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ORIGINAL ARTICLE

Existence of Solution of Nonlinear Fuzzy Fredholm Integro-differential Equations

M. Mosleh · M. Otadi

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Abstract In this paper, we prove some results concerning the existence of solution of a class of nonlinear fuzzy Fredholm integro-differential equations. Also an iterative approach is proposed to obtain approximate solution of a class of nonlinear fuzzy Fredholm integro-differential equation of the second kind. A numerical example is presented to illustrate the proposed method.

Keywords Fuzzy numbers · Fuzzy integral · Nonlinear fuzzy integro differential equations · Numerical methods

1. Introduction

The topics of fuzzy integral equations (FIE) and fuzzy differential equations have been rapidly growing in recent years [1-13]. The fuzzy mapping function was introduced by Chang and Zadeh [14]. Later, Dubois and Prade [15] presented an elementary fuzzy calculus based on the extension principle. Also the concept of integration of fuzzy functions was first introduced by them. Then the fuzzy integration is discussed by Allahviranloo [16], Allahviranloo and Otadi [17, 18] and Mosleh and Otadi [19].

M. Mosleh (✉) · M. Otadi
Department of Mathematics, Firoozkooh Branch, Islamic Azad University, Firoozkooh, Iran
email: maryammosleh79@yahoo.com
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In the existence of the solution of fuzzy integral equations, the Ascoli's theorem or metric fixed point theorems are used. For the existence and uniqueness, the main tool is the Banach fixed point principle. Such discussions can be found in [20-24].

Babolian et al. and Abbasbandy et al. [25, 26] obtained a numerical solution of linear Fredholm fuzzy integral equations of the second kind. Then Otadi and Mosleh [27] considered fuzzy nonlinear integral equations of the second kind and obtained an approximate solution to the fuzzy nonlinear integral equations. In [23] the author considered nonlinear fuzzy Fredholm integral equations such as

$$F(s) = f(s) \oplus \int_a^b K(s,t,F(t))dt,$$

therefore, in this paper, we generalize the nonlinear fuzzy integral equations to the nonlinear fuzzy integro-differential equations

$$F'(s) = f(s) \oplus \int_a^b K(s,t,F(t))dt, \ F(a) = F_0.$$

In this paper, we present a simple numerical method to nonlinear fuzzy Fredholm integro-differential equations of the second kind.

2. Preliminaries

In this section, the basic notations used in fuzzy operations are introduced. We start by defining the fuzzy number.

Definition 2.1 *A fuzzy number is a function* $u : \mathbb{R} \longrightarrow I = [0,1]$ *having the properties* [28]*:*

(i) *u is normal, that is* $\exists x_0 \in \mathbb{R}$ *such that* $u(x_0) = 1$*;*

(ii) *u is a fuzzy convex set;*

(iii) *u is upper semicontinuous on* $\mathbb{R}$*;*

(iv) *The support* $\overline{\{x \in \mathbb{R} \mid u(x) > 0\}}$ *is a compact set.*

The set of all the fuzzy numbers is denoted by E. An alternative definition which yields the same E is given by Kaleva [29].

Definition 2.2 *A fuzzy number u is a pair* $(\underline{u}, \overline{u})$ *of functions* $\underline{u}(r)$ *and* $\overline{u}(r)$*,* $0 \le r \le 1$*, which satisfies the following requirements* [29]*:*

(i) $\underline{u}(r)$ *is a bounded monotonically non-decreasing, left continuous function on* $(0,1]$ *and right continuous at* 0*;*

(ii) $\overline{u}(r)$ *is a bounded monotonically non-increasing, left continuous function on* $(0,1]$ *and right continuous at* 0*;*

(iii) $\underline{u}(r) \le \overline{u}(r), 0 \le r \le 1.$

A crisp number r is simply represented by $\underline{u}(\alpha) = \overline{u}(\alpha) = r, 0 \le \alpha \le 1$. This fuzzy number space as shown in [30], can be embedded into the Banach space $B = \overline{C}[0,1] \times \overline{C}[0,1]$.

For arbitrary $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r))$ and $k \in \mathbb{R}$, we define addition and multiplication by k as

$$\underline{(u+v)}(r) = (\underline{u}(r) + \underline{v}(r)),$$

$$\overline{(u+v)}(r) = (\overline{u}(r) + \overline{v}(r)),$$

$$\underline{ku}(r) = k\underline{u}(r), \overline{ku}(r) = k\overline{u}(r) \text{ if } k \geq 0,$$

$$\underline{ku}(r) = k\overline{u}(r), \overline{ku}(r) = k\underline{u}(r) \text{ if } k < 0.$$

Definition 2.3 *For arbitrary fuzzy numbers u, v, we use the Hausdorff distance $D : E \times E \to \mathbb{R}_+ \cup \{0\}$, as in* [29]*:*

$$D(u, v) = \sup_{0 \leq r \leq 1} \max\{|\overline{u}(r) - \overline{v}(r)|, |\underline{u}(r) - \underline{v}(r)|\}.$$

We denote $\| \cdot \|_E = D(\cdot, \widetilde{0})$, where $\widetilde{0} \in E, \widetilde{0} = \chi_{\{0\}}$.

Theorem 2.1 [6]

(i) *The pair $(E, \oplus)$ is a commutative semigroup with $\widetilde{0} = \chi_{\{0\}}$ zero element;*

(ii) *For fuzzy numbers which are not crisp, there is no opposite element (that is, $(E, \oplus)$ cannot be a group);*

(iii) *For any $a, b \in \mathbb{R}$ with $a, b \geq 0$ or $a, b \leq 0$, and for any $u \in E$, we have $(a + b) \odot u = a \odot u \oplus b \odot u$;*

(iv) *For any $\lambda \in \mathbb{R}$ and $u, v \in E$, we have $\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot v$;*

(v) *For any $\lambda, \mu \in \mathbb{R}$ and $u \in E$, we have $\lambda \odot (\mu \odot u) = (\lambda.\mu) \odot u$;*

(vi) *The function $\| \cdot \|_E$: $E \to \mathbb{R}_+ \cup \{0\}$ has the usual properties of the norm, that is, $\| u \|_E = 0$ if and only if $u = \widetilde{0}$, $\| \lambda \odot u \|_E = |\lambda| \, \| u \|_E$ and $\| u \oplus v \|_E = \| u \|_E \oplus \| v \|_E$ for any $u, v \in E$;*

(vii) *$| \| u \|_E - \| v \|_E | \leq D(u, v)$ and $D(u, v) \leq \| u \|_E + \| v \|_E$ for any $u, v \in E$.*

Since $(E, \oplus)$ is not a group, but only a commutative monoid, the structure $(E, \oplus, \odot, \| \cdot \|_E)$ is not normed space. Some properties of the above distance are the following:

Theorem 2.2 [32]

(i) *(E, D) is complete metric space;*

(ii) *$D(u \oplus v, v \oplus w) = D(u, w)$ for all $u, v, w \in E$;*

(iii) *$D(k \odot u, k \odot v) = |k| D(u, v)$ for all $u, v \in E$ and $k \in \mathbb{R}$;*

(iv) *$D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e)$ for all $u, v, w, e \in E$.*

Definition 2.4 *Let $f : [a, b] \longrightarrow E^1$, for each partition $P = \{t_0, t_1, \cdots, t_n\}$ of $[a, b]$ and for arbitrary $\xi_i \in [t_{i-1}, t_i], 1 \leq i \leq n$ suppose* [15].

$$R_p = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}),$$

$$\int_a^b f(x)dx = \lim_{\Delta\to 0} R_p,$$

where

$$\Delta := \max\{|t_i - t_{i-1}|, i = 1, 2, \cdots, n\}$$

provided that this limit exists in the metric D.

If the fuzzy function $f(t)$ is continuous in the metric D, its definite integral exists [31] *and also,*

$$\underline{(\int_a^b f(t;r)dt)} = \int_a^b \underline{f}(t;r)dt,$$

$$\overline{(\int_a^b f(t;r)dt)} = \int_a^b \overline{f}(t;r)dt.$$

Lemma 2.1 [32] *If f and g are Henstock integrable functions and if the function given by $D(f(t), g(t))$ is Lebesgue integrable, then*

$$D((FH)\int_a^b f(t)dt, (FH)\int_a^b g(t)dt) \le (L)\int_a^b D(f(t), g(t))dt.$$

Definition 2.5 [34] *Let $f : [a, b] \to E$ be a bounded function. Then the function $\omega_{[a,b]}(f, \delta) : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+$*

$$\omega_{[a,b]}(f, \delta) = \sup\{D(f(x), f(y)); x, y \in [a, b], |x - y| \le \delta\}$$

is called the modulus of continuity of f on $[a, b]$.

Some properties of the modulus of continuity are presented below:

Theorem 2.3 [34] *The following properties holds:*

(i) $D(f(x), f(y)) \le \omega_{[a,b]}(f, |x - y|)$ *for any $x, y \in [a, b]$;*

(ii) $\omega_{[a,b]}(f, \delta)$ *is increasing function of δ;*

(iii) $\omega_{[a,b]}(f, 0) = 0$;

(iv) $\omega_{[a,b]}(f, \delta_1 + \delta_2) \le \omega_{[a,b]}(f, \delta_1) + \omega_{[a,b]}(f, \delta_2)$ *for any $\delta_1, \delta_2 \ge 0$;*

(v) $\omega_{[a,b]}(f, n\delta) \le n\omega_{[a,b]}(f, \delta)$ *for any $\delta \ge 0$ and $n \in \mathbb{N}$;*

(vi) $\omega_{[a,b]}(f, \lambda\delta) \le (\lambda + 1)\omega_{[a,b]}(f, \delta)$ *for any $\delta, \lambda \ge 0$;*

(vii) *If $[c, d] \subseteq [a, b]$, then $\omega_{[a,b]}(f, \delta) \le \omega_{[a,b]}(f, \delta)$.*

Definition 2.6 [35] *For $L \ge 0$, a function $f : [a, b] \to E$ is L-Lipschitz if*

$$D(f(x), f(y)) \le L|x - y|$$

for any $x, y \in [a, b]$.

Definition 2.7 [36] *Let* $f : [a, b] \to E$. *Fix* $s_0 \in [a, b]$. *We say X is differentiable at* s_0, *if there exists an element* $f'(s_0) \in E$ *such that, the H-differences* $f(s_0 + h) \ominus f(s_0)$, $f(s_0) \ominus f(s_0 - h)$ *exist and the limits (in the metric D) presents as follows:*

$$\lim_{h\to 0^+} \frac{f(s_0 + h) \ominus f(s_0)}{h} = \lim_{h\to 0^+} \frac{f(s_0) \ominus f(s_0 - h)}{h} = f'(s_0).$$

Theorem 2.4 [35] *Let* $f : [a, b] \to E$ *be a bounded and Henstock integrable function. Then for any partition* $a = s_0 < s_1 < \cdots < s_n = b$ *and* $\zeta_i \in [s_{i-1}, s_i]$, *we have*

$$D((FH)\int_a^b f(t)dt, \sum_{i=1}^{n}(s_i - s_{i-1}) \odot f(\zeta_i)) \leq \sum_{i=1}^{n}(s_i - s_{i-1})\omega_{[s_{i-1},s_i]}(f, s_i - s_{i-1}).$$

Particular election of the point ζ_i leads to the following result.

Here we present the quadrature rules obtained in [35], which contain as particular cases with the three point, middle point and trapezoidal rules.

Corollary 2.1 [35] *Let* $f : [a, b] \to E$ *be a bounded and Henstock integrable function. Then:*

(i) $D((FH)\int_a^b f(t)dt, (b - a) \odot f((a + b)/2)) \leq ((b - a)/2)\omega_{[a,b]}(f, (a - b)/2)$;

(ii) $D((FH)\int_a^b f(t)dt, (a - b)/2 \odot [f(a) \oplus f(b)]) \leq ((b - a)/2)\omega_{[a,b]}(f, (b - a)/2)$;

(iii) $D((FH)\int_a^b f(t)dt, (b-a)/6\odot[f(a)\oplus 4\odot f((a+b)/2)\oplus f(b)]) \leq 3(b-a)\omega_{[a,b]}(f, (b-a)/6)$.

3. Fuzzy Integro-differential Equations

We consider the nonlinear Fredholm integro-differential equations of the second kind

$$F'(s) = f(s) \oplus \int_a^b K(s, t, F(t))dt, \quad F(a) = F_0, \tag{1}$$

where

$$f : [a, b] \to E \text{ and } K : [a, b] \times [a, b] \times E \to E$$

are continuous. Moreover, K is uniformly continuous with respect to s.

Theorem 3.1 *Let* $f : [a, b] \to E$ *and* $K : [a, b]\times[a, b]\times E \to E$ *are continuous. Consider the nonlinear fuzzy Fredholm integro-differential* (1). *A mapping* $F : [a, b] \to E$ *is a solution to* (1) *if and only if F is continuous and satisfies the integral equation*

$$F(s) = F_0 \oplus \int_a^s f(z)dz \oplus \int_a^s \int_a^b K(z, t, F(t))dtdz, \ s \in [a, b]. \tag{2}$$

Proof Since f and K are continuous, by [29], it must be integrable. So, for

$$F'(s) = f(s) \oplus \int_a^b K(s,t,F(t))dt,\ s \in [a,b],$$

we have equivalently [37]

$$F(s) = F(a) \oplus \int_a^s f(z) \oplus \int_a^b K(z,t,F(t))dtdz;$$

equivalently [29]

$$F(s) = F(a) \oplus \int_a^s f(z)dz \oplus \int_a^s \int_a^b K(z,t,F(t))dtdz.$$

Since $F(a) = F_0$, we have

$$F(s) = F_0 \oplus \int_a^s f(z)dz \oplus \int_a^s \int_a^b K(z,t,F(t))dtdz.$$

Consider the space of functions

$$X = \{f : [a,b] \to E \mid f \text{ continuous}\}$$

with the metric $D^*(f,g) = \sup_{a \le s \le b} D(f(s),g(s))$. Recall the fact that (X,D^*) is complete metric space [29].

Define the operator A by

$$A(F)(s) = F_0 \oplus \int_a^s f(z)dz \oplus \int_a^s \int_a^b K(z,t,F(t))dtdz,\ s \in [a,b],\ \forall f \in X. \quad (3)$$

Theorem 3.2 *Suppose that the functions f and K are continuous. In addition, K is uniformly continuous with respect to s and there exist $L > 0, M_1 > 0, M_2 > 0$ such that*

$$\|K(z,t,u)\|_E \le M_2, \quad \forall z,t \in [a,b],\ \forall u \in E,$$

$$\|f(z)\|_E \le M_1, \quad \forall z \in [a,b]$$

and

$$D(k(z,t,u),k(z,t,v)) \le LD(u,v), \quad \forall z,t \in [a,b],\ \forall u,v \in E.$$

Moreover, for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $s_1, s_2 \in [a,b]$ and $|s_2 - s_1| \le \delta$, the following inequalities are satisfied:

$$D(\tilde{0}, \int_{s_1}^{s_2} f(z)dz) < \epsilon,$$

$$D(\tilde{0}, \int_{s_1}^{s_2} \int_a^b K(z,t,F(t))dtdz) < \epsilon,$$

$$L(b-a)(S-a) < 1,$$

then Eq. (1) has a unique solution F^* in X, which can be obtained through the method of successive approximations starting by any element of X. Moreover, in the approximation of solution by terms of sequence of successive approximations, $(F_m)_{m\in\mathbb{N}}, F_1(s) = F_0$,

$$F_{m+1}(s) = F_0 \oplus \int_a^s f(z)dz \oplus \int_a^s \int_a^b K(z,t,F_m(t))dtdz,\ \ s \in [a,b], m = 1,2,\cdots, \tag{4}$$

the prior error estimate is

$$D(F^*(s), F_{m+1}(s)) \leq \frac{[L(b-a)(S-a)]^m}{1-L(b-a)(S-a)}[(S-a)M_1 + (S-a)(b-a)M_2], \tag{5}$$

$$s \in [a,b],\ \ m = 1,2,\cdots.$$

Proof Firstly, we prove that $A(X) \subset X$. In this aim, we see that for all

$$\begin{aligned} D(A(F)(s_1), A(F)(s_2)) &= D(F_0 \oplus \int_a^{s_1} f(z)dz \oplus \int_a^{s_1}\int_a^b K(z,t,F(t))dtdz, \\ &\quad F_0 + \int_a^{s_2} f(z)dz \oplus \int_a^{s_2}\int_a^b K(z,t,F(t))dtdz) \\ &\leq D(\int_a^{s_1} f(z)dz, \int_a^{s_2} f(z)dz) + \\ &\quad D(\int_a^{s_1}\int_a^b K(z,t,F(t))dtdz, \int_a^{s_2}\int_a^b K(z,t,F(t))dtdz) \\ &\leq D(\int_a^{s_1} f(z)dz \oplus \tilde{0}, \int_a^{s_1} f(z)dz \oplus \int_{s_1}^{s_2} f(z)dz) + \\ &\quad D(\int_a^{s_1}\int_a^b K(z,t,F(t))dtdz \oplus \tilde{0}, \\ &\quad \int_a^{s_1}\int_a^b K(z,t,F(t))dtdz \oplus \int_{s_1}^{s_2}\int_a^b K(z,t,F(t))dtdz) \\ &= D(\tilde{0}, \int_{s_1}^{s_2} f(z)dz) + D(\tilde{0}, \int_{s_1}^{s_2}\int_a^b K(z,t,F(t))dtdz) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

So, $A(F)$ is uniformly continuous for any $F \in X$, and consequently continuous on $[a,b]$. Then, $A(X) \subset X$.

For $F, G \in X$ and $s \in [a, b]$ follows:

$$\begin{aligned}
D(A(F)(s), G(F)(s)) &= D(F_0 \oplus \int_a^s f(z)dz \oplus \int_a^s \int_a^b K(z,t,F(t))dtdz, \\
&\quad F_0 + \int_a^s f(z)dz \oplus \int_a^s \int_a^b K(z,t,G(t))dtdz) \\
&\leq D(\int_a^s \int_a^b K(z,t,F(t))dtdz, \int_a^s \int_a^b K(z,t,G(t))dtdz) \\
&\leq D(\int_a^s \int_a^b L.D(F(t),G(t))dtdz \\
&\leq LD^*(F,G)(b-a)(s-a) \\
&= LD^*(F,G)(b-a)\varphi(s).
\end{aligned}$$

Let $\varphi(S) = \sup_{s\in[a,b]} \{\varphi(s)\} = (S-a)$. Consequently,

$$D(A(F)(s), G(F)(s)) \leq LD^*(F,G)(b-a)\varphi(S), \ \forall F, G \in X. \tag{6}$$

Since, $L(b-a)\varphi(S) < 1$, the operator A is a contraction. Using the Banach's fixed point principle we infer that (1) has a unique solution F^* in X and the following inequality holds:

$$D(F^*(s), F_{m+1}(s)) \leq D^*(F^*, F_{m+1}) \leq \frac{[L(b-a)(S-a)]^m}{1-L(b-a)(S-a)} D^*(F_1, F_2), \tag{7}$$

$$m = 1, 2, \cdots .$$

On the other hand,

$$\begin{aligned}
D^*(F_1, F_2) &= \sup_{a\leq z\leq b} D(F_0 \oplus \tilde{0}, F_0 \oplus \int_a^s f(z)dz \oplus \int_a^s \int_a^b K(z,t,F_0(t))dtdz) \\
&\leq \sup_{a\leq z\leq b} [D(\tilde{0}, \int_a^s f(z)dz) + D(\tilde{0}, \int_a^s \int_a^b K(z,t,F_0(t))dtdz)] \\
&\leq \sup_{a\leq z\leq b} [\int_a^s \|f(z)\|_E dz) + (\int_a^s \int_a^b \|K(z,t,F_0(t))\|_E dtdz)] \\
&\leq M_1(S-a) + M_2(S-a)(b-a).
\end{aligned}$$

In this way, we obtain the inequality (5).

Theorem 3.2 states the existence and uniqueness of the solution to Eq. (1) and the sequence of successive approximations $(F_m)_{m\in\mathbb{N}}$, converges to this solution in (X, D^*). To approximate, this solution by terms of the sequence of successive approximations must compute the integral and differential.

4. The Numerical Approach

We replace the interval $[a, b]$ by a set of discrete equally spaced grid points

$$a = s_0 < s_1 < \cdots < s_{n-1} < s_n = b$$

at which the exact solution $F^*(s)$ is approximated by some $x(s)$. The exact and approximate solutions at $s_i,\ 0 \leq i \leq n$ are denoted by $F(s_i)$ and $x(s_i)$, respectively. The grid points at which the solution is calculated are

$$s_i = s_0 + ih,\ \ h = \frac{(b-a)}{n};\ \ 1 \leq i \leq n. \tag{8}$$

The first-order approximation of $F'(s)$ is given by [38]

$$F'(s) \approx \frac{F(s+h) \ominus F(s)}{h}. \tag{9}$$

By virtue of Eq. (9) and the quadrature formula we obtain

$$\begin{aligned} x_{m+1}(s_{i+1}) = x_{m+1}(s_i) \oplus h[f(s_i) \oplus \sum_{j=0}^{n-1} \frac{b-a}{2n} \\ [K(s_i, s_j, x_m(t_j)) \oplus K(s_i, s_{j+1}, x_m(t_{j+1}))]]; \end{aligned} \tag{10}$$

$$x_1(s_i) = x_{m+1}(s_0) = F_0;\ i = 0, 1, \cdots, n;\ m = 1, 2, \cdots. \tag{11}$$

By Theorem 5.2 in [29] we may replace (10) by the equivalent system

$$\begin{aligned} \underline{x}_{m+1}(s_{i+1}; r) = \underline{x}_{m+1}(s_i; r) + h[\underline{f}(s_i; r) + \sum_{j=0}^{n-1} \frac{b-a}{2n} \\ [\underline{K}(s_i, s_j, \underline{x}_m(t_j), \overline{x}_m(t_j)) + \underline{K}(s_i, s_{j+1}, \underline{x}_m(t_{j+1}), \overline{x}_m(t_{j+1}))]]; \end{aligned} \tag{12}$$

$$\begin{aligned} \overline{x}_{m+1}(s_{i+1}; r) = \overline{x}_{m+1}(s_i; r) + h[\overline{f}(s_i; r) + \sum_{j=0}^{n-1} \frac{b-a}{2n} \\ [\overline{K}(s_i, s_j, \underline{x}_m(t_j), \overline{x}_m(t_j)) + \overline{K}(s_i, s_{j+1}, \underline{x}_m(t_{j+1}), \overline{x}_m(t_{j+1}))]]; \\ \underline{x}_1(s_i; r) = \underline{x}_{m+1}(s_0; r) = \underline{F}_0; \overline{x}_1(s_i; r) = \overline{x}_{m+1}(s_0; r) = \overline{F}_0; \\ i = 0, 1, \cdots, n;\ m = 1, 2, \cdots. \end{aligned}$$

Let $\underline{K}(s, t, u, v)$ and $\overline{K}(s, t, u, v)$ be functions $\underline{K}$ and $\overline{K}$ of (12) where u and v are constants and $u \leq v$. In other words, $\underline{K}(s, t, u, v)$ and $\overline{K}(s, t, u, v)$ are obtained by substituting $x = (u, v)$ in (12). The domain where $\underline{K}$ and $\overline{K}$ are defined

$$B = \{(s, t, u, v) \mid a \leq s, t \leq b, -\infty < v < +\infty, -\infty < u \leq v\}.$$

Theorem 4.1 *Let $\underline{K}(s, t, u, v)$ and $\overline{K}(s, t, u, v)$ belong to $C^1(B)$. Let the partial derivatives of $\underline{K}, \overline{K}$ be bounded over B also*

$$D(F_m(s_p), x_m(s_p)) = \max_{0 \leq i \leq n}\{D_m(F_m(s_i), x_m(s_i))\},$$

$$D(F_{m+1}(s_{k+1}), x_{m+1}(s_{k+1})) = \max_{0 \leq i \leq n}\{D_{m+1}(F_{m+1}(s_i), x_{m+1}(s_i))\}.$$

Then, for arbitrary fixed $r : 0 \leq r \leq 1$,

$$\lim_{h\to 0} D(F_{m+1}(t_k), x_{m+1}(t_k)) = 0.$$

Proof Let

$$\begin{aligned}\underline{F}_{m+1}(s_{k+1}) = \underline{F}_{m+1}(s_k) + h[\underline{f}(s_i;r) + \sum_{j=0}^{n-1}\frac{b-a}{2n}[\underline{K}(s_i,s_j,\underline{F}_m(t_j),\overline{F}_m(t_j))+\\ \underline{K}(s_i,s_{j+1},\underline{F}_m(t_{j+1}),\overline{F}_m(t_{j+1}))]] + O(h^2),\\ \overline{F}_{m+1}(s_{k+1}) = \overline{F}_{m+1}(s_k) + h[\overline{f}(s_i;r) + \sum_{j=0}^{n-1}\frac{b-a}{2n}[\overline{K}(s_i,s_j,\underline{F}_m(t_j),\overline{F}_m(t_j))+\\ \overline{K}(s_i,s_{j+1},\underline{F}_m(t_{j+1}),\overline{F}_m(t_{j+1}))]] + O(h^2),\end{aligned}$$

and

$$\begin{aligned}\underline{x}_{m+1}(s_{k+1}) = \underline{x}_{m+1}(s_k) + h[\underline{f}(s_i;r) + \sum_{j=0}^{n-1}\frac{b-a}{2n}[\underline{K}(s_i,s_j,\underline{x}_m(t_j),\overline{x}_m(t_j))+\\ \underline{K}(s_i,s_{j+1},\underline{x}_m(t_{j+1}),\overline{x}_m(t_{j+1}))]],\\ \overline{x}_{m+1}(s_{k+1}) = \overline{x}_{m+1}(s_k) + h[\overline{f}(s_i) + \sum_{j=0}^{n-1}\frac{b-a}{2n}[\overline{K}(s_i,s_j,\underline{x}_m(t_j),\overline{x}_m(t_j))+\\ \overline{K}(s_i,s_{j+1},\underline{x}_m(t_{j+1}),\overline{x}_m(t_{j+1}))]].\end{aligned}$$

Consequently,

$$\begin{aligned}\underline{F}_{m+1}(s_{k+1}) - \underline{x}_{m+1}(s_{k+1}) = \underline{F}_{m+1}(s_k) - \underline{x}_{m+1}(s_k) + h[\sum_{j=0}^{n-1}\frac{b-a}{2n}\\ [\underline{K}(s_i,s_j,\underline{F}_m(t_j),\overline{F}_m(t_j)) - \underline{K}(s_i,s_j,\underline{x}_m(t_j),\overline{x}_m(t_j))\\ +\underline{K}(s_i,s_{j+1},\underline{F}_m(t_{j+1}),\overline{F}_m(t_{j+1})) - \underline{K}(s_i,s_{j+1},\\ \underline{x}_m(t_{j+1}),\overline{x}_m(t_{j+1}))]] + O(h^2),\\ \overline{F}_{m+1}(s_{k+1}) - \overline{x}_{m+1}(s_{k+1}) = \overline{F}_{m+1}(s_k) - \overline{x}_{m+1}(s_k) + h[\sum_{j=0}^{n-1}\frac{b-a}{2n}\\ [\overline{K}(s_i,s_j,\underline{F}_m(t_j),\overline{F}_m(t_j)) - \overline{K}(s_i,s_j,\underline{x}_m(t_j),\overline{x}_m(t_j))\\ +\overline{K}(s_i,s_{j+1},\underline{F}_m(t_{j+1}),\overline{F}_m(t_{j+1})) - \overline{K}(s_i,s_{j+1},\underline{x}_m(t_{j+1}),\\ \overline{x}_m(t_{j+1}))]] + O(h^2).\end{aligned}$$

Denote $W_{m+1}(s_{k+1}) = \underline{F}_{m+1}(s_{k+1}) - \underline{x}_{m+1}(s_{k+1})$, $V_{m+1}(s_{k+1}) = \overline{F}_{m+1}(s_{k+1}) - \overline{x}_{m+1}(s_{k+1})$. Then

$$\begin{aligned}|W_{m+1}(s_{k+1})| \le |W_{m+1}(s_k)| + h[\sum_{j=0}^{n-1}\frac{b-a}{2n}[2L\max\{|W_m(t_j)|,|V_m(t_j)|\}\\ +2L\max\{|W_m(t_{j+1})|,|V_m(t_{j+1})|\}] + O(h^2),\\ |V_{m+1}(s_{k+1})| \le |V_{m+1}(s_k)| + h[\sum_{j=0}^{n-1}\frac{b-a}{2n}[2L\max\{|W_m(t_j)|,|V_m(t_j)|\}\\ +2L\max\{|W_m(t_{j+1})|,|V_m(t_{j+1})|\}] + O(h^2),\end{aligned}$$

where $L > 0$ is a bound for the partial derivatives of $\underline{K}, \overline{K}$. Thus, we have

$$|W_{m+1}(s_{k+1})| \le |W_{m+1}(s_k)| + 2nLh\frac{b-a}{n}D(F_m(t_p), x_m(t_p) + O(h^2),$$

$$|V_{m+1}(s_{k+1})| \le |V_{m+1}(s_k)| + 2nLh\frac{b-a}{n}D(F_m(t_p), x_m(t_p) + O(h^2).$$

Since $W_{m+1}(t_0) = V_{m+1}(t_0) = 0$, we obtain

$$|W_{m+1}(s_{k+1})| \leq 2(k+1)nLh\frac{b-a}{n}D(F_m(t_p), x_m(t_p) + O(h^2),$$

$$|V_{m+1}(s_{k+1})| \leq 2(k+1)nLh\frac{b-a}{n}D(F_m(t_p), x_m(t_p) + O(h^2)$$

and if $h \to 0$, we get $D(F_{m+1}(t_k), x_{m+1}(t_k)) \to 0$.

5. Numerical Example

Consider the following nonlinear fuzzy Fredholm integro-differential equation

$$F'(s) = (r - \frac{s^2r^2}{40}, 2 - r - \frac{s^2(2-r)^2}{40}) \oplus \int_0^1 (\frac{s^2t}{10}) \odot F^2(t)dt,$$

$$F(0) = 0; \quad 0 \leq r \leq 1.$$

The exact solution in this case is given by

$$F(s) = (r, 2-r)s.$$

By using of (10) and the quadrature formula we obtain

$$x_{m+1}(s_{i+1}) = x_{m+1}(s_i) \oplus h[(r - \frac{s_i^2r^2}{40}, 2 - r - \frac{s_i^2(2-r)^2}{40}) \oplus \sum_{j=0}^{29} \frac{1}{60}[(\frac{s_i^2s_j}{10} \odot x_m^2(s_j)) \oplus (\frac{s_i^2s_{j+1}}{10} \odot x_m^2(s_{j+1}))]],$$

also by using of (8) and (11) we have

$$x_1(s_i) = x_{m+1}(s_0) = 0; s_i = ih, i = 0, 1, \cdots, 30; \; m = 1, 2, \cdots, 10,$$

where

$$x_m^2(s_j) = (\min\{\underline{x_m}^2(s_j; r), \overline{x_m}^2(s_j; r), \underline{x_m}(s_j; r)\overline{x_m}(s_j; r)\},$$
$$\max\{\underline{x_m}^2(s_j; r), \overline{x_m}^2(s_j; r), \underline{x_m}(s_j; r)\overline{x_m}(s_j; r)\}).$$

Comparison between the exact solution and the approximate solution of nonlinear fuzzy Fredholm integro-differential equation in this example given by the numerical approach in Section 4 are drawn in Fig. 1 and Fig. 2.

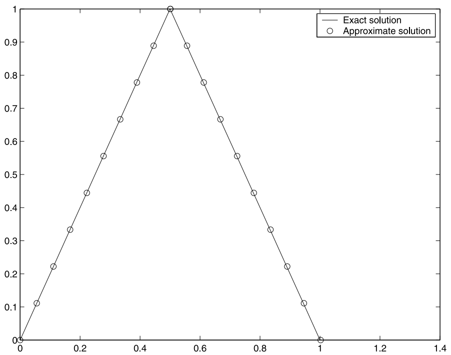

Fig. 1 Compares the exact solution and obtained solution at $s = 0.5$

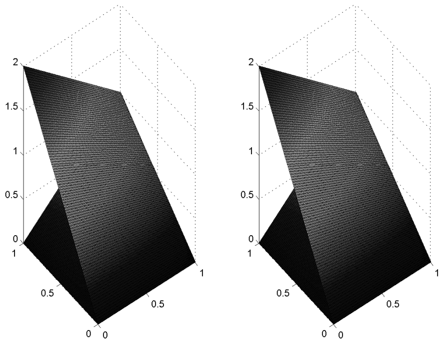

Fig. 2 Comparison between the exact solution and approximate solution: left: The exact solution and right: The approximate solution

6. Conclusion

This paper aims at proposing a numerical method to nonlinear fuzzy Fredholm integro-differential equations. In this paper, the standard Newton-Cotes method is designed for approximating integral. Also we can execute this method in a computer simply.

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