Unsteady unidirectional micropolar fluid flow

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(Received 18 August 2011; accepted 29 September 2011; published online 10 November 2011)

Abstract This paper considers the unsteady unidirectional flow of a micropolar fluid, produced by the sudden application of an arbitrary time dependent pressure gradient, between two parallel plates. The no-slip and the no-spin boundary conditions are used. Exact solutions for the velocity and microrotation distributions are obtained based on the use of the complex inversion formula of Laplace transform. The solution of the problem is also considered if the upper boundary of the flow is a free surface. The particular cases of a constant and a harmonically oscillating pressure gradient are then examined and some numerical results are illustrated graphically. [doi:10.1063/2.1106205]

Keywords Laplace transform, micropolar fluid, unidirectional flow

The theory of micropolar fluids, introduced by Eringen,[1] gives a mathematical model for a type of fluids with microstructure. This type of fluids consists of rigid, randomly oriented (or spherical) particles with their own spins and microrotations, suspended in a viscous medium. It exhibits micro-rotational effects and micro-rotational inertia. This model includes the classical Navier-Stokes equations as a special case, but can cover, both in theory and applications, many more phenomena than the classical model. Physically, the mathematical model underlying micropolar fluids may represent the behavior of polymeric additives, animal blood with rigid cells, liquid crystals, dirty oils, bubbly fluids and other biological fluids.[1,2] A comprehensive review of micropolar fluids was provided by Ariman et al.[3,4] kaszewicz[5] has presented the mathematical theory of equations of micropolar fluids and some of its applications.

The steady micropolar Poiseuille flow between two stationary parallel plates, due to a constant pressure gradient, was presented in Ref. 6. Cvetkovié[7] discussed the problem of steady micropolar flow between two parallel plates with couple stress boundary conditions. The micropolar fluid flow over a semi-infinite flat plate has been described by using the parabolic coordinates and the method of series truncation for low to large Reynold numbers in Ref. 8. The problem of steady micropolar fluid flow through a wavy tube is studied in Ref. 9. The three dimensional micropolar fluid flow in a straight pipe with variable cross section has been discussed in Ref. 10. The unsteady micropolar fluid flow has attracted the attention of a low number of researchers. Debnath[11] investigated the unsteady blood flow through rigid circular tubes. Ashmawy[12] discussed the unsteady Couette micropolar fluid flow with the effect of slip boundary condition. The unsteady motion of a micropolar fluid filling a half space has been investigated in Ref. 13.

Delhommelle et al.[14] used non-equilibrium molecular dynamics to study the flow of a micropolar fluid and tested an extended Navier-Stokes theory for such fluids. They found that the angular streaming velocity and the translational streaming velocity are in good agreement with the predictions of extended Navier-Stokes theory, they found also that the translational streaming velocity profile deviates from the classical parabolic profile. Pięal[15] studied Poiseuille flows in microchannel in detail and confirmed that micropolar model is applicable for small characteristic geometrical dimension of the flow

The field equations governing an incompressible micropolar fluid flow in the absence of body forces and body couples are given in vector forms as

$$\operatorname{div} \boldsymbol{u} = 0, \tag{1}$$

$$-(\mu+\kappa)\,\mathrm{curl}\,\mathrm{curl}\boldsymbol{u} + \kappa\mathrm{curl}\,\boldsymbol{\nu} - \mathrm{grad}p = \rho\dot{\boldsymbol{u}}, \tag{2}$$

$$(\alpha+\beta+\gamma)\,\mathrm{grad}\ \mathrm{div}\boldsymbol{\nu} - \gamma\mathrm{curlcurl}\boldsymbol{\nu} + \kappa\mathrm{curl}\boldsymbol{u} - 2\kappa\boldsymbol{\nu} = \rho\, j\, \dot{\boldsymbol{\nu}}, \tag{3}$$

The field unknowns are the velocity vector $\boldsymbol{u}$, the microrotation vector $\boldsymbol{\nu}$, and the fluid pressure p, while the fluid density ρ, and the microinertia j are assumed to be constants. The material constants $(\mu,\ \kappa)$ are viscosity coefficients and $(\alpha,\ \beta,\ \gamma)$ are gyro-viscosity coefficients. The dot denotes the material time derivative.

The constitutive equations for the stresses t_{ij} and couple stresses m_{ij} can be written as

$$t_{ij} = -p\,\delta_{ij} + \mu\,(u_{j,i} + u_{i,j}) + \kappa\,(u_{j,i} - \varepsilon_{ijk}\nu_k)\,, \tag{4}$$

$$m_{ij} = \alpha\nu_{r,r}\delta_{ij} + \beta\nu_{i,j} + \gamma\nu_{j,i}, \tag{5}$$

where the comma denotes partial differentiation, δ_{ij} and ε_{ijk} are the Kronecker delta and the alternating tensor, respectively.

Initially the fluid is assumed to be at rest. It is then set in motion by a sudden application of a time dependent pressure gradient. Using the Cartesian coordinates (x, y, z) with the x-axis coincident with the central line, the y-axis is normal to the plates (see Fig. 1). Since the fluid flow extends to infinity and the motion is due to the pressure gradient only, then the field unknowns are

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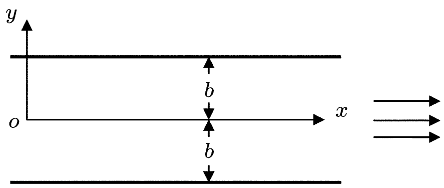

Fig. 1. Poiseuille flow between fixed parallel plates.

functions of y and t only, thus the velocity and micro-rotation components become

$\boldsymbol{u} = [u(y,t), 0, 0]$ and $\boldsymbol{\nu} = [0, 0, \nu(y,t)]$

The nonlinear terms of the field equations vanish automatically.

The equation of continuity (1) is satisfied identically and the field Eqs. (2), (3) reduce to

$$(\mu+\kappa)\frac{\partial^2 u}{\partial y^2} + \kappa\frac{\partial \nu}{\partial y} - \rho\frac{\partial u}{\partial t} - \frac{\partial p}{\partial x} = 0, \tag{6}$$

$$\gamma\frac{\partial^2 \nu}{\partial y^2} - \kappa\frac{\partial u}{\partial y} - 2\kappa\nu - \rho j\frac{\partial \nu}{\partial t} = 0. \tag{7}$$

The boundary and initial conditions of the problem have the forms

$$u(\pm b, t) = 0, \nu(\pm b, t) = 0, \text{ for all } t, \tag{8}$$

$$u(y,0) = 0, \ \nu(y,0) = 0, \text{ for all } -b \le y \le b. \tag{9}$$

The spin gradient viscosity coefficient j is assumed to be a constant and is given by[2]

$$j = \frac{2\gamma}{2\mu+\kappa}. \tag{10}$$

We now introduce the following non-dimensional variables

$$u^* = \frac{\rho\, b}{\mu+\kappa}u, \quad \nu^* = \frac{\rho\,\kappa\, b^2}{(\mu+\kappa)^2}\nu, \quad x^* = \frac{x}{b},$$
$$y^* = \frac{y}{b}, \quad t^* = \frac{\mu+\kappa}{\rho b^2}t, \quad p^* = \frac{\rho b^2}{(\mu+\kappa)^2}p,$$
$$m_{ij}^* = \frac{\rho\kappa b^3}{\gamma(\mu+\kappa)^2}m_{ij}, \quad t_{ij}^* = \frac{\rho b^2}{\mu(\mu+\kappa)}t_{ij}\,. \tag{11}$$

In terms of these variables, Eqs. (6), (7), take the following forms (dropping asterisks for convenience)

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial \nu}{\partial y} - \frac{\partial u}{\partial t} - \frac{\partial p}{\partial x} = 0, \tag{12}$$

$$\frac{\partial^2 \nu}{\partial y^2} - f\frac{\partial u}{\partial y} - g\,\nu - h\frac{\partial \nu}{\partial t} = 0. \tag{13}$$

Also, the non-vanishing shear stresses and couple stresses take the forms

$$t_{xy} = \tilde{k}\frac{\partial \mathrm{u}}{\partial \mathrm{y}} - \nu, \quad t_{yx} = \frac{\partial \mathrm{u}}{\partial \mathrm{y}} + \nu\ ,$$

$$m_{yz} = \frac{1}{\tilde{\beta}}m_{zy} = \frac{\partial \nu}{\partial y}, \tag{14}$$

where

$$f = \frac{\kappa^2 b^2}{\gamma(\mu+\kappa)}, \quad g = \frac{2\kappa b^2}{\gamma}, \quad \lambda = g - f,$$
$$h = \frac{(\mu+\kappa)\, j}{\gamma} = \frac{g}{\lambda}, \quad \tilde{\kappa} = \frac{\mu}{\mu+\kappa}, \quad \tilde{\beta} = \frac{\beta}{\gamma}. \tag{15}$$

Introducing the Laplace transform (denoted by an over bar) defined by the formula

$$\bar{F}(y,s) = \int_0^\infty \mathrm{e}^{-st} F(y,t)\mathrm{d}t, \tag{16}$$

into Eqs. (12) and (13) and using the initial conditions Eq. (9), we obtain the general solutions of the field equations in the Laplace domain as

$$\bar{u} = A_1\cosh(m_1 y) + A_2\sinh(m_1 y) + A_3\cosh(m_2 y) + A_4\sinh(m_2 y) + \frac{1}{s}\bar{\phi}(s)\,, \tag{17}$$

$$\bar{\nu} = B_1\cosh(m_1 y) + B_2\sinh(m_1 y) + B_3\cosh(m_2 y) + B_4\sinh(m_2 y), \tag{18}$$

where

$$\bar{\phi}(s) = -\frac{\partial \bar{p}}{\partial x}, \quad m_1^2 = \lambda + s\ , \quad m_2^2 = h\, s.$$

Applying the boundary conditions Eq. (8) in the Laplace transform domain, we get

$$A_2 = A_4 = B_1 = B_3 = 0\,, \tag{19}$$

$$A_1 = \frac{-\bar{\phi}}{sQ(s)}(h-1)\sqrt{s}\sqrt{\lambda+s}\sinh(m_2),$$

$$A_3 = \frac{\bar{\phi}}{sQ(s)}\sqrt{h}\lambda\sinh(m_1), \tag{20}$$

$$B_2 = \frac{\bar{\phi}}{sQ(s)}\lambda(h-1)\sqrt{s}\sinh(m_2)\,,$$

$$B_4 = \frac{-\bar{\phi}}{sQ(s)}\lambda(h-1)\sqrt{s}\sinh(m_1), \tag{21}$$

where

$$Q(s) = (h-1)\sqrt{s}\sqrt{\lambda+s}\sinh(m_2)\cosh(m_1) - \lambda\sqrt{h}\,\sinh(m_1)\cosh(m_2). \tag{22}$$

Using the complex inversion formula of the Laplace transform,[16] given by

$$F(y,t) = \frac{1}{2\pi\,\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \mathrm{e}^{st}\bar{F}(y,s)\mathrm{d}s, \tag{23}$$

where $\mathrm{i} = \sqrt{-1}$ and σ is a constant greater than all the real parts of the singularities of $\bar{F}(y,s)$ together with Cauchy residue theorem and convolution theorem, we get the velocity and microrotation in the physical domain as

$$u(y,t)=\sum_{n=1}^{\infty}\frac{2\,\alpha_n}{\beta_n\,\Delta}[(h-1)\,\alpha_n\beta_n\sin\beta_n\cos\alpha_n y+\lambda h\sin\alpha_n\cos\beta_n y]\,\psi_1(t)\,, \tag{24}$$

$$\nu(y,t)=2\lambda(h-1)\sum_{n=1}^{\infty}\frac{\alpha_n}{\Delta}(\sin\alpha_n\sin\beta_n y-\sin\beta_n\sin\alpha_n y)\,\psi_1(t)\,, \tag{25}$$

where

$$\Delta=\alpha_n\left[(h-1)\left(\lambda+\alpha_n^2\right)+\lambda h\right]\sin\alpha_n\sin\beta_n-(h-1)\left(\lambda+2\alpha_n^2\right)\sin\beta_n\cos\alpha_n-\beta_n\left[(h-1)\,\alpha_n^2+\lambda\right]\cos\alpha_n\cos\beta_n\,. \tag{26}$$

The shear stress and couple stress thus take the forms

$$t_{yx}=\sum_{n=1}^{\infty}\frac{-2\alpha_n}{\Delta}\left[\,(h-1)\left(\alpha_n^2+\lambda\right)\sin\beta_n\sin\alpha_n y+\lambda\sin\alpha_n\sin\beta_n y\right]\psi_1(t)\,, \tag{27}$$

$$m_{yz}=\frac{1}{\bar{\beta}}m_{zy}=2\lambda(h-1)\sum_{n=1}^{\infty}\frac{\alpha_n}{\Delta}\left(\beta_n\sin\alpha_n\cos\beta_n y-\alpha_n\sin\beta_n\cos\alpha_n y\right)\psi_1(t)\,, \tag{28}$$

where

$$\psi_1(t)=\int_0^t\phi(t-\tau)\,\mathrm{e}^{-\left(\lambda+\alpha_n^2\right)\tau}\mathrm{d}\tau\,. \tag{29}$$

The volume flux across a plane normal to the flow can be obtained from the relation

$$F(t)=\int_{-1}^{1}u(y,t)\mathrm{d}y=\sum_{n=1}^{\infty}\frac{4\alpha_n\left[\lambda h+(h-1)\,\beta_n^2\right]}{\beta_n^2\;\Delta}\psi_1(t)\sin\alpha_n\sin\beta_n. \tag{30}$$

Here, we assume that the fluid region is bounded by a fixed infinite plate at $y=-b$ and a free surface at $y=b$. The boundary conditions for this case are given by

$$u(-b,t)=0,\quad t_{yx}(b,t)=0,\quad \nu(\pm b,t)=0,\quad \text{for all}\ \ t. \tag{31}$$

Following the same procedure of the previous sections, we get the corresponding solutions for this case as

$$u(y,t)=\sum_{n=1}^{\infty}\frac{2\alpha_n}{\beta_n\Delta_1}\left[\,(h-1)\,\alpha_n\beta_n\sin 2\beta_n\cdot\cos\alpha_n(1-y)+\lambda h\sin 2\alpha_n\cos\beta_n(1-y)\right]\psi_1(t)\,, \tag{32}$$

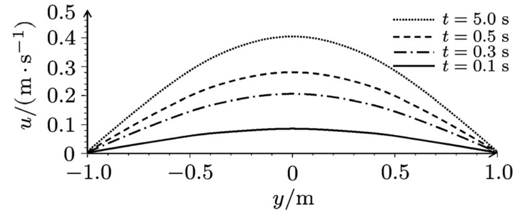

Fig. 2. Velocity distribution for $\kappa/\mu=2.0$.

$$\nu(y,t)=\sum_{n=1}^{\infty}\frac{2\lambda(h-1)\,\alpha_n}{\Delta_1}\left[\,\sin 2\alpha_n\sin\beta_n(1-y)-\sin 2\beta_n\sin\alpha_n(1-y)\right]\psi_1(t)\,, \tag{33}$$

$$\Delta_1=2\alpha_n\left[(h-1)\left(\lambda+\alpha_n^2\right)+\lambda h\right]\sin 2\alpha_n\sin 2\beta_n+(h-1)\left(\lambda+2\alpha_n^2\right)\sin 2\beta_n\cos 2\alpha_n-2\beta_n\left[(h-1)\,\alpha_n^2+\lambda\right]\cos 2\alpha_n\cos\,2\beta_n\,. \tag{34}$$

The volume flux across a plane normal to the flow is then

$$F(t)=\sum_{n=1}^{\infty}\frac{4\,\alpha_n\left[\lambda h+(h-1)\;\beta_n^2\right]}{\beta_n^2\,\Delta_1}\psi_1(t)\cdot\sin 2\alpha_n\sin 2\beta_n. \tag{35}$$

We consider some special cases of physical interest as follows:

(1) Constant pressure gradient: In this case the non-dimensional pressure gradient function is $-\partial p/\partial x=\phi(t)=1$.

Figure 2 indicates that the velocity increases at any point with the increase of time. As expected the velocity is an even function and reaches its maximum value at the central line between the two plates. We observe also in this graph that the steady state is established in the limit as $t\to\infty$. We observe from Fig. 3 that the magnitude of microrotation also increases at any point with the increase of time. It is an odd function and vanishes at the central line. It is clear from Figs. 4 and 5 that the velocity decreases and the magnitude of microrotation increases monotonically with the increase of the micropolarity parameter. Figure 6 shows that the flux decreases with the increase of the micropolarity parameter.

(2) Harmonically oscillating pressure gradient: in this case the non-dimensional pressure gradient function is $-\partial p/\partial x=\phi(t)=\sin t$. Therefore

$$\psi_1(t)=\frac{\mathrm{e}^{-\left(\lambda+\alpha_n^2\right)\,t}-\cos t+\left(\lambda+\alpha_n^2\right)\sin t}{\left(\lambda+\alpha_n^2\right)^2+1}.$$

It is clear from the Figs. 7 and 8 that, the flow has no steady state, since the pressure gradient function has an oscillating behavior.

Table 1. Numerical values of the flux versus time for some different values of κ/μ.

t/s	Flux			
	$\kappa/\mu = 0$	$\kappa/\mu = 1$	$\kappa/\mu = 4$	$\kappa/\mu = 6$
0.05	0.159 664	0.070 839	0.056 912	0.051 231
0.10	0.296 604	0.130 129	0.103 162	0.092 198
0.50	0.944 554	0.406 320	0.316 214	0.278 172
1.00	1.218 600	0.523 160	0.408 879	0.359 957
2.00	1.322 482	0.567 686	0.446 643	0.394 647
3.00	1.331 410	0.571 546	0.450 315	0.398 260
4.00	1.332 178	0.571 881	0.450 673	0.398 637
5.00	1.332 244	0.571 910	0.450 707	0.398 676

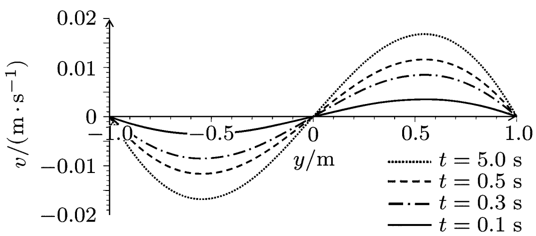

Fig. 3. Microrotation distribution for $\kappa/\mu = 2.0$.

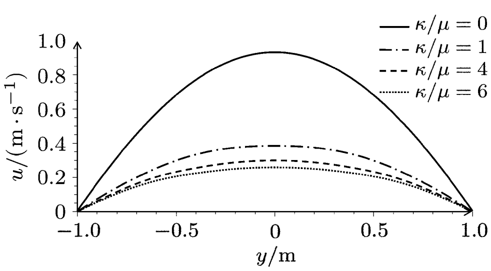

Fig. 4. Velocity distribution for t.

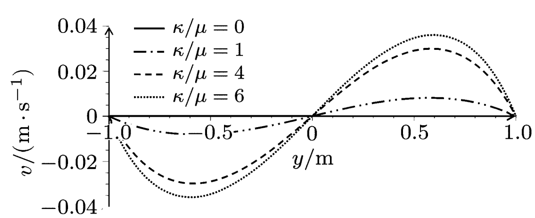

Fig. 5. Microrotation distribution for $t = 1$.

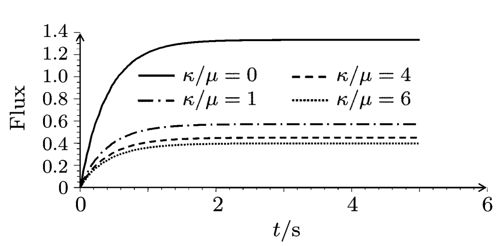

Fig. 6. Flux versus time.

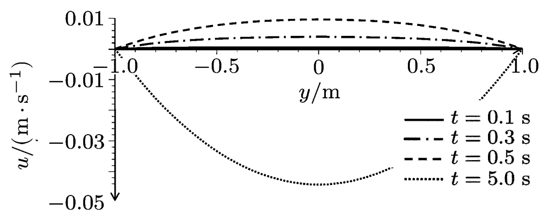

Fig. 7. Velocity distribution for $\kappa/\mu = 2.0$.

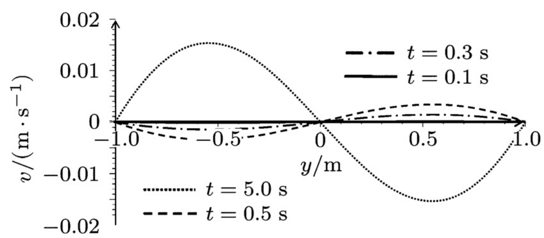

Fig. 8. Microrotation distribution for $\kappa/\mu = 2.0$.

Figures 9–11 show the distributions of the velocity, microrotation and total flux for different values of time with $\kappa/\mu = 2.0$ in the case of one plate with a free surface for a constant pressure gradient. The graphs show similar features as those in the previous discussions. A comparison between Tables 1 and Table 2 shows that the volume flux in the case of two plates is less than its corresponding value in the case of a free surface.

Exact solution has been found for the unsteady micropolar fluid flow between two parallel plates. The no slip and no spin boundary conditions are used at the surface of each plate. The flow is produced by the sudden application of an arbitrary time dependent pressure

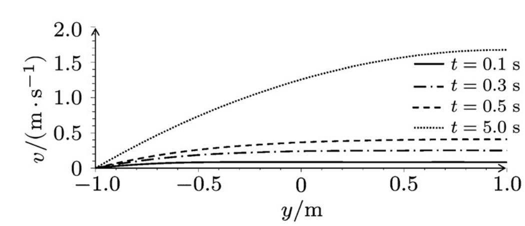

Fig. 9. Velocity distribution for $\kappa/\mu = 2.0$.

Table 2. Flux versus time for different values of micropolarity parameter.

t	Flux			
	$\kappa/\mu = 0$	$\kappa/\mu = 1$	$\kappa/\mu = 4$	$\kappa/\mu = 6$
0.05	0.183 179	0.129 586	0.109 237	0.076 248
0.10	0.352 423	0.249 421	0.206 882	0.141 597
0.50	1.468 079	1.025 408	0.733 723	0.457 302
1.00	2.496 625	1.733 285	1.052 245	0.612 780
2.00	3.802 661	2.636 826	1.271 260	0.693 923
3.00	4.507 317	3.132 156	1.321 389	0.706 339
4.00	4.887 578	3.404 500	1.333 141	0.708 327
5.00	5.092 782	3.554 334	1.335 927	0.708 654

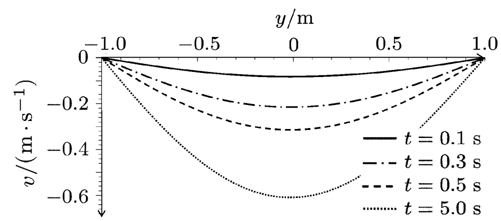

Fig. 10. Microrotation distribution for $\kappa/\mu = 2.0$.

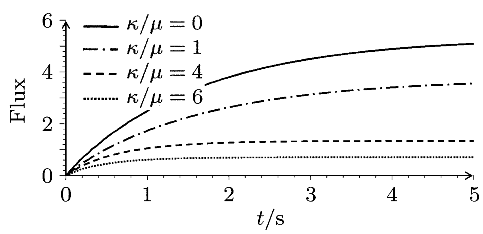

Fig. 11. Total flux against time t.

gradient function. The total flux, in general, is an increasing function of time and decreases as the micropolarity parameter increases. Exact solution is also found if the upper boundary of the flow is a free surface.

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