Available online at www.sciencedirect.com
ScienceDirect
Discrete Applied Mathematics 155 (2007) 2394 – 2400
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

Note

On the 2-rainbow domination in graphs

Boštjan Brešar[a,1], Tadeja Kraner Šumenjak[b]

[a] *FEECS, University of Maribor, Smetanova 17, 2000 Maribor, Slovenia*
[b] *FA, University of Maribor, Vrbanska 30, 2000 Maribor, Slovenia*

Received 6 February 2007; received in revised form 20 June 2007; accepted 13 July 2007
Available online 4 September 2007

Abstract

The concept of 2-rainbow domination of a graph G coincides with the ordinary domination of the prism $G\Box K_2$. In this paper, we show that the problem of deciding if a graph has a 2-rainbow dominating function of a given weight is NP-complete even when restricted to bipartite graphs or chordal graphs. Exact values of 2-rainbow domination numbers of several classes of graphs are found, and it is shown that for the generalized Petersen graphs $GP(n,k)$ this number is between $\lceil 4n/5 \rceil$ and n with both bounds being sharp.

MSC: 05C85; 05C69

Keywords: Complexity; NP-completeness; Domination; Cartesian product; Generalized Petersen graph

1. Introduction

Domination and its variations in graphs have been extensively studied, cf. [5,6]. For a graph $G=(V(G),E(G))$, a set S is a *dominating set* if every vertex in $V(G)\backslash S$ is adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. Domination presents a model for situations in which vertices from S guard neighboring vertices that are not in S. A generalization was proposed in [2] where different types of guards are used, and vertices not in S must have all types of guards in their neighborhoods.

Let G be a graph and $v \in V(G)$. The *open neighborhood of* v is the set $N(v)=\{u \in V(G) \mid uv \in E(G)\}$, and its *closed neighborhood* is the set $N[v]=N(v)\cup\{v\}$. Let f be a function that assigns to each vertex a set of colors chosen from the set $\{1,\ldots,k\}$; that is, $f\colon V(G)\to \mathscr{P}(\{1,\ldots,k\})$. If for each vertex $v \in V(G)$ such that $f(v)=\emptyset$ we have

$$\bigcup_{u\in N(v)} f(u)=\{1,\ldots,k\},$$

then f is called a *k-rainbow dominating function* (kRDF) of G. The *weight*, $w(f)$, of a function f is defined as $w(f)=\sum_{v\in V(G)}|f(v)|$. Given a graph G, the minimum weight of a kRDF is called the *k-rainbow domination number of* G,

E-mail addresses: bostjan.bresar@uni-mb.si (B. Brešar), tadeja.kraner@uni-mb.si (T. Kraner Šumenjak).
[1] Supported by the Ministry of Education, Science and Sport of Slovenia under the Grant P1-0297.

0166-218X/$ - see front matter
doi:10.1016/j.dam.2007.07.018

which we denote by $\gamma_{rk}(G)$. For a set X of vertices in G we denote by $f(X)=\bigcup_{x\in X} f(x)$. Clearly when $k=1$ this concept coincides with the ordinary domination.

Rainbow domination of a graph G coincides with the ordinary domination of the Cartesian product of G with the complete graph, in particular $\gamma_{rk}(G)=\gamma(G\Box K_k)$ for any graph G [2]. In the language of domination of Cartesian products, Hartnell and Rall obtained several observations about rainbow domination, for instance:

$$\min\{|G|, \gamma(G)+k-2\}\leqslant\gamma_{rk}(G)\leqslant k\gamma(G)$$

for any $k\geqslant 2$ and any graph G [4]. The attempt in [4] to characterize graphs with $\gamma(G)=\gamma_{r2}(G)$ was inspired by the following famous open problem [7].

Vizing's Conjecture. *For any graphs G and H*, $\gamma(G\Box H)\geqslant\gamma(G)\gamma(H)$.

One of the related problems posed in [3] is to find classes of graphs that achieve the equality. There it was shown that $\gamma(G\Box H)=\gamma(G)\gamma(H)$, if G is any graph with $\gamma(G)=\gamma_{r2}(G)$ and H is a so-called generalized comb.

In [2] rainbow domination was introduced and studied in relation with paired-domination of Cartesian products of graphs. In addition, a linear-time algorithm for determining a minimum weight 2-rainbow dominating function of an arbitrary tree was presented. In this paper we concentrate on the case $k=2$, that is, the 2-rainbow domination of graphs, and show that the decision version of this problem is NP-complete. In addition, some bounds and exact results for several standard classes of graphs are proven.

In the next section, we prove that the problem of determining whether a graph has a 2-rainbow dominating function of a given weight is NP-complete even when restricted to chordal graphs (or bipartite graphs). Then in Section 3 we study the 2-rainbow domination parameter for some classes of graphs. We present exact results for paths, cycles and suns, and upper and lower bounds for the generalized Petersen graphs. In the last section we add some more observations and pose a few open problems, concentrating mostly on algorithmic issues.

2. Complexity of 2-rainbow domination problem

It is well-known that the domination problem is NP-complete when restricted to chordal graphs (resp. bipartite graphs) [6]. We use these two results in showing that the same holds for the 2-rainbow domination problem. First, we pose it as a decision problem.

2-RAINBOW DOMINATING FUNCTION

INSTANCE: A graph G and a positive integer k.

QUESTION: Does G have a 2-rainbow dominating function of weight k?

Theorem 2.1. 2-*RAINBOW DOMINATING FUNCTION is NP-complete*.

Proof. First note that 2-RAINBOW DOMINATING FUNCTION is in NP. Indeed, given a function $f: V(G)\rightarrow\mathscr{P}(\{1,2\})$ of weight k one can clearly check in linear time whether it is a 2-rainbow dominating function (notably, for each vertex u with $f(u)=\emptyset$ one has to check whether in the neighborhood of u both colors appear).

Let G be an arbitrary graph, an instance of the problem DOMINATING SET. We shall construct a graph G' from it such that for any positive integer k: G' has a 2-rainbow dominating function of weight $k+|V(G)|$ if and only if G has a dominating set of size k. Namely, let G' be obtained from G by adding a leaf to each vertex of G. That is, for $V(G)=\{v_1,\ldots,v_n\}$ we have $V(G')=V(G)\cup\{u_1,\ldots,u_n\}$, and $E(G')=E(G)\cup\{v_iu_i|i=1,\ldots,n\}$.

Let D be a dominating set of G of size k. Then by setting $f(v_i)=\{1\}$ precisely when $v_i\in D$, and $f(u_i)=\{2\}$ for all i we get a function of weight $n+k$ which is clearly a 2-rainbow dominating function of G'. Indeed, every vertex v_j from $V(G)\backslash D$ has a neighbor u_j with $f(u_j)=\{2\}$, and a neighbor $v_i\in D$ with $f(v_i)=\{1\}$.

Let f be a 2-rainbow dominating function of G'. Since every leaf must be dominated, we easily derive that $w(f)>n$. Then $w(f)=n+k$ for some positive integer k. Let f' be a function obtained from f by setting $f'(u_i)=\{2\}$ and $f'(v_i)=f(v_i)\cup\{1\}$ whenever $f(u_i)=\{1,2\}$. It is obvious that f' is also a 2-rainbow dominating function of G' and $w(f')\leqslant w(f)$ (the weight of f' is smaller when $\{1\}\subseteq f(v_i)$ and $f(u_i)=\{1,2\}$ for some i). Let D be the set of vertices from $V(G)$ such that $f'(v_i)\neq\emptyset$. We claim that D is a dominating set of G. Suppose D is not a dominating set of G. Then there is a vertex $v_j\in V(G)$ such that $f'(v_j)=\emptyset$ and for every vertex $v_i\in V(G)$ adjacent to v_j we have $f'(v_i)=\emptyset$. Since

f' is a 2-rainbow dominating function of G' we infer $f'(u_j) = \{1, 2\}$ which is in a contradiction with the construction of f'. We derive D is a dominating set of G. Let D' be the set of vertices v_i from D such that $f(v_i) = \{1, 2\}$. It is easy to see that for every j such that $v_j \notin D'$ we have $f'(u_j) \neq \emptyset$. Thus $w(f') \geqslant 2|D'| + (|D| - |D'|) + (n - |D'|) = n + |D|$. Hence the size of D is at most k.

Hence $\gamma(G) \leqslant k$ if and only if $\gamma_{r2}(G') \leqslant n + k$ for any positive integer k, in particular $\gamma_{r2}(G') = \gamma(G) + n$. One can construct G' from G in linear time, and given a 2-rainbow dominating function of G' one can construct a dominating set of G in linear (that is polynomial) time. This can be done by first constructing f' from f, and then selecting all vertices from $V(G)$ with nonempty weight which yields a dominating set. This implies that 2-RAINBOW DOMINATING FUNCTION is NP-complete. □

Since the graph G' from the above proof is chordal (resp. bipartite) if G is chordal (resp. bipartite) we infer two corollaries by using the fact that domination problem is NP-complete when restricted to chordal graphs (resp. bipartite graphs).

Corollary 2.2. *2-RAINBOW DOMINATING FUNCTION is NP-complete even when restricted to chordal graphs.*

Corollary 2.3. *2-RAINBOW DOMINATING FUNCTION is NP-complete even when restricted to bipartite graphs.*

3. Classes of graphs

3.1. Paths and cycles

The following observation is an easy exercise and is left to the reader.

Observation 3.1. $\gamma_{r2}(P_n) = \lfloor n/2 \rfloor + 1$.

In [2, Observation 2.6] a 2RDF of C_n was found showing $\gamma_{r2}(C_n) \leqslant \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$. We prove that this inequality is in fact equality for all cycles.

Proposition 3.2. *For* $n \geqslant 3$, $\gamma_{r2}(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

Proof. Clearly $\gamma_{r2}(C_n) \geqslant \gamma_{r2}(P_{n-1}) = \lfloor (n-1)/2 \rfloor + 1$, since for every optimal 2RDF of C_n there is a vertex x with $f(x) = \emptyset$, and f restricted to $C_n - x$ is obviously a 2RDF of $C_n - x$.

Note that $\lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor = \lfloor (n-1)/2 \rfloor + 1$ except in case $n \equiv 2 \pmod 4$. Hence the proof is complete except in this case.

Let $n \equiv 2 \pmod 4$, and let f be a 2-rainbow dominating function of C_n of minimum weight. Suppose there is a vertex $x \in C_n$ with $f(x) = \{1, 2\}$. Then we get $w(f) \geqslant 2 + \gamma_{r2}(P_{n-3}) = 2 + \lfloor (n-3)/2 \rfloor + 1 = n/2 + 1$. Finally suppose that $|f(x)| \leqslant 1$ for all $x \in C_n$. Then we derive that for any pair of adjacent vertices x and y to at least one of them f assigns a nonempty value. Hence $w(f) \geqslant n/2$, and note that $w(f) = n/2$ implies that there must be two vertices x and y with a common neighbor in C_n such that $f(x) = f(y)$. This is a contradiction with f being a 2RDF. We derive that also in this case $w(f) \geqslant n/2 + 1$. Since $n/2 + 1 = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$ we infer that also in this case $\gamma_{r2}(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$. □

Interestingly, $\gamma_{r2}(C_n) = \gamma_t(C_n)$ for $n \geqslant 3$ where γ_t denotes the total domination number.

3.2. Suns

Recall that a graph is *chordal* if it contains no induced cycles of length at least 4. A *sun* S_n is a chordal graph on $2n$ vertices ($n \geqslant 3$) whose vertex set can be partitioned into sets $W = \{w_1, w_2, \ldots, w_n\}$ and $U = \{u_1, u_2, \ldots, u_n\}$ such that W is independent and u_i is adjacent to w_j if and only if $i = j$ or $i \equiv j + 1 \pmod n$, cf. [1]. If U induces a complete graph then the graph is called the *complete sun*.

Proposition 3.3. *For* $n \geqslant 3$, $\gamma_{r2}(S_n) = n$.

Proof. Let $f : V(S_n) \to \mathscr{P}(\{1, 2\})$ be defined as follows: For even n let $f(w_i)=\{1\}$ if $i \equiv 1 \pmod 2$ and $f(w_i)=\{2\}$ if $i \equiv 0 \pmod 2$. For odd n let $f(u_n) = \{1, 2\}$, $f(w_i) = \{2\}$ if $i \equiv 0 \pmod 2$ for $2 \leqslant i \leqslant n-1$ and $f(w_i) = \{1\}$ if $i \equiv 1 \pmod 2$ for $3 \leqslant i \leqslant n-2$. Then, f is a 2RDF of S_n. This implies $\gamma_{r2}(S_n) \leqslant w(f) = n$.

It remains to show that $\gamma_{r2}(S_n) \geqslant n$. Let f be a 2-rainbow dominating function of S_n of minimum weight. Let W' be the subset of W, of vertices w_i such that $f(w_i) \neq \emptyset$ and $W'' = W \backslash W'$. If $W' = W$, then $\gamma_{r2}(S_n) \geqslant n$. Otherwise, for every vertex $w \in W''$ we have $f(N(w)) = \{1, 2\}$. From the definition of suns we get $|N(w_i) \cap N(w_j)| \leqslant 1$, for every distinct i and j. It is then easy to derive that $\sum_{u\in U} |f(u)| \geqslant |W''|$. Hence $\gamma_{r2}(S_n) \geqslant |W'| + \sum_{u\in U} |f(u)| \geqslant |W'| + |W''| = |W| = n$. $\square$

3.3. Generalized Petersen graphs

The domination invariants of generalized Petersen graphs were studied in [8]. Let us recall what a generalized Petersen graph is, cf. also [8].

Let $n \geqslant 3$ and k be relatively prime natural numbers and $k < n$. The *generalized Petersen graph* $GP(n, k)$ is defined as follows. Let C_n, C'_n be two disjoint cycles of length n. Let the vertices of C_n be $u_1, \ldots, u_n$ and edges $u_i u_{i+1}$ for $i = 1, \ldots, n-1$ and $u_n u_1$. Let the vertices of C'_n be $v_1, \ldots, v_n$ and edges $v_i v_{i+k}$ for $i = 1, \ldots, n$, the sum $i + k$ being taken modulo n (throughout this section). The graph $GP(n, k)$ is obtained from the union of C_n and C'_n by adding the edges $u_i v_i$ for $i = 1, \ldots, n$. The graph $GP(5, 2)$ is the well-known Petersen graph.

Proposition 3.4. *For a generalized Petersen graph $GP(n, k)$ we have $\gamma_{r2}(GP(n, k) \leqslant n$.*

Proof. Clearly for the proof it suffices to find a 2RDF of $GP(n, k)$ of weight n. We distinguish two cases with respect to the parity of k.

Case 1: k odd. We define $f : V(GP(n, k)) \to \mathscr{P}(\{1, 2\})$ as follows. Let $f(v_i) = \{2\}$ and $f(u_i) = \emptyset$ if i is odd, and $f(u_i) = \{1\}$ and $f(v_i) = \emptyset$ if i is even. For $u_j \in C_n$ with $f(u_j) = \emptyset$ (implying j is odd) we have $f(v_j) = \{2\}$ and $f(u_{j+1}) = \{1\}$, hence $\cup_{x\in N[u_j]} f(x) = \{1, 2\}$. For $v_j \in C'_n$ with $f(v_j) = \emptyset$ (implying j is even), we have $f(u_j) = \{1\}$. Since k is odd, $j + k$ is odd, and for the vertex v_{j+k} that is adjacent to v_j we have $f(v_{j+k}) = \{2\}$. We infer f is a 2RDF of $GP(n, k)$ with $w(f) = n$.

Case 2: k even. Since n and k are relatively prime, n is odd. Also in this case we define f in such a way that for all i, $f(v_i) = \emptyset$ if and only if $f(u_i) \neq \emptyset$. In addition, let $f(v_i) \neq \emptyset$ imply $f(v_i) = \{2\}$, and let $f(u_i) \neq \emptyset$ imply $f(u_i) = \{1\}$. Clearly this yields $w(f) = n$, and to define f it is enough to specify $f(v_i)$ for all i.

Let i be a natural number between 1 and n. Let $d = \lfloor n/k \rfloor$. For i with $i/k \leqslant d$ we define f as follows. If $\lfloor (i-1)/k \rfloor$ is even we set $f(v_i) = \{2\}$ if and only if i is odd (for instance $f(v_1) = \{2\}$ and $f(v_k) = \emptyset$). If $\lfloor (i-1)/k \rfloor$ is odd we set $f(v_i) = \{2\}$ if and only if i is even (for instance $f(v_{k+1}) = \emptyset$ and $f(v_{2k}) = \{2\}$). For definition of f-values on vertices with largest indices we distinguish three cases.

Case 2.1: $n \equiv 3 \pmod 4$. Then $n = dk + 3$ and we set $f(v_{dk+1}) = f(v_{dk+3}) = \emptyset$, and $f(v_{dk+2}) = \{2\}$. It is easy to check that f is a 2RDF.

Case 2.2: $n \equiv 1 \pmod 4$ and d is odd. Then $n = dk + 1$ and we set $f(v_n) = \emptyset$. Again it is easy to check that f is a 2RDF.

Case 2.3: $n \equiv 1 \pmod 4$ and d is even, then $n = dk + 1$. In this case by setting $f(v_n) = \emptyset$, v_n would not have a neighbor with $f(x) = \{2\}$. On the other hand by setting $f(v_n) = \{2\}$ we have three consecutive v's, that is v_{dk}, v_{dk+1}, v_1 to which f assigns $\{2\}$. By definition of f, for all their neighbors in C_n we have $f(u_i) = \emptyset$. But then u_n does not have neighbor with $f(x) = \{1\}$. The solution is that we reassign $f(v_{dk}) = \emptyset$ and let $f(v_n) = \{2\}$. Note that v_{dk} is adjacent to v_{k-1} and $f(v_{k-1}) = \{2\}$ hence v_{dk} is 2-rainbow dominated. For all other vertices one checks similarly as earlier that f is 2RDF. $\square$

We believe that this bound is close to the exact result for many classes of generalized Petersen graphs. One argument for this is demonstrated by the following lower bound for domination number of an arbitrary graph G:

$$\gamma(G) \geqslant \left\lceil \frac{|V(G)|}{1 + \Delta(G)} \right\rceil,$$

which implies that for the generalized Petersen graph $GP(n,k)$

$$\gamma(GP(n,k)) \geqslant \left\lceil \frac{n}{2} \right\rceil .$$

Hence for any 2RDF f with $|f(x)| \in \{0,2\}$ for all $x \in V(GP(n,k))$, we have $w(f) \geqslant n$.

In addition, let us present a general lower bound for $\gamma_{r2}(GP(n,k))$.

Proposition 3.5. *For any relatively prime numbers n and k, with $k<n$, we have*

$$\gamma_{r2}(GP(n,k)) \geqslant \lceil \tfrac{4}{5}n \rceil .$$

Proof. Let n and k be relatively prime numbers, and denote $H = GP(n,k)$. Let f be a 2RDF of H of minimum weight. Let $S = \{x \in V(H) : f(x) \neq \emptyset\}$. Then for every $u \in V(H) \backslash S$ we have $|f(N(u))| \geqslant 2$. By summing this up for all vertices of $V(H) \backslash S$ we get

$$\sum_{u \in V(H)\backslash S} |f(N(u))| \geqslant 2(|V(H)| - |S|) \geqslant 2(|V(H)| - \gamma_{r2}(H)).$$

Since every vertex from S is adjacent to at most three vertices (from $V(H)\backslash S$) we find that on the left-hand side of the above inequality each weight is counted at most 3 times. Thus

$$3\gamma_{r2}(H) \geqslant 2(|V(H)| - \gamma_{r2}(H)),$$

which readily implies

$$\gamma_{r2}(H) \geqslant \tfrac{2}{5}|V(H)| = \tfrac{4}{5}n.$$

Since $\gamma_{r2}(H)$ is an integer we derive $\gamma_{r2}(H) \geqslant \lceil \frac{4}{5}n \rceil$. □

Now we show that there are several classes of generalized Petersen graphs that achieve the lower bound from Proposition 3.5. Below we present 2RDFs of $GP(n,2)$ for different odd n. We use two lines where in the first line there are values of vertices of $C_n = \{u_1, \ldots, u_n\}$, and in the second line of the vertices of $C_n' = \{v_1, \ldots, v_n\}$, such that u_i lies above v_i for all i. Note that the only 2RDFs values used are {1}, {2} and ∅ (which we denote by 1, 2, and 0, respectively). Since k is even, n is odd, and we distinguish the following five cases:

$n \equiv 1 \pmod{10}$:
10010 20020 . . . 10010 20022 1
02200 01100 . . . 02200 01100 0
$n \equiv 3 \pmod{10}$:
10010 20020 . . . 10010 20020 100
02200 01100 . . . 02200 01100 022
$n \equiv 5 \pmod{10}$:
10010 20020 . . . 10010 20020 10011
02200 01100 . . . 02200 01100 02200
$n \equiv 7 \pmod{10}$:
10010 20020 . . . 10010 20020 10010 20
02200 01100 . . . 02200 01100 02200 11
$n \equiv 9 \pmod{10}$:
10010 20020 . . . 10010 20020 10010 2002
02200 01100 . . . 02200 01100 02200 0110.

Note that for $n \equiv 3 \pmod{10}$ and $n \equiv 9 \pmod{10}$ we have $\gamma_{r2}(GP(n,2)) = \lceil \frac{4}{5}n \rceil$, and for other odd n we have $\gamma_{r2}(GP(n,2)) = \lceil \frac{4}{5}n \rceil + 1$.

On the other hand, there are generalized Petersen graphs that achieve the upper bound from Proposition 3.4. In particular this holds for the Petersen graph $GP(5,2)$.

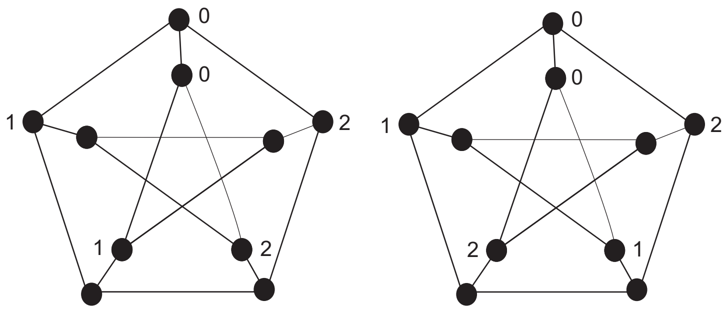

Fig. 1. The first case from the proof of Proposition 3.6.

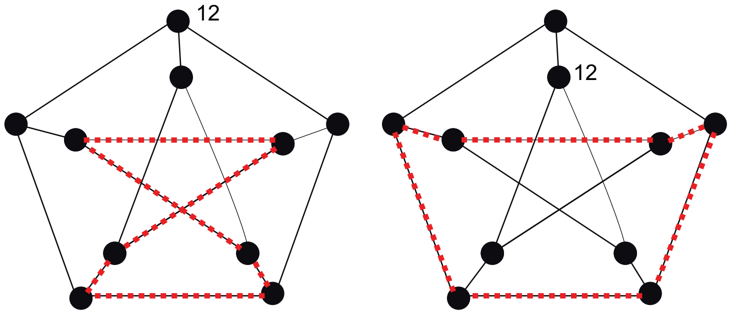

Fig. 2. The second case from the proof of Proposition 3.6.

Proposition 3.6. $\gamma_{r2}(GP(5,2)) = 5$.

Proof. For the proof it suffices to show that one cannot construct a 2RDF function f of $GP(5,2)$ with $w(f) = 4$. Suppose that there exists such a function f. First let $|f(v)| \leqslant 1$ for every vertex $v \in V(GP(5,2))$. Then clearly there exist adjacent vertices $u \in V(C_5)$ and $v \in V(C_5')$ such that $f(u) = \emptyset$ and $f(v) = \emptyset$. Because of symmetry there are essentially two possible ways how to dominate u and v (see Fig. 1, where values on u and v are marked by 0), but in both cases f is not a 2RDF.

Now let $f(v) = \{1, 2\}$ for some vertex v of $GP(5,2)$. Then there exists an induced cycle C_6 which is not dominated by v (see Fig. 2). It is easy to see that we cannot dominate this cycle to obtain a 2RDF f of weight at most 4. □

4. Concluding remarks

1. We suspect there are infinite families of graphs that achieve the bound n from Proposition 3.4. Our candidate families arise from the Petersen graph as the first graph in the sequence. We pose two questions:

Question 1. *Is* $\gamma_{r2}(GP(2k+1,k)) = 2k+1$ *for all* $k \geqslant 2$?

Question 2. *Is* $\gamma_{r2}(GP(n,3)) = n$ *for all* $n \geqslant 7$ *where* n *is not divisible by* 3?

2. A linear algorithm for determining a 2RDF of minimum weight of an arbitrary tree was presented in [2]. The algorithm was based on the related concept of so-called *weak* 2*-domination*. Intuitively, we could call it a monochromatic version of 2-rainbow domination.

   Let $G = (V, E)$ be a graph and let f be a function that assigns to each vertex a number chosen from $\{0, 1, 2\}$ called its weight; that is, $f: V \to \{0, 1, 2\}$. For $v \in V$, we define

$$f[v] = \sum_{u \in N[v]} f(u)$$

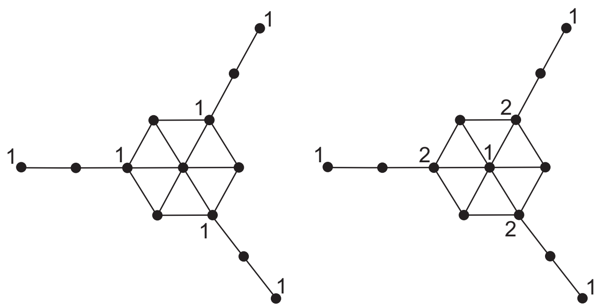

Fig. 3. Dually chordal graph.

for notational convenience. We call a vertex $v \in V$ a *bad vertex* with respect to f if $f(v)=0$ and $f[v] \leqslant 1$; otherwise, we say that v is a *good vertex* with respect to f. Note that if v is a good vertex with respect to f and $f(v)=0$, then $f[v] \geqslant 2$. If every vertex of T is a good vertex with respect to f, then f is called a *weak* $\{2\}$*-dominating function* (W2DF) of G. The *weight* $w(f)$ of f is defined as $w(f)=\sum_{v \in V} f(v)$. The minimum weight of a W2DF in G is called the *weak* $\{2\}$*-domination number* of G, which we denote by $\gamma_{\mathrm{w2}}(G)$.

Using analogous arguments as in Theorem 2.1 one can prove

Corollary 4.1. *The decision version of the weak* $\{2\}$*-dominating function* ($W2DF$) *is NP-complete* (*even when restricted to chordal graphs*, *resp. bipartite graphs*).

The main reason for introducing this concept was the following relation.

Observation 4.2 (*Brešar et al.* [2]). *For every tree* T, $\gamma_{\mathrm{r2}}(T)=\gamma_{\mathrm{w2}}(T)$.

The following question is thus relevant.

Question 3. *For which classes of graphs is* $\gamma_{\mathrm{r2}}(G)=\gamma_{\mathrm{w2}}(G)$ *for every graph G of a class*?

If for a class of graphs the Question 3 is negative, this seems to reduce the chances of an efficient algorithm for the 2-rainbow domination of that class.

Beside trees, one can easily see that interval graphs also have the property $\gamma_{\mathrm{r2}}(G)=\gamma_{\mathrm{w2}}(G)$ for any graph G. However, we were not able to design a desired algorithm even for the subclass of proper interval graphs.

Question 4. *Is there a polynomial algorithm to find an optimum* 2*RDF* (*or W2DF*) *of an arbitrary* (*proper*) *interval graph?*

Question 3 has a negative answer in the class of dually chordal graphs. On the left-hand side of Fig. 3 there is a dually chordal graph with a minimum W2DF, while on the right-hand side a minimum 2RDF of the same graph is shown.

It is also clear that cycles and circular arc graphs have graphs G with $\gamma_{\mathrm{w2}}(G)<\gamma_{\mathrm{r2}}(G)$ while this is open for instance for strongly chordal and doubly chordal graphs.

References

[1] A. Brandstaet, V.B. Le, J.P. Spinrad, Graphs Classes: A Survey, SIAM, Philadelphia, 1999.
[2] B. Brešar, M.A. Henning, D.F. Rall, Rainbow domination in graphs, Taiwanese J. Math., to appear (March, 2008).
[3] B. Hartnell, D.F. Rall, Domination in Cartesian products: Vizing's Conjecture, in: T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1998, pp. 163–189.
[4] B. Hartnell, D.F. Rall, On dominating the Cartesian product of a graph and K_2, Discuss. Math. Graph Theory 24 (2004) 389–402.
[5] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.
[6] T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1998.
[7] V.G. Vizing, Some unsolved problems in graph theory, Uspekhi Mat. Nauk 23 (1968) 117–134.
[8] B. Zelinka, Domination in generalized Petersen graphs, Czechoslovak Math. J. 52 (2002) 11–16.