it has a terminal object id $: X \to X$ and any morphism $X \to Y$ induces a functor $\mathscr{A}/X \to \mathscr{A}/Y$. Consequently, the cellular chain complex $C_*(\mathscr{A}/X)$ is an acyclic carrier from $C_*(\mathscr{A})$ to itself, i.e., a functor from $\mathscr{A}$ into the category of augmented acyclic chain complexes over $C_*(\mathscr{A})$. We call this the *identity carrier* since it carries the identity morphism on $C_*(\mathscr{A})$.

In order to show that $\psi \circ \phi$ is carried by the identity carrier we need to show that the chain map ψ is covered by a mapping from the forest carrier to the identity carrier, i.e., we need a natural commuting diagram as follows for all objects Γ in $\mathscr{F}in$.

$$\begin{array}{ccc} F_*(\Gamma)\otimes\mathbb{Q} & \xrightarrow{\widetilde{\psi}} & C_*(\mathscr{F}in/\Gamma;\mathbb{Q}) \\ p\downarrow & & \downarrow q \\ \mathscr{G}_*\otimes\mathbb{Q} & \xrightarrow{\psi} & C_*(\mathscr{F}in;\mathbb{Q}) \end{array}$$

Recall that the generators of $F_*(\Gamma)$ are isomorphism classes $[f:\Gamma'\to\Gamma]$ of oriented ribbon graphs over Γ. The projection p sends this to $\langle\Gamma'\rangle\in\mathscr{G}_*\otimes\mathbb{Q}$ which then goes to the average dual cell $\bar{D}(\Gamma')\in C_*(\mathscr{F}in;\mathbb{Q})$. But the morphism $f:\Gamma'\to\Gamma$ makes all the terms in the definition (8) of $\bar{D}(\Gamma')$ into simplices in $\mathscr{F}in/\Gamma$ and therefore defines a lifting

$$\widetilde{\psi}[f:\Gamma'\to\Gamma]=f_*(\bar{D}(\Gamma'))$$

of $\bar{D}(\Gamma')$ to $C_*(\mathscr{F}in/\Gamma;\mathbb{Q})$ as required. □

2. Stasheff associahedra

We use several different versions of the Stasheff associahedron. The convex n-dimensional Stasheff polyhedron is usually called K_{n+2}, but we denote it K^n to emphasize its dimension. We also consider the category $\mathscr{A}_n=\operatorname{simp}K^{n-3}$ of simplices in K^{n-3} and the simplicial nerve of $\mathscr{A}_n$ which is a triangulation of the polyhedron K^{n-3}.

An outline of this section:

(1) Stasheff polyhedron K^n.
(2) The category $\mathscr{A}_{n+3}$.
(3) Orientation of K^n.
(4) Orientation of K^{odd}.
(5) Proof of Proposition 1.21.

2.1. Stasheff polyhedron K^n

The Stasheff polyhedron K^n (usually written K_{n+2} although it is n dimensional) originates in [19]. The application of this polyhedron to the present context (the moduli space of curves) originates in [7]. The simplicial decomposition described in the following theorem also appear in [7] but originally is due to Boardman and Vogt [2].

Theorem 2.1. *There is an n-dimensional convex polyhedron K^n whose points correspond to isomorphism classes of planar metric trees with $n+3$ leaves of fixed length and up to n internal edges of variable length $\leqslant 1$. Two planar metric trees lie in the same open face of K^n if and only if, after collapsing all internal edges of length <1, they become isomorphic fixing the leaves.*

Let $C_*(K^n)$ be the cellular chain complex of K^n. Then $C_0(K^n)$ is freely generated by isomorphism classes of trivalent planar trees with $n+3$ given leaves. Recall that any such tree has a natural orientation. More generally we have the following.

Proposition 2.2. *For all $0 \leqslant k \leqslant n$, $C_k(K^n)$ is generated by isomorphism classes $[T]$ of oriented planar trees T with $n+3$ fixed leaves and $n-k$ internal edges. The boundary map is given by*

$$d[T] = \sum_{[T',e]} [T'],$$

where the sum is taken over all isomorphism classes of pairs (T', e) where e is an edge in an oriented tree T' so that $T \cong T'/e$ with the induced orientation.

Remark 2.3. This is more or less a tautology since we choose the geometric orientation of the faces to make this algebraic statement true. We also note that trees with fixed leaves have no nontrivial automorphisms. Therefore,

$$\langle T \rangle = |\mathrm{Aut}(T)|[T] = [T].$$

Proof. Let T_0 be a trivalent planar tree with $n+3$ fixed leaves $h_0, \ldots, h_{n+2}$ (in cyclic order) and n internal edges $e_1, \ldots, e_n$. Since T_0 is trivalent, it has an intrinsic orientation. If we collapse the edges $e_1, \ldots, e_k$ in that order we get a tree T_k of codimension k with the induced orientation. The trees T_k for various k are related by

$$T_k = T_{k-1}/e_k.$$

The k-dimensional face of K^n corresponding to the tree T_k consists of isomorphism class of trees T having edges $e_1, \ldots, e_k$ of variable length < 1 and the other edges of length equal to 1. Let $x_1, \ldots, x_k$ be the lengths of $e_1, \ldots, e_k$. Then we choose the geometric orientation of this face of K^n (in a neighborhood of the vertex T_0) by taking these coordinated in opposite order:

$$(x_k, \ldots, x_1).$$

When x_k reaches 1 we get to the face corresponding to T_{k-1}. The orientation of the $k-1$ face is therefore related to that of the k-face by the "first vector points outward" rule which is standard. □

2.2. The category $\mathscr{A}_n$

Let $\mathscr{A}_{n+3}$ be the category of faces of K^n with inclusion maps as morphisms. Then the geometric realization of the nerve of $\mathscr{A}_{n+3}$ is homeomorphic to K^n. A homeomorphism

$$\phi : |\mathscr{A}_{n+3}| \to K^n$$

is given by sending each object (=vertex in the nerve) to some point in the interior of the corresponding face of K_n and extending linearly over the simplices.

Take an n-simplex

$$T_* = (T_0 \to T_1 \to \cdots \to T_n) \tag{9}$$

in $\mathscr{A}_{n+3}$ which is *nondegenerate* in the sense that T_0 is trivalent and each T_i is obtained from T_{i-1} by collapsing one edge. Then, by induction, each tree T_i obtains an induced orientation from the intrinsic orientation of T_0. The orientation of the tree T_n gives a geometric orientation of the polyhedron K^n as explained in the proof of Proposition 2.2.

Suppose that an orientation for the tree T_n representing the top cell of K^n is given. (For example, when n is even, we can take the intrinsic orientation of T_n.) Then the *algebraic orientation* $o(T_*)$ of the n-simplex T_* in (9) is defined to be ± 1 depending on whether the given orientation of T_n is equal to the one induced from T_0. The *dual cell* $D(T_n)$ is defined by

$$D(T_n) = \sum o(T_*)T_* \in C_n(\mathscr{A}_{n+3}),$$

where the sum is over all nondegenerate n-simplices T_*. This is an n-chain in the cellular chain complex of $\mathscr{A}_{n+3}$. It represents a triangulation of the top cell of K^n with algebraic orientation which does not agree with the geometric orientation.

2.3. Orientation of K^n

Lemma 2.4. *The embedding*

$$\Delta^n \to K^n$$

given by the n simplex (9) *has degree* $(-1)^{n(n+1)/2}$ *with respect to the orientation of K^n corresponding to the orientation of T_n induced from that of T_0.*

Remark 2.5. This implies that, given any orientation of T_n, the corresponding geometric orientation of K^n agrees with

$$\psi[T_n] := (-1)^{n(n+1)/2} D(T_n).$$

Proof. For any $0 \leqslant k \leqslant n$ we claim that the map

$$\sigma^k : \Delta^k \to F_k$$

given by $T_0 \to \cdots \to T_k$ has degree $(-1)^{k(k+1)/2}$ where F_k is the face of K^n corresponding to T_k with the induced orientation.

This statement holds for $k = 0$. Suppose it holds for $k-1$. Since the front $k-1$ face Δ^{k-1} is opposite the kth vertex T_k, its orientation is equal to $(-1)^k$ times the induced orientation from Δ^k. Whereas, we are orienting the faces to make Proposition 2.2 true, i.e., the orientation of F_{k-1} is the one induced from F_k. Consequently, the degree of σ^k is

$$(-1)^{k(k-1)/2}(-1)^k = (-1)^{k(k+1)/2}.$$

Putting $k = n$ we get the lemma. □

Now suppose that $n = 2k$. Then the tree T_{2k} has an intrinsic orientation which determines an *intrinsic orientation* for the polyhedron K^{2k}. We can ask what is the sign of each $2k$ simplex $\sigma^{2k} : \Delta^{2k} \to K^{2k}$.

In [8], the sign of the $2k$ simplex $T_* = (T_0 \to T_1 \to \cdots \to T_{2k})$ is defined to be the sign of the permutation of $2k+3$ letters

$$\operatorname{sgn}(T_*) := \operatorname{sgn}(a_1, a_2, a_3, b_1, b_2, \ldots, b_{2k}),$$

where a_i, b_j are the *regions* (components of the complement of the tree in the disk in which it can be embedded with leaves on the boundary) given as follows. The regions which bound the edge e_1

(so that $T_1 = T_0/e_1$) are a_1, a_3 (in either order). Let the two other regions which touch e_1 be a_2, b_1 so that they are a_1, a_2, a_3, b_1 in cyclic order. For $i \geqslant 2$ let v_i be the vertex of e_i furthest away from e_1 and let b_i be the region which touches e_i at the point v_i. Then $\mathrm{sgn}(T_*)$ is defined to be the sign of the permutation of $a_1, a_2, a_3, b_1, \ldots, b_{2k}$ which puts these regions into the correct cyclic order.

Theorem 2.6. *With respect to the intrinsic orientation of K^{2k}, the sign of the embedding*

$$\sigma^{2k} : \Delta^{2k} \to K^{2k}$$

given by the $2k$ simplex $(T_0 \to T_1 \to \cdots \to T_{2k})$ is equal to $\mathrm{sgn}(T_*)$ as defined above.

Proof. We have three definitions of the orientation of K^{2k}:

(a) the intrinsic orientation of K^{2k} induced from the intrinsic orientation of T_{2k};
(b) the orientation of K^{2k} induced from that of T_0 by the sequence of maps $T_0 \to \cdots \to T_{2k}$;
(c) the orientation of K^{2k} induced from Δ^{2k} by the map σ^{2k}.

The statement we are trying to prove is that (a) and (c) differ by $\mathrm{sgn}(T_*)$. We know, by Lemma 2.4, that (b) and (c) differ by the sign

$$(-1)^{2k(2k+1)/2} = (-1)^k.$$

Therefore, it suffices to show that the sign difference between (a) and (b), which is equal to $o(T_*)$ by definition, is given by

$$o(T_*) = (-1)^k \mathrm{sgn}(T_*) = \mathrm{sgn}(a_1, a_2, a_3, b_{2k}, \ldots, b_1). \tag{10}$$

But this is a special case of the following lemma. □

Lemma 2.7. *Suppose that T_0 is a planar tree with internal edges $e_1, \ldots, e_{2k}$ and $2k+1$ internal vertices all trivalent except for v_0 which has valence $2n+1$. Let v_i be the vertex on e_i furthest away from v_0. Let $T_1, \ldots, T_{2k}$ be given by $T_i = T_{i-1}/e_i$. Then the difference $o(T_*)$ between the intrinsic orientation of T_{2k} and the one induced from T_0 is given by*

$$o(T_*) = (-1)^k \mathrm{sgn}(a_1, \ldots, a_{2n+3}, b_1, \ldots, b_{2k}), \tag{11}$$

where a_i are the regions around v_0 and b_i is the region which touched e_i only at v_i.

Proof. First, we claim that the truth value of this statement remains unchanged if we permute the edges $e_1, \ldots, e_{2k}$. To see this suppose we change the order of e_i, e_{i+1}. Then T_i will become a different tree and the orientations of $T_{i+1}, \ldots, T_{2k}$ will be reversed. This comes from the proof that $\partial^2 = 0$ in graph homology. Switching e_i, e_{i+1} will also transpose the labels b_i, b_{i+1}. So sign (11) will also change so the relative sign remains unchanged.

By permuting the order of the edges we may assume that each tree T_i has only one vertex v_0 which is not trivalent and e_{i+1} becomes an edge in T_i connecting v_0 to v_{i+1}. Consequently, T_{2k-2} has an intrinsic orientation and, by induction on k, this orientation differs from the one induced from T_0 by

$$o(T_0, \ldots, T_{2k-2}) = (-1)^{k-1} \mathrm{sgn}(a_1, \ldots, a_{2n+3}, b_1, \ldots, b_{2k-2}). \tag{12}$$

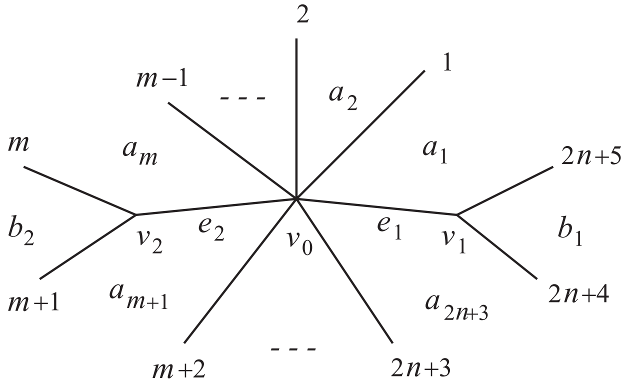

Fig. 2. Case 1: $\mathrm{sgn}(a_1,\ldots,a_{2n+3},b_1,b_2)=(-1)^m$ where $1\leqslant m\leqslant 2n+2$.

To prove that (12) implies (11) we look at the difference between the two signs and the difference between the intrinsic orientation of T_{2k} and the one induced from the intrinsic orientation of T_{2k-2}. The statement that we need is exactly the statement of the lemma in the case $k=1$.

Now we assume that $k=1$. As before, we may assume that e_1 connects v_0 to v_1 but there are two possibilities for e_2.

Case 1: e_2 connects v_0 and v_2.

If we let $a_1,\ldots,a_m$ be the regions at v_0 from e_1 to e_2 as shown in Fig. 2 then the intrinsic orientation of T_0 and induced orientations of T_1,T_2 are

$$
\begin{aligned}
&(T_0)\quad \mathrm{Sgn}(\mathbf{v_0e_1^-}h_1h_2\cdots h_{m-1}e_2^-h_{m+2}\cdots h_{2n+3}\ \mathbf{v_1e_1^+}h_{2n+4}h_{2n+5}\ v_2e_2^+h_mh_{m+1}),\\
&(T_1)\quad -\mathrm{Sgn}(\mathbf{v_0}h_1h_2\cdots h_{m-1}\mathbf{e_2^-}h_{m+2}\cdots h_{2n+5}\ \mathbf{v_2e_2^+}h_mh_{m+1}),\\
&\qquad\quad =(-1)^m\mathrm{Sgn}(\mathbf{v_0e_2^-}\mathbf{v_2e_2^+}h_1\cdots h_{2n+5}),\\
&(T_2)\quad (-1)^{m-1}\,\mathrm{Sgn}(v_0h_1h_2\cdots h_{2n+5}),
\end{aligned}
$$

where the leaves are labelled $h_1,\ldots,h_{2n+5}$ in counterclockwise order and e_i^-,e_i^+ are the halves of e_i closer/further from v_0. We see that the induced orientation of T_1 differs from the intrinsic orientation by $(-1)^{m-1}$. However, the regions $a_1,\ldots,a_{2n+3},b_1,b_2$ are arranged in the cyclic order

$$[b_1,a_1,\ldots,a_m,b_2,a_{m+1},\ldots,a_{2n+3}],$$

so the permutation sign is

$$\mathrm{sgn}(a_1,\ldots,a_{2n+3},b_1,b_2)=(-1)^m.$$

Multiplying by $(-1)^k=-1$ we get (11).

Case 2: e_2 connects v_1 and v_2.

Here there are two subcases as shown in Fig. 3.

(2a) b_1 is clockwise from b_2.
(2b) b_2 is clockwise from b_1.

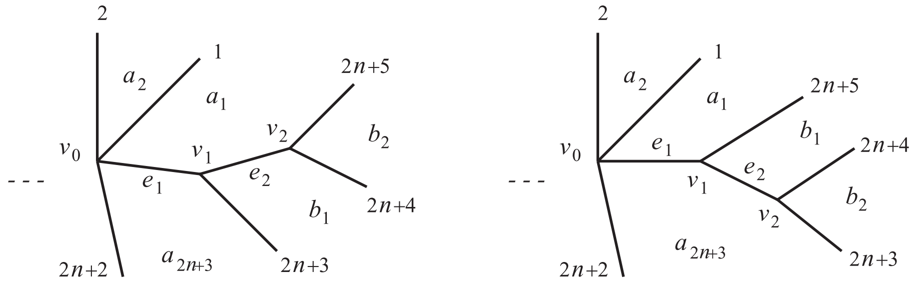

Fig. 3. Case 2a: b_1b_2 (on left) and Case 2b: b_2b_1 (on right).

In subcase (2a), we have the following orientations on T_0, T_1, T_2 induced from the intrinsic orientation of T_0.

$$(T_0)\quad \mathrm{Sgn}(\mathbf{v_0e_1^-}h_1h_2\cdots h_{2n+2}\ \mathbf{v_1e_1^+}h_{2n+3}e_2^-\ v_2e_2^+h_{2n+4}h_{2n+5}),$$

$$(T_1)\quad -\mathrm{Sgn}(\mathbf{v_0}h_1h_2\cdots h_{2n+3}\ \mathbf{e_2^-v_2e_2^+}h_{2n+4}h_{2n+5})$$

$$=\mathrm{Sgn}(\mathbf{v_0e_2^-v_2e_2^+}h_1\cdots h_{2n+5}),$$

$$(T_2)\quad -\mathrm{Sgn}(v_0h_1h_2\cdots h_{2n+5}).$$

The induced orientation of T_2 is negative the natural orientation. But

$$\mathrm{sgn}(a_1,\dots,a_{2n+3},b_1,b_2)=+1,$$

since the regions are in the correct cyclic order. Thus, the lemma holds in this case.

In subcase (2b), the induced orientations on T_0, T_1, T_2 are

$$(T_0)\quad \mathrm{Sgn}(\mathbf{v_0e_1^-}h_1h_2\cdots h_{2n+2}\ \mathbf{v_1e_1^+}e_2^-h_{2n+5}\ v_2e_2^+h_{2n+3}h_{2n+4}),$$

$$(T_1)\quad -\mathrm{Sgn}(\mathbf{v_0}h_1h_2\cdots h_{2n+2}\mathbf{e_2^-}h_{2n+5}\ \mathbf{v_2e_2^+}h_{2n+3}h_{2n+4}),$$

$$=-\mathrm{Sgn}(\mathbf{v_0e_2^-v_2e_2^+}h_1\cdots h_{2n+5}),$$

$$(T_2)\quad \mathrm{Sgn}(v_0h_1h_2\cdots h_{2n+5}).$$

The induced orientation on T_2 is equal to the natural orientation. But

$$\mathrm{sgn}(a_1,\dots,a_{2n+3},b_1,b_2)=-1$$

so the lemma holds in this final case. □

2.4. Orientation of K^{odd}

Suppose that n is odd. Then the planar trees in K^n have an even number $(n+3)$ of leaves. An orientation of these trees is given by taking a fixed cyclic ordering of these leaves, starting at one point and going counterclockwise. This gives an orientation of a tree with one vertex and thus of the top cell of K^n. For any other tree $T\in K^n$ an orientation is given by choosing an ordering of the internal edges of T. The orientation given by this ordering is the one which induces the chosen orientation on the one vertex tree by collapsing the edges in order.

We can now prove Proposition 1.21.

2.5. *Proof of Proposition 1.21*

Recall that we have a ribbon graph Γ_0 all of whose vertices are trivalent except for $v_1,\dots,v_r$ which have valence $n_1+3,\dots,n_r+3$. For each i, we choose a cyclic ordering of the half-edges at v_i and take the orientation on Γ_0 given by v_1 followed by its half-edges, v_2 followed by its half-edges, etc.

Not suppose that for each i, T_i is a planar tree with n_i+3 leaves representing an m_i-face of K^{n_i}. Thus T_i has n_i-m_i internal edges. Choose an ordering of these edges. Let $\Gamma_0(T_1,\dots,T_r)$ denote the graph obtained from Γ_0 by replacing v_i by T_i. Then the T_i will form a forest in Γ_0. Let $e_{i1},e_{i2},\dots$ be the internal edges of T_i. Take the orientation on $\Gamma_0(T_1,\dots,T_r)$ so that, if the edges e_{ij} are collapsed in lexicographic order, we get the chosen orientation on Γ_0.

Let

$$\phi : C_{m_1}(K^{n_1})\otimes\cdots\otimes C_{m_r}(K^{n_r})\to F_m(\Gamma_0),$$

where $m=m_1+\cdots+m_r$ be given by

$$\phi([T_1]\otimes\cdots\otimes[T_r])=[\Gamma_0(T_1,\dots,T_r)\to\Gamma_0].$$

We claim that this gives a chain isomorphism

$$\phi : C_*(K^{n_1})\otimes\cdots\otimes C_*(K^{n_r})\to F_*(\Gamma_0).$$

Since ϕ sends basis elements to basis elements, it suffices to show that ϕ is a chain map. But this is straightforward.

The boundary of $[T_1]\otimes\cdots\otimes[T_r]$ is, by Proposition 2.2, equal to

$$\sum_{i=1}^{r}(-1)^{m_1+\cdots+m_{i-1}}\sum\,[T_1]\otimes\cdots\otimes[T_{i-1}]\otimes[T_i']\otimes[T_{i+1}]\otimes\cdots\otimes[T_r],$$

where the second sum is over all pairs (T_i',e_{i0}) so that $T_i'/e_{i0}\cong T_i$. But $(-1)^{m_1+\cdots+m_{i-1}}$ is also the sign of the permutation which brings the edge e_{i0} to the beginning in the ordering of all edges of $\Gamma_0(T_1,\dots,T_i',\dots,T_r)$. So

$$d\phi([T_1]\otimes\cdots\otimes[T_r])=\sum(-1)^{m_1+\cdots+m_{i-1}}\phi([T_1]\cdots[T_i']\cdots[T_r])$$

as required.

3. MMM classes

We review the definition of the adjusted MMM classes. Morita [17] explains these cohomology classes in more detail. Arbarello and Cornalba [1] explain why the adjusted version is more suitable. We will pull these cohomology classes back to the graph cohomology complex $\mathscr{G}_*$ and describe what happens when we evaluate them on the dual Kontsevich cycles.

(1) Cyclic set cocycle.
(2) Adjusted MMM classes in $H^{2k}(\mathscr{G}_*;\mathbb{Q})$.
(3) Cup products of adjusted MMM classes.
(4) Computing the numbers $b^{k_*}_{n_*}$.
(5) Kontsevich cycles in terms of MMM classes.
(6) Computing a^{μ}_{λ}.

Suppose that

$$\Sigma^s_g \to E \xrightarrow{p} B$$

is a compact manifold bundle where Σ^s_g is an oriented connected surface of genus g and s unordered distinguished points. Let

$$\pi : \widetilde{B} \to B$$

be the s-fold covering space given by the s distinguished point in each fiber of E. Then the vertical tangent bundle of E is an oriented 2-plane bundle and therefore has an Euler class $e(E) \in H^2(E;\mathbb{Z})$. The push-down of the $(k+1)$th power of this class is the *MMM class*

$$\kappa_k(E) = p_*(e(E)^{k+1}) \in H^{2k}(B;\mathbb{Z}).$$

The restriction of the Euler class $e(E)$ to $\widetilde{B} \subseteq E$ gives another Euler class $e(\widetilde{B}) \in H^2(\widetilde{B};\mathbb{Z})$. The push-down of the kth power of this second class is the *boundary class*

$$\gamma_k(E) = \pi_*(e(\widetilde{B})^k) \in H^{2k}(B;\mathbb{Z}).$$

The *adjusted* or *punctured* MMM classes $\widetilde{\kappa}_k(E)$ are given by

$$\widetilde{\kappa}_k(E) = \kappa_k(E) - \gamma_k(E).$$

The surface bundle $\Sigma^s_g \to E \to B$ is classified by a map $f : B \to BM^s_g$ and all three cohomology classes defined above are pull-backs of universal classes

$$\kappa_k, \gamma_k, \widetilde{\kappa}_k \in H^{2k}(M^s_g;\mathbb{Z}) \cong H^{2k}(\mathscr{F}at^s_g;\mathbb{Z}).$$

By the fundamental results of Morita [16] and Miller [14] the MMM classes κ_k and the first s boundary classes $\gamma_1,\ldots,\gamma_s$ are algebraically independent over $\mathbb{Q}$ in the stable range (given by Harer [6] stability). The classes γ_r for $r > s$ are polynomials in $\gamma_1,\ldots,\gamma_s$. For example,

$$\gamma_3 = 3\gamma_1\gamma_2 - \tfrac{1}{2}\gamma_1^3$$

if $s=2$. Since the category $\mathscr{F}at$ is the disjoint union of $\mathscr{F}at^s_g$ for all g,s, this also extends to $\widetilde{\kappa}_0 = 1-2g-s$ and we have the following.

Theorem 3.1. *The universal adjusted MMM classes*

$$\widetilde{\kappa}_k \in H^*(\mathscr{F}at;\mathbb{Q})$$

for $k \geqslant 0$ are algebraically independent.

In [8], a combinatorial rational cocycle is constructed for the punctured class $\widetilde{\kappa}_k$. It is called the “cyclic set cocycle.”

3.1. Cyclic set cocycle

Let $\mathscr{Z}$ be the category of cyclically ordered sets and cyclic order preserving monomorphisms. Then it is well known that

$$|\mathscr{Z}| \simeq BU(1).$$

The kth power of the first Chern class of the canonical complex line bundle over $\mathscr{Z}$ is given by the *unadjusted cyclic set cocycle* $c^k_{\mathscr{Z}}$ whose value on a $2k$-simplex

$$C_* = (C_0 \to C_1 \to \cdots \to C_{2k})$$

is given by

$$c^k_{\mathscr{Z}}(C_*) = \frac{\sum \operatorname{sgn}(a_0, a_1, \ldots, a_{2k})}{(-2)^k (2k-1)!! |C_0| \cdots |C_{2k}|}, \tag{13}$$

where the sum is taken over all choices of elements $a_i \in C_i - C_{i-1}$. (In our notation, we pretend that the maps in C_* are inclusion maps. Strictly speaking, a_i should be an element of C_{2k} which lies in the image of C_i but not in the image of C_{i-1}.)

The unadjusted cyclic set cocycle has the following obvious property.

Proposition 3.2. *In order for the unadjusted cyclic set cocycle $c^k_{\mathscr{Z}}$ to be nonzero on C_* it is necessary* (*but not sufficient*) *for each C_i to be larger than C_{i-1}, i.e.,*

$$|C_0| < |C_1| < \cdots < |C_{2k}|.$$

In [8] it is shown that the adjusted MMM class $\tilde{\kappa}_k$ is given on the category of ribbon graphs by evaluating the unadjusted cyclic set cocycle (13) at each vertex and dividing by -2. Since there is already a factor of $(-2)^k$ in the denominator it seems reasonable to define the (adjusted) *cyclic set cocycle* $\tilde{c}^k_{\mathscr{Z}}$ by

$$\tilde{c}^k_{\mathscr{Z}}(C_*) = \frac{1}{-2} c^k_{\mathscr{Z}}(C_*) = \frac{\sum \operatorname{sgn}(a_0, a_1, \ldots, a_{2k})}{(-2)^{k+1} (2k-1)!! |C_0| \cdots |C_{2k}|}. \tag{14}$$

Theorem 3.3 (Igusa [8]). *The adjusted rational MMM class $\tilde{\kappa}_k \in H^{2k}(\mathscr{F}at; \mathbb{Q})$ is given on a 2k-simplex*

$$\Gamma_* = (\Gamma_0 \to \cdots \to \Gamma_{2k}) \tag{15}$$

in $\mathscr{F}at$ by evaluating the cyclic set cocycle $\tilde{c}^k_{\mathscr{Z}}$ on every vertex of Γ_0 counted with multiplicity where the multiplicity *of a vertex is defined to be its valence minus* 2.

We use the notation $\tilde{c}^k_{\mathscr{F}at}$ to denote this rational cocycle on the cellular chain complex of $\mathscr{F}at$ and we refer to it by the same name, i.e., $\tilde{c}^k_{\mathscr{F}at}$ is the *cyclic set cocycle* on the category of ribbon graphs.

By definition, $\tilde{c}^k_{\mathscr{F}at}$ satisfies the following important condition.

Proposition 3.4. *The value of $\tilde{c}^k_{\mathscr{F}at}$ on a $2k$-simplex* (15) *can be nonzero only if there is at least one vertex v_0 of Γ_0 whose image in each Γ_i has greater valence than its image in Γ_{i-1}.*

Using dual cells we can pull back the adjusted MMM classes to the rational cohomology of the graph cohomology complex $\mathscr{G}_*$.

3.2. Adjusted MMM classes in $H^{2k}(\mathscr{G}_*;\mathbb{Q})$

Recall that there is a chain homotopy equivalence

$$\psi:\mathscr{G}_*\otimes\mathbb{Q}\to C_*(\mathscr{F}at;\mathbb{Q})$$

given up to sign by sending each rational generator $\langle\Gamma\rangle\in\mathscr{G}_n$ to the average dual cell of Γ inside a finite model $\mathscr{F}in$:

$$\psi\langle\Gamma\rangle=(-1)^{\binom{n+1}{2}}\bar{D}(\Gamma).$$

Since the cyclic set cocycle $\widetilde{c}^k_{\mathscr{F}at}$ has the same value on isomorphic simplices, its value on any dual cell of Γ is the same. Therefore, its value of the average dual cell $\bar{D}(\Gamma)$ is equal to its value on any particular dual cell $D(\Gamma)$.

Definition 3.5. The *cyclic set cocycle*

$$\widetilde{c}^k_{\mathscr{G}}:\mathscr{G}_{2k}\to\mathbb{Q}$$

on the graph cohomology complex $\mathscr{G}_*$ is given by

$$\widetilde{c}^k_{\mathscr{G}}\langle\Gamma\rangle=(-1)^k\widetilde{c}^k_{\mathscr{F}at}(D(\Gamma))$$

for any choice of dual cell $D(\Gamma)$.

Theorem 3.6. *The cyclic set cocycle $\widetilde{c}^k_{\mathscr{G}}$ represents the adjusted MMM class*

$$[\widetilde{c}^k_{\mathscr{G}}]=\psi^*(\widetilde{\kappa}_k)\in H^{2k}(\mathscr{G}_*;\mathbb{Q}).$$

It is clear from the definition of the cyclic set cocycle that it can only be nonzero on the Witten cycle W_k and it has the same value on $\langle\Gamma\rangle$ for every element Γ of W_k with natural orientation. Therefore, $\widetilde{c}^k_{\mathscr{G}}$ is proportional to the dual Witten cycle W_k^*. The proportionality constant was computed in [8].

Theorem 3.7. *We have the following equation of rational cocycles on the associative graph cohomology complex $\mathscr{G}_*$:*

$$W_k^*=(-2)^{k+1}(2k+1)!!\widetilde{c}^k_{\mathscr{G}}.$$

Proof. As we showed in Theorem 2.6 and Remark 2.5, the sign convention used in [8] for every simplex in the dual cell of any $\Gamma\in W_k$ is given by $(-1)^k o(\Gamma_*)$ which agrees with the coefficient of the same term in

$$\psi\langle\Gamma\rangle=(-1)^k\sum o(\Gamma_*)(\Gamma_0,\ldots,\Gamma_{2k}).\qquad\square$$

Putting these together, we get the following which can be interpreted as statement about the relationship between rational cohomology classes in graph cohomology, the category of ribbon graphs or the mapping class group.

Corollary 3.8 (Igusa [8]). *The dual Witten cycle* $[W_n^*]$ *is a multiple of the adjusted MMM class*

$$[W_n^*] = (-2)^{n+1}(2n+1)!!\widetilde{\kappa}_n.$$

For example, we have

$$[W_0^*] = -2\widetilde{\kappa}_0,$$

$$[W_1^*] = 12\widetilde{\kappa}_1,$$

$$[W_2^*] = -120\widetilde{\kappa}_2,$$

$$[W_3^*] = 1680\widetilde{\kappa}_3,$$

$$[W_4^*] = -30240\widetilde{\kappa}_4.$$

3.3. Cup products of adjusted MMM classes

We now consider cup products of the $\widetilde{\kappa}_k$'s for $k \geqslant 1$. If

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$$

is a *partition* of n in the sense that $\lambda_i > 0$ and $n = \sum \lambda_i$ then let $\widetilde{\kappa}_\lambda$ denote the cup product

$$\widetilde{\kappa}_{\lambda_1} \cup \cdots \cup \widetilde{\kappa}_{\lambda_s} \in H^{2n}(\mathscr{F}at; \mathbb{Z}).$$

Since the $\widetilde{\kappa}_{\lambda_i}$ are even degree classes, this cup product does not depend on the order of the λ_i's. However, at the chain level, the order does matter. (The order matters in $\mathscr{F}at$ but not in the graph cohomology complex by Corollary 3.22.) Let $\widetilde{c}^{\lambda}_{\mathscr{F}at}$ denote the cup product

$$\widetilde{c}^{\lambda}_{\mathscr{F}at} = \widetilde{c}^{\lambda_1}_{\mathscr{F}at} \cup \cdots \cup \widetilde{c}^{\lambda_s}_{\mathscr{F}at} \in C^{2n}(\mathscr{F}at; \mathbb{Q}) \tag{16}$$

and let

$$\widetilde{c}^{\lambda}_{\mathscr{G}} = \psi^* \widetilde{c}^{\lambda}_{\mathscr{F}at} \in \mathrm{Hom}(\mathscr{G}_{2n}, \mathbb{Q})$$

denote the pull-back of $\widetilde{c}^{\lambda}_{\mathscr{F}at}$ to $\mathscr{G}_*$. Then, by Theorem 3.3, we have

$$[\widetilde{c}^{\lambda}_{\mathscr{G}}] = \psi^* \widetilde{\kappa}_\lambda.$$

Definition 3.9. $\lambda = (\lambda_1, \ldots, \lambda_s)$ is a *refinement* of $\mu = (\mu_1, \ldots, \mu_r)$ and we write

$$\lambda \leqslant \mu$$

if each μ_i is a sum of λ_j's so that, if we let $\pi(i)$ be the set of these indices j, then π is a partition of the set $\{1, 2, \ldots, s\}$ into r parts and

$$\lambda_{\pi(i)} := \sum_{j \in \pi(i)} \lambda_j = \mu_i.$$

We say that π *represents* λ as a refinement of μ.

Lemma 3.10. *If $\widetilde{c}^{\lambda}_{\mathscr{G}}$ is nonzero on $\langle\Gamma\rangle \in \mathscr{G}_{2n}$ then Γ must lie in some Kontsevich cycle W_μ where λ is a refinement of μ.*

Proof. In order for the cup product (16) to be nonzero on a dual cell $D(\Gamma)$ where $\Gamma \in \mathscr{G}_{2n}$ there must be a *nondegenerate* $2n$-simplex (i.e., where none of the morphisms are isomorphisms)

$$\Gamma_0 \to \Gamma_1 \to \cdots \to \Gamma_{2n} \cong \Gamma,$$

so that

$$\widetilde{c}^{\lambda_1}_{\mathscr{F}at}(\Gamma_0 \to \cdots \to \Gamma_{2\lambda_1}) \neq 0,$$

$$\widetilde{c}^{\lambda_2}_{\mathscr{F}at}(\Gamma_{2\lambda_1} \to \cdots \to \Gamma_{2\lambda_1+2\lambda_2}) \neq 0, \text{ etc.}$$

By Proposition 3.4, $\Gamma_{2\lambda_1}$ must have a vertex of multiplicity $2\lambda_1$ and it must have a vertex which increases in multiplicity by $2\lambda_2$ by the time it gets to $\Gamma_{2\lambda_1+2\lambda_2}$. This implies that $\Gamma_{2\lambda_1+2\lambda_2}$ must have either a vertex of multiplicity $2\lambda_1 + 2\lambda_2$ or two vertices of multiplicity $2\lambda_1, 2\lambda_2$, resp.

By induction, $\Gamma = \Gamma_{2\lambda_1+\cdots+2\lambda_{s-1}} = \Gamma_{2n-2\lambda_s}$ must lie in a Kontsevich cycle W_ν so that $(\lambda_1, \ldots, \lambda_{s-1})$ is a refinement of ν as a partition of $n - \lambda_s$. In order for $\widetilde{c}^{n-\lambda_s}_{\mathscr{F}at}$ to be nonzero on the back λ_s face of Γ_*, the graph Γ must have a vertex which increases in multiplicity by $2\lambda_s$ by the time it gets to Γ_{2n}. Thus, Γ_{2n} must lie in W_μ where either $\mu = (\nu, \lambda_s)$ or μ is equal to ν with one of the ν_i increased by λ_s. In either cases

$$\lambda \leqslant (\nu, \lambda_s) \leqslant \mu$$

as claimed. □

Lemma 3.11. *Suppose that $\Gamma \in W_\mu$ where $\mu = (\mu_1, \ldots, \mu_r)$ is a partition of n and λ is a refinement of μ. Then the value of $\widetilde{c}^{\lambda}_{\mathscr{G}}$ on $\langle\Gamma\rangle$ depends only on the ordered partition λ and the unordered partition μ (and is independent of the choice of $\Gamma \in W_\mu$).*

Remark 3.12. We will denote this number by

$$b^{\mu}_{\lambda} := \widetilde{c}^{\lambda}_{\mathscr{G}}\langle\Gamma\rangle = \psi^*\widetilde{c}^{\lambda}_{\mathscr{F}at}\langle\Gamma\rangle = (-1)^n\widetilde{c}^{\lambda}_{\mathscr{F}at}D(\Gamma) \in \mathbb{Q}. \tag{17}$$

It is obvious that the order of the μ_i's is not important. We will later show (Corollary 3.22) that the order of the λ_j's is also not important.

Proof. If Γ lies in W_μ then Γ has vertices $v_1, \ldots, v_r$ of codimension $2\mu_1, \ldots, 2\mu_r$. In any dual cell for Γ, each of these vertices is expanded to a tree in all possible ways (up to isomorphism). The rest of the graph is left fixed. However, the cyclic set cocycle is only evaluated on the vertices of these trees. Since the orientation of the simplices in the dual cell depend only on these trees and the value of the cocycle depends only on the trees, which in turn depend only on the numbers $\mu_1, \ldots, \mu_r$ the value of $\widetilde{c}^{\lambda}_{\mathscr{F}at}$ on $D(\Gamma)$ depends only on μ. □

Putting these two lemmas together we get the following.

Theorem 3.13. *Any cup product of cyclic set cocycles*

$$\widetilde{c}^{\lambda}_{\mathscr{G}} = \psi^*(\widetilde{c}^{\lambda_1}_{\mathscr{F}at} \cup \cdots \cup \widetilde{c}^{\lambda_s}_{\mathscr{F}at})$$

can be expressed as a rational linear combinations of dual Kontsevich cycles by

$$\widetilde{c}^{\lambda}_{\mathscr{G}} = \sum b^{\mu}_{\lambda} W^{*}_{\mu},$$

where the sum is over all partitions μ *of* $n = \sum \lambda_j$ *so that* λ *is a refinement of* μ *and* b^{μ}_{λ} *is given by* (17).

Remark 3.14. At the level of cohomology this theorem implies that

$$\widetilde{\kappa}_{\lambda} = \sum b^{\mu}_{\lambda} [W^{*}_{\mu}].$$

Once we show that the cohomology classes $[W^{*}_{\mu}]$ are linearly independent (Corollary 3.21) then this equation can be used to define b^{μ}_{λ}. We can then conclude that b^{μ}_{λ} and $\widetilde{c}^{\lambda}_{\mathscr{G}}$ are independent of the order of λ (Corollary 3.22).

3.4. Computing the numbers b^{μ}_{λ}

We first show that the computation of the numbers b^{μ}_{λ} can be reduced to the case when $\mu = n$ is the trivial partition of n.

Lemma 3.15. *If* $\mu = (\mu_1, \dots, \mu_r)$ *and* $\lambda = (\lambda_1, \dots, \lambda_s)$ *are partitions of* n *then*

$$b^{\mu}_{\lambda} = \sum_{\pi} \prod_{i=1}^{r} b^{\mu_i}_{\lambda_{\pi(i)}}, \tag{18}$$

where we sum over all partitions π *of the set* $\{1, \dots, s\}$ *into* r *parts* $\pi(i)$ *representing* λ *as a refinement of* μ *in the sense that* $\mu_i = \sum_{j \in \pi(i)} \lambda_j$.

Example 3.16. Take the partition $\mu = (5,3)$ of $n = 8$ and the refinement $\lambda = (3,2,1,1,1)$. Then

$$b^{5,3}_{3,2,1,1,1} = b^{5}_{3,2} b^{3}_{1,1,1} + 3 b^{5}_{3,1,1} b^{3}_{2,1} + b^{5}_{2,1,1,1} b^{3}_{3}.$$

These terms come from the five ways in which $(5,3)$ can be refined to $(3,2,1,1,1)$. They are

$$(\{3,2\},\{1,1,1\}), 3 \times (\{3,1,1\},\{2,1\}), (\{2,1,1,1\},\{3\}).$$

There are three ways to do the second refinement depending on which 1 goes to the right (to 3 in $(5,3)$).

Proof. The number b^{μ}_{λ} from (17) times $(-1)^n$ is given by evaluating the cup product $\widetilde{c}^{\lambda}_{\mathscr{F}at}$ of the cocycles $\widetilde{c}^{\lambda_j}_{\mathscr{F}at}$ on every term of the dual cell $D(\Gamma)$ of any ribbon graph Γ in W_{μ}.

The dual cell is given by choosing graphs over Γ and taking nondegenerate $2n$-simplices

$$\Gamma_* = (\Gamma_0 \to \cdots \to \Gamma_{2n} \cong \Gamma)$$

(times $o(\Gamma_*)$) where each Γ_i is a chosen representative.

Look at all the terms in the dual cell $D(\Gamma)$ which begin with a fixed Γ_0 (fixed as an object over Γ). We will see that the sum of the values of the cup product $\widetilde{c}^{\lambda}_{\mathscr{F}at}$ on these terms is a sum of products corresponding to the sum of products on the right-hand side of (18).

Let $v_1,\dots,v_r$ be the nontrivalent vertices of Γ and let $T^1,\dots,T^r$ be the trees in Γ_0 which collapse to these vertices. Thus, T^i has $2\mu_i$ edges which are naturally ordered up to even permutation.

If the cup product $\tilde{c}^{\lambda}_{\mathscr{F}at}=\cup\tilde{c}^{\lambda_p}_{\mathscr{F}at}$ is nontrivial on a $2n$-simplex Γ_* beginning with Γ_0 then to each index p the cocycle $\tilde{c}^{\lambda_p}_{\mathscr{F}at}$ must be nontrivial on

$$\Gamma_{2n_p}\to\Gamma_{2n_p+1}\to\cdots\to\Gamma_{2n_p+2\lambda_p}, \tag{19}$$

where $n_p=\lambda_1+\lambda_2+\cdots+\lambda_{p-1}$. This means the edges which collapse in this sequence must all lie in the same tree T^i. Let $i=f(p)$. Then f is an epimorphism

$$f:\{1,\dots,s\}\twoheadrightarrow\{1,\dots,r\}, \tag{20}$$

so that μ_i is equal to the sum of the λ_p for all $p\in f^{-1}(i)$. In other words, $\pi=f^{-1}$ represents λ as a refinement of μ.

The value of the cocycle $\tilde{c}^{\lambda_p}_{\mathscr{F}at}$ on the middle $2\lambda_p$-simplex (19) depends only on the sequence of edges in T^i which are collapsing. Thus, if we fix epimorphism (20), then, for each index i, the order in which the edges of T^i collapse in (19) varies independently of the edges of T^j for $j\neq i$. Consequently, the sum of products becomes a product of sums ($\pi=f^{-1}$ still being fixed).

$$\sum_{\Gamma_*}\prod_{i=1}^{r}\prod_{p\in\pi(i)}\tilde{c}^{\lambda_p}_{\mathscr{F}at}(\Gamma_{2n_p},\dots,\Gamma_{2n_p+2\lambda_p})=\prod_{i=1}^{r}\sum_{T^i_*}\prod_{p\in\pi(i)}\tilde{c}^{\lambda_p}_{\mathscr{F}at}(T^i_{2n_p},\dots,T^i_{2n_p+2\lambda_p}),$$

where T^i_k is the inverse image in Γ_k of the vertex v_i of Γ.

But a different example with $r=1$ gives the same sum

$$(-1)^{\mu_i}b^{\mu_i}_{\lambda_{pi(i)}}=\sum_{T^i_*}\prod_{p\in\pi(i)}\tilde{c}^{\lambda_p}_{\mathscr{F}at}(T^i_{2n_p},\dots,T^i_{2n_p+2\lambda_p}).$$

Taking the product over all i and the sum over all π we get

$$(-1)^n b^{\mu}_{\lambda}=\sum_{\pi}\prod_{i=1}^{r}(-1)^{\mu_i}b^{\mu_i}_{\lambda_{\pi(i)}},$$

which is the same as (18) since $n=\sum\mu_i$. □

We can now compute some of the numbers b^{μ}_{λ}. We start with the following case which follows from Theorem 3.7.

Lemma 3.17. *In the case of the trivial partitions $\lambda=\mu=n$ we have*

$$b^n_n=\frac{1}{(-2)^{n+1}(2n+1)!!}.$$

By Lemma 3.15 this give the following.

Proposition 3.18. *If the partition λ of n has m_i terms equal to λ_i for $i=1,\dots,r$ (so that $\sum m_i\lambda_i=n$) then*

$$b^{\lambda}_{\lambda}=\prod_{i=1}^{r}m_i!(b^{\lambda_i}_{\lambda_i})^{m_i}=\prod_{i=1}^{r}\frac{m_i!}{((-2)^{\lambda_i+1}(2\lambda_i+1)!!)^{m_i}}.$$

Proof. The factor of $\prod m_i!$ is equal to the number of ways that the partition λ refines itself. $\square$

3.5. Kontsevich cycles in terms of MMM classes

We are now ready to show that the dual Konsevich cycles W_μ^* represent polynomials in the adjusted MMM classes $\widetilde{\kappa}_k$ (as we claimed in [8]). We will then conclude that their cohomology classes $[W_\mu^*]$ are linearly independent as promised in Remark 3.14.

To avoid circular reasoning, we must assume at this point that the numbers b_λ^μ may depend on the order of the parts of λ. Therefore, we take the ordering of both λ and μ to be nonincreasing: $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0$. If λ is a refinement of μ then $\lambda \leqslant \mu$ in lexicographic order. Consequently, the matrix

$$B_n = (b_\lambda^\mu)$$

is upper triangular. (A priori this uses only some of the numbers sb_λ^μ.) The diagonal entries b_λ^λ are nonzero by Proposition 3.18 so B_n is invertible. Let A_n be the inverse matrix

$$A_n = B_n^{-1} = (a_\mu^\lambda).$$

The entries of this matrix are rational numbers uniquely determined by the equation

$$\sum_\nu a_\lambda^\nu b_\nu^\mu = \delta_\lambda^\mu. \tag{21}$$

The main theorem is the following.

Theorem 3.19. *The cohomology classes of the dual Kontsevich cycles are polynomials in the adjusted MMM classes*

$$[W_\mu^*] = \sum_\lambda a_\mu^\lambda \widetilde{\kappa}_\lambda.$$

Remark 3.20. This formula holds in the rational cohomology of $\mathcal{G}_*$, and in the integral cohomology modulo torsion of the category of ribbon graphs and the mapping class group M_g^s for all $g, s \geqslant 1$ (and for $g = 0, s \geqslant 3$).

Proof. Since the matrix B_n is invertible, the system of linear equations given in Theorem 3.13 has the unique solution

$$W_\mu^* = \sum_\lambda a_\mu^\lambda \widetilde{c}_{\mathcal{G}}^\lambda$$

provided that λ is in nonincreasing order. $\square$

Since the coefficients a_μ^λ form an invertible matrix and the monomials $\widetilde{\kappa}_\lambda$ are linearly independent by Theorem 3.1 we have the following.

Corollary 3.21. *The cohomology classes $[W_\mu^*]$ are linearly independent over $\mathbb{Q}$.*

This, in turn, implies the following as explained in Remark 3.14.

Corollary 3.22. *The cocycles $\widetilde{c}^{\lambda}_{\mathscr{G}}$ and the numbers b^{μ}_{λ} are independent of the ordering of the partitions λ, μ.*

Combining these with Theorem 3.13, we get the following.

Corollary 3.23. *The cohomology classes $[W^*_{\mu}]$ form a $\mathbb{Q}$-basis for the polynomial algebra generated by the adjust MMM classes $\widetilde{\kappa}_k$ for $k \geqslant 1$.*

3.6. Computing a^{μ}_{λ}

We will use the defining equation (21) to determine the numbers a^{μ}_{λ} in some simple cases.

Proposition 3.24. *When $\lambda = \mu = n$ we have*

$$a^n_n = \frac{1}{b^n_n} = (-2)^{n+1}(2n+1)!!.$$

Proposition 3.25. *When $\lambda = \mu = (\lambda_1^{m_1}, \dots, \lambda_r^{m_r})$ with $\sum m_i\lambda_i = n$ then*

$$a^{\lambda}_{\lambda} = \frac{1}{b^{\lambda}_{\lambda}} = \prod_{i=1}^{r} \frac{((-2)^{k_i+1}(2k_i+1)!!)^{m_i}}{m_i!}.$$

For example, $a^{2,2}_{2,2} = 120^2/2 = 7200$ and $a^{1,1,1}_{1,1,1} = 12^3/3! = 288$. These numbers are not always integers. For example,

$$a^{1^5}_{1^5} = \frac{12^5}{5!} = \frac{10368}{5}.$$

Corollary 3.26. *The dual Kontsevich cycle $[W^*_{\lambda}]$ is a polynomial of degree $\sum m_i$ in the adjusted MMM classes with leading term*

$$\prod_{i=1}^{r} \frac{((-2)^{\lambda_i+1}(2\lambda_i+1)!!\widetilde{\kappa}_{\lambda_i})^{m_i}}{m_i!}.$$

In order to compute the remaining terms, we need to compute a^{μ}_{λ} for $\lambda \neq \mu$. The first case is $a^{n+k}_{n,k}$ which occurs in the equation

$$[W^*_{n,k}] = a^{n,k}_{n,k}\widetilde{\kappa}_n\widetilde{\kappa}_k + a^{n+k}_{n,k}\widetilde{\kappa}_{n+k}. \tag{22}$$

The defining equation for $a^{n+k}_{n,k}$ is

$$a^{n+k}_{n,k}b_{n+k} + a^{n,k}_{n,k}b^{n+k}_{n,k} = 0.$$

From this and Proposition 3.24 we get

$$a^{n,k}_{n,k} = (-2)^{2n+2k+2}(2n+1)!!(2k+1)!!,$$

$$\begin{aligned} a^{n+k}_{n,k} &= -a_{n+k}a^{n,k}_{n,k}b^{n+k}_{n,k} \\ &= -(-2)^{2n+2k+3}(2n+2k+1)!!(2n+1)!!(2k+1)!!b^{n+k}_{n,k} \end{aligned}$$

if $n \neq k$ and

$$a^{n,n}_{n,n} = -(-2)^{4n+3}((2n+1)!!)^2,$$

$$a^{2n}_{n,n} = (-2)^{4n+2}(4n+1)!!((2n+1)!!)^2 b^{2n}_{n,n}.$$

In the next section we will compute $b^{n+k}_{n,k}$ in the special case $k = 1$.

4. Some computations

(1) The degenerate case $n = 0$.
(2) Computation of $b^{n+1}_{n,1}$.
(3) Conjectures.

We will compute the numbers $b^{n+1}_{n,1}$ and obtain $a^{n+1}_{n,1}$ and the coefficients of $[W^*_{n,1}]$ when expressed as a polynomial in the adjusted MMM classes. To simplify the notation, we write a_n and b_n instead of a^n_n and b^n_n in this section.

4.1. The degenerate case $n = 0$

First, we consider the degenerate case $n = 0$. In this case, it is easy to compute $W^*_{k,0}$ and work backwards since this cocycle counts the number of pairs of vertices, one of codimension $2k$ and the other of codimension 0 (i.e., trivalent). The number of trivalent vertices of a graph with one vertex of multiplicity $2k+1$ is t where

$$t + 2k + 1 = -2\chi = -2\widetilde{\kappa}_0.$$

So, $t = -2\widetilde{\kappa}_0 - (2k+1)$ and

$$[W^*_{k,0}] = t[W^*_k] = (-2\widetilde{\kappa}_0 - (2k+1))(-2)^{k+1}(2k+1)!!\widetilde{\kappa}_k,$$

$$[W^*_{k,0}] = (-2)^{k+2}(2k+1)!!\widetilde{\kappa}_k\widetilde{\kappa}_0 - (2k+1)(-2)^{k+1}(2k+1)!!\widetilde{\kappa}_k. \tag{23}$$

The right-hand side should be divided by 2 when $k = 0$.

For the case $k = 1$ this gives

$$[W^*_{1,0}] = [W^*_{0,1}] = -24\widetilde{\kappa}_0\widetilde{\kappa}_1 - 36\widetilde{\kappa}_1.$$

Consequently, $a^1_{0,1} = -36$ and

$$b^1_{0,1} = \frac{1}{2^5 9}\, a^1_{0,1} = -\frac{1}{8}.$$

4.2. Computation of $b^{n+1}_{n,1}$

To obtain $b^{n+1}_{n,1}$ in general we use the formula from Lemma 3.11

$$b^{n+1}_{n,1} = (-1)^{n+1}(\widetilde{c}^{\,n}_{\mathscr{F}at} \cup \widetilde{c}^{\,1}_{\mathscr{F}at})D(\Gamma),$$

where Γ is any graph in the Witten cycle W_{n+1}, i.e., a ribbon graph which is trivalent except at one vertex of multiplicity $2n+2$.

The dual cell $D(\Gamma)$ is a sum

$$D(\Gamma)=\sum_{\Gamma_*} o(\Gamma_*)(\Gamma_0,\dots,\Gamma_{2n+2}=\Gamma).$$

When we evaluate the cup product $(-1)^{n+1}(\tilde{c}^n_{\mathscr{F}at}\cup\tilde{c}^1_{\mathscr{F}at})$ we get

$$b^{n+1}_{n,1}=(-1)^{n+1}\sum_{\Gamma_*} o(\Gamma_*)\tilde{c}^n_{\mathscr{F}at}(\Gamma_0,\dots,\Gamma_{2n})\tilde{c}^1_{\mathscr{F}at}(\Gamma_{2n},\Gamma_{2n+1},\Gamma).$$

There are three possible configurations for Γ_{2n}. They are Case 1, given in Fig. 2 and Cases 2a, 2b, given in Fig. 3 from Section 2. In all cases, Γ_{2n} is odd-valent so it has a natural orientation. The orientation of the sequence Γ_* can then be expressed as a product of two orientations

$$o(\Gamma_0,\dots,\Gamma_{2n},\Gamma',\Gamma)=o(\Gamma_0,\dots,\Gamma_{2n})o(\Gamma_{2n},\Gamma',\Gamma),$$

where we write $\Gamma'=\Gamma_{2n+1}$. Consequently, for any fixed Γ_{2n}, we get

$$(-1)^n\sum o(\Gamma_0,\dots,\Gamma_{2n})\tilde{c}^n_{\mathscr{F}at}(\Gamma_0,\dots,\Gamma_{2n})\left(-\sum_{\Gamma'} o(\Gamma_{2n},\Gamma',\Gamma)\tilde{c}^1_{\mathscr{F}at}(\Gamma_{2n},\Gamma',\Gamma)\right).$$

The first factor is b_n regardless of Γ_{2n} so

$$\frac{b^{n+1}_{n,1}}{b_n}=-\sum_{\Gamma_{2n},\Gamma'} o(\Gamma_{2n},\Gamma',\Gamma)\tilde{c}^1_{\mathscr{F}at}(\Gamma_{2n},\Gamma',\Gamma).$$

If we denote the sequence $\Gamma_{2n}\to\Gamma'\to\Gamma$ by Γ_* and substitute $1/b_n=a_n$ we get

$$a_n b^{n+1}_{n,1}=-\sum_{\Gamma_*} o(\Gamma_*)\tilde{c}^1_{\mathscr{F}at}(\Gamma_*). \tag{24}$$

Recall that

$$\tilde{c}^1_{\mathscr{F}at}(\Gamma_*)=\sum\frac{\mu(v_0)}{4}\frac{\sum\operatorname{sgn}(a,b,c)}{|C_0||C_1||C_2|}$$

with the first sum being over all vertices v_0 of Γ_{2n} where $\mu(v_0)$ is the multiplicity of v_0 and $|C_i|$ is the valence of the image of v_i in Γ_{2n+i}. The sign sum

$$\sum\operatorname{sgn}(a,b,c)=\operatorname{sgn}(x_1\cdots x_{2n+5})$$

is the number of times that the letters a,b,c occur in the correct cyclic order in the word $w=x_1\cdots x_{2n+5}$ minus the number of times it occurs in the other cyclic order in w where the jth letter x_j of w is equal to b_i ($b_0=a, b_1=b, b_2=c$) if the jth region in the complement of the graph reaches v_0 at the ith step (Γ_{2n+i}).

The sum (24) breaks up into three parts depending on the graph Γ_{2n}.

Case 1: Suppose the graph Γ_{2n} is given by Fig. 2 and $T'=T_{2n}/e_1$. Then

$$o(\Gamma_*)=(-1)^m.$$

The number of times this same configuration (with fixed $1\leqslant m\leqslant 2n+2$) occurs is

$$2n+5.$$

The cyclic set cocycle $\widetilde{c}^1_{\mathscr{F}at}(\Gamma_*)$ has two terms

(1) The center vertex has multiplicity $2n+1$ and average sign

$$\frac{\operatorname{sgn}(a^m c a^{2n-m+3} b)}{(2n+3)(2n+4)(2n+5)} = \frac{2n-2m+3}{(2n+3)(2n+4)(2n+5)}$$

for a contribution of

$$\frac{(2n+1)(2n-2m+3)}{4(2n+3)(2n+4)(2n+5)}.$$

(2) The vertex v_1 has multiplicity 1 and average sign

$$\frac{\operatorname{sgn}(a^3 b^{m-1} c b^{2n-m+2})}{3(2n+4)(2n+5)} = -\frac{2n-2m+3}{(2n+4)(2n+5)}.$$

The total value of the cocycle is

$$\widetilde{c}^1_{\mathscr{F}at}(\Gamma_*) = \frac{(2n-2m+3)(2n+1-(2n+3))}{4(2n+3)(2n+4)(2n+5)} = \frac{-(2n-2m+3)}{2(2n+3)(2n+4)(2n+5)}.$$

Multiply this by $(-1)^m(2n+5)$ and sum over all $1 \leqslant m \leqslant 2n+2$ to get

$$\frac{-1}{2(2n+3)(2n+4)} \sum_{m=1}^{2n+2} (-1)^m (2n-2m+3) = \frac{n+1}{(2n+3)(2n+4)}.$$

Case 2a: Suppose Γ_{2n} is given by Fig. 3a and $\Gamma' = \Gamma_{2n}/e_1$. Then

$$o(\Gamma_*) = \operatorname{sgn}(a_1, \ldots, a_{2n+3}, b_1, b_2) = 1.$$

This configuration occurs $2n+5$ times and the cyclic set cocycle has two terms.

(1) The center vertex has multiplicity $2n+1$ and average sign

$$\frac{\operatorname{sgn}(a^{2n+3} bc)}{(2n+3)(2n+4)(2n+5)} = \frac{1}{(2n+4)(2n+5)}$$

for a contribution of

$$\frac{2n+1}{4(2n+4)(2n+5)}.$$

(2) The vertex v_1 has multiplicity 1 and average sign

$$\frac{\operatorname{sgn}(a^2cab^{2n+1})}{3(2n+4)(2n+5)} = \frac{-(2n+1)}{3(2n+4)(2n+5)}.$$

So

$$\widetilde{c}^{1}_{\mathscr{F}at}(\Gamma_*) = \frac{2n+1}{6(2n+4)(2n+5)}.$$

Multiply this by $2n+5$ for a total of

$$\frac{2n+1}{6(2n+4)}.$$

Case 2a′: Suppose that T_{2n} is the same (Fig. 3a) but $\Gamma' = \Gamma_{2n}/e_2$. Then $o(\Gamma_*) = -1$. The configuration still occurs $2n+5$ times and the cyclic set cocycle again has two terms.

(1) At v_1 we have

$$\frac{1}{4}\,\frac{\operatorname{sgn}(a^2bac^{2n+1})}{3\cdot 4(2n+5)} = \frac{2n+1}{48(2n+5)}.$$

(2) At v_2 we get

$$\frac{1}{4}\,\frac{\operatorname{sgn}(ba^3c^{2n+1})}{3\cdot 4(2n+5)} = \frac{-3(2n+1)}{48(2n+5)}.$$

So,

$$\widetilde{c}^{1}_{\mathscr{F}at}(\Gamma_*) = \frac{-(2n+1)}{24(2n+5)},$$

making the total in this case

$$\frac{2n+1}{24}.$$

Case 2b is the same as Case 2a. (The sign changes twice.) So

$$a_n b^{n+1}_{n,1} = \frac{n+1}{(2n+3)(2n+4)} + \frac{2n+1}{3(2n+4)} + \frac{2n+1}{12}.$$

Simplifying this expression we get

$$a_n b^{n+1}_{n,1} = \frac{2n+5}{12} - \frac{1}{2(2n+3)}.$$

Multiplying by $a_1 a_{n+1} = 12(-2)^{n+2}(2n+3)!!$ we get

$$a_{n+1}a_1a_n b^{n+1}_{n,1} = (-2)^{n+2}(2n+5)!! + 3(-2)^{n+3}(2n+1)!!.$$

Theorem 4.1. *For $n \neq 1$ we have*

$$[W^*_{n,1}] = 3(-2)^{n+3}(2n+1)!!(\widetilde{\kappa}_n\widetilde{\kappa}_1 - \widetilde{\kappa}_{n+1}) - (-2)^{n+2}(2n+5)!!\widetilde{\kappa}_{n+1}.$$

For $n = 1$ we divide the right-hand side by 2.

For example, we have

$$[W^*_{0,1}] = -24\widetilde{\kappa}_0\widetilde{\kappa}_1 - 36\widetilde{\kappa}_1,$$

$$[W^*_{1,1}] = 72\widetilde{\kappa}_1^2 + 348\widetilde{\kappa}_2,$$

$$[W^*_{2,1}] = -1440\widetilde{\kappa}_2\widetilde{\kappa}_1 - 13680\widetilde{\kappa}_3,$$

$$[W^*_{3,1}] = 20160\widetilde{\kappa}_3\widetilde{\kappa}_1 + 312480\widetilde{\kappa}_4,$$

$$[W^*_{4,1}] = -362880\widetilde{\kappa}_4\widetilde{\kappa}_1 - 8285760\widetilde{\kappa}_5.$$

This agrees with calculation (23) when $n=0, k=1$ and also agrees with the calculation of Arbarello and Cornalba [1]. The sign difference comes from the fact that they use the opposite sign for all $\widetilde{\kappa}_{\text{even}}$. (What we call $\widetilde{\kappa}_k$ is what they would call $(-1)^{k+1}\kappa_k$ restricted to the open moduli space of curves.) The calculations of [1] give cohomology classes which they showed act as Poincaré duals of the Kontsevich cycles with respect to products of boundary cycles. Therefore, Theorems 3.1 and 3.19 already suggests that they must be correct.

4.3. Conjectures

The formula in Theorem 4.1 has an apparent symmetry which also appears in formula (23) when it is rephrased as follows:

$$[W^*_{k,0}] = (-2)^{k+2}(2k+1)!!(\widetilde{\kappa}_k\widetilde{\kappa}_0 - \widetilde{\kappa}_k) - (-2)^{k+1}(2k+3)!!\widetilde{\kappa}_k. \tag{25}$$

This leads to the following conjecture.

Conjecture 4.2.

$$[W^*_{n,k}] = (-2)^{n+k+2}(2n+1)!!(2k+1)!!(\widetilde{\kappa}_n\widetilde{\kappa}_k - \widetilde{\kappa}_{n+k}) - (-2)^{n+k+1}(2n+2k+3)!!\widetilde{\kappa}_{n+k}.$$

The right-hand side should be divided by 2 if $n = k$.

Remark 4.3. First, note that this conjectured formula can be simplified using the numbers $a_n = (-2)^{n+1}(2n+1)!!$:

$$[W^*_{n,k}] = a_n a_k(\widetilde{\kappa}_n\widetilde{\kappa}_k - \widetilde{\kappa}_{n+k}) + \frac{1}{2}\, a_{n+k+1}\widetilde{\kappa}_{n+k}.$$

Next, we also note that it is symmetrical in n, k. And finally, in the first new case when $n = k = 2$ it gives

$$[W^*_{2,2}] = \frac{120^2}{2}(\widetilde{\kappa}_2^2 - \widetilde{\kappa}_4) + \frac{665280}{4}\widetilde{\kappa}_4 = 7200\widetilde{\kappa}_2^2 + 159120\widetilde{\kappa}_4, \tag{26}$$

which agrees with Arbarello and Cornalba [1].

Remark 4.4. Michael Kleber and I have made further progress on calculation and interpretation of the coefficients b^n_λ. So far we verified Conjecture 4.2 for all $k \leqslant 7$. In particular this proves (26). We also found that

$$[W^*_{1,1,1}] = 288\widetilde{\kappa}_1^3 + 4176\widetilde{\kappa}_1\widetilde{\kappa}_2 + 20736\widetilde{\kappa}_3.$$

This, together with Theorem 4.1 and Corollary 3.8, verifies all calculations of the coefficients a_λ^μ given by Arbarello and Cornalba. Details will be given in a subsequent joint paper.

Remark 4.5. Conjecture 4.2 was proved 6 days after it was announced by Mondello [15] and 9 days after that by Michael Kleber and the author [10]. Both papers give general formulas for the coefficients $a_\lambda^\mu, b_\lambda^\mu$.

References

[1] E. Arbarello, M. Cornalba, Combinatorial and algebro-geometric cohomology classes on the moduli spaces of curves, J. Algebraic Geom. 5 (1996) 705–749.
[2] J.M. Boardman, R.M. Vogt, Homotopy Invariant Algebraic Structures on Topological Spaces, in: Lecture Notes in Mathematics, Vol. 347, Springer, Berlin, 1973.
[3] J. Conant, K. Vogtmann, On a theorem of Kontsevich, math.QA/0208169, 2002.
[4] M. Culler, K. Vogtmann, Moduli of graphs and automorphisms of free groups, Invent. Math. 84 (1) (1986) 91–119.
[5] E. Getzler, J.D.S. Jones, A_∞-algebras and the cyclic bar construction, Illinois J. Math. 34 (1989) 256–283.
[6] J.L. Harer, Stability of the homology of the mapping class groups of orientable surfaces, Ann. Math. (2) 121 (2) (1985) 215–249.
[7] J. Hubbard, H. Masur, Quadratic differentials and foliations, Acta Math. 142 (1979) 221–273.
[8] K. Igusa, Combinatorial Miller–Morita–Mumford classes and Witten cycles, math.GT/0207042, 2002.
[9] K. Igusa, Higher Franz–Reidemeister Torsion, in: AMS/IP Studies in Advance Mathematics, Vol. 31, International Press, Somerville, MA, 2002.
[10] K. Igusa, M. Kleber, Increasing trees and Kontsevich cycles, math.AT/0303353, 2003, preprint.
[11] M. Kontsevich, Intersection theory on the moduli space of curves and the matrix Airy function, Commun. Math. Phys. 147 (1) (1992) 1–23.
[12] M. Kontsevich, Formal (non)commutative symplectic geometry, The Gelfand Mathematical Seminars, 1990–1992, Birkhäuser, Boston, 1993, pp. 173–187.
[13] M. Kontsevich, Feynman diagrams and low-dimensional topology, First European Congress of Mathematics, Vol. II (Paris, 1992), Birkhäuser, Basel, 1994, pp. 97–121.
[14] E.Y. Miller, The homology of the mapping class group, J. Differential Geom. 24 (1) (1986) 1–14.
[15] G. Mondello, Combinatorial classes on the moduli space of curves are tautological, math.AT/0303207, 2003.
[16] S. Morita, Characteristic classes of surface bundles, Invent. Math. 90 (3) (1987) 551–577.
[17] S. Morita, Geometry of Characteristic Classes, in: Translations of Mathematical Monographs, Vol. 199, American Mathematical Society, Providence, RI, 2001.
[18] R.C. Penner, The decorated Teichmüller space of punctured surfaces, Commun. Math. Phys. 113 (2) (1987) 299–339.
[19] J.D. Stasheff, Homotopy associativity of H-spaces I, II, Trans. AMS 108 (1963) 275–312.
[20] K. Strebel, Quadratic Differentials, Springer, Berlin, 1984.