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Degree-constrained spanners for multidimensional grids

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Abstract

A spanning subgraph $S = (V, E')$ of a connected simple graph $G = (V, E)$ is a $f(x)$ *-spanner* if for any pair of nodes u and v, $d_S(u, v) \leqslant f(d_G(u, v))$ where d_G and d_S are the usual distance functions in graphs G and S, respectively. The *delay* of the $f(x)$ -spanner is $f(x) - x$. We construct four spanners with maximum degree 4 for infinite d-dimensional grids with delays $2d - 4$, $2\lceil d/2 \rceil + 2\lceil (d-2)/4 \rceil + 2$, $2\lceil (d-6)/8 \rceil + 4\lceil (d+1)/4 \rceil + 6$, and $\lceil (\lceil d/2 \rceil + l)/(l+1) \rceil + 2\lceil d/2 \rceil + 2l + 2$. All of these constructions can be modified to produce spanners of finite d-dimensional grids with essentially the same delay. We also construct a $(5d + 4 + x)$ -spanner with maximum degree 3 for infinite d-dimensional grids. This construction can be used to produce spanners of finite d-dimensional grids where all dimensions are even with the same delay. We prove an $\Omega(d)$ lower bound for the delay of maximum degree 3 or 4 spanners of finite or infinite d-dimensional grids. For the particular cases of infinite 3- and 4-dimensional grids, we construct $(6 + x)$ -spanners and $(14 + x)$ -spanners, respectively. The former can be modified to construct a $(6 + x)$ -spanner of a finite 3-dimensional grid where all dimensions are even or where all dimensions are odd and a $(8 + x)$ -spanner of a finite 3-dimensional grid otherwise. The latter yields $(14 + x)$ -spanners of finite 4-dimensional grids where all dimensions are even.

1. Introduction and definitions

There are several popular topologies used for constructing parallel computers. Our goal is to determine substructures of such topologies with smaller maximum degree. We require also that these substructures, called *spanners*, have the property that the distance between two vertices in the substructure is not significantly larger than the corresponding distance in the original structure.

A network is represented by a connected simple graph $G = (V, E)$. We use the notation $d_G(u, v)$ to denote the distance between vertices u and v in G. The maximum

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degree of G is denoted Δ_G and the average degree is denoted δ_G. The subscript G may be omitted if it is clear from context. Liestman and Shermer introduced a general definition of graph spanner in [2]: A spanning subgraph S of a connected simple graph G is an $f(x)$ *-spanner* if for any pair of nodes u and v, $d_S(u, v) \leqslant f(d_G(u, v))$. We call $d_S(u, v) - d_G(u, v)$ the *delay between vertices u and v* in S, denoted $d'_S(u, v)$. For an $f(x)$ -spanner S, we let $f'(x) = f(x) - x$ and refer to $f'(x)$ as the *delay* of the spanner. Note that $f'(x)$ is an upper bound (but not necessarily a tight bound) on the maximum delay in S between any pair of vertices at distance x in G.

It may be possible to express the delay $f'(x)$ in several ways. Although any spanner S of a finite graph G is an $(x + c)$ -spanner where c is the maximum delay between any pair of vertices in S, a more careful analysis of S may reveal a closer relationship between the distance in G and the delay in S. For example, the $(x + c)$ -spanner mentioned above may also be determined to be a $2x$-spanner. In general, we prefer to express $f(x)$ in a manner that bounds the delay as clearly as possible.

Spanners were introduced by Peleg and Ullman [8], who used these structures in order to simulate efficiently synchronous networks on asynchronous networks. Various aspects of spanners have been investigated in recent papers [1–3, 7, 9]. Our particular interest here is in spanners of multidimensional grids.

Liestman and Shermer [4] described how to construct low average degree $(t + x)$ -spanners for 2-dimensional grids, X-trees, and pyramids. In particular, for 2-dimensional grids, they constructed spanners with maximum degree 4 and low average degree. They introduced the concept of *highways* and constructed spanners of infinite grids which could be modified to produce spanners of finite grids. By including all of the edges in every lth row and every kth column, *highways* are created which allow long distances to be covered with no delay. By specifying the detailed connections of other vertices to the highways, they obtained spanners for both infinite and finite grids with delays $2l - 4 + 2\lfloor (k-1)/2 \rfloor$ and $2k + 2\lceil l/2 \rceil - 2 + x$ and average degree $\approx 2 + 2/((l-1)(k-1))$. In addition, they showed that, for any integer $t \geqslant 1$, to determine whether a given graph G and integer m, whether G has a $(t + x)$ -spanner with m or fewer edges is NP-complete.

In [5], Liestman and Shermer studied spanners of 2-dimensional grids, X-trees, and pyramids with smaller maximum degree. In the particular case of 2-dimensional grids, they showed how to construct two different types of $(1.25\sqrt{3x+6} + 6 + x)$-spanners with maximum degree 3 (and different average degrees) for two-dimensional grids. Further, they established a lower bound that shows that the delay of these spanners is within a constant factor of optimal.

In this paper, we show how to construct spanners with maximum degrees 3 and 4 of both infinite and finite d-dimensional grids for $d > 2$. We begin with a simple lower bound on the delay of such spanners in Section 2. In Section 3, we show how to construct four different spanners of the infinite d-dimensional grid ($d > 2$) with maximum degree 4. These can all be modified to yield spanners for finite grids with the same (or approximately the same) delay. In each case, the delay is a function of d but not of x. In Section 4, we describe a general method to construct grid spanners with maximum degree 3 and we give a specific construction based on this general method. This construction can be modified to yield spanners for finite grids where all of the dimensions are even with the same delay. Again, this delay is a function of d but not of x. In Section 5, we describe two specific constructions which give spanners with maximum degree 3 for 3- and 4-dimensional infinite grids. These constructions yield lower delay than the construction of Section 4. The construction for 3-dimensional grids can be modified to produce spanners for finite grids with approximately the same delay. The construction for 4 dimensional grids can be modified to yield spanners for finite grids where all of the dimensions are even with the same delay. The delay of each of these spanners is within a constant factor of the lower bound.

An infinite d-dimensional grid, denoted $G_{(d)}$, has the vertex set $V = \boldsymbol{Z}^d$. Its edges are between pairs of vertices whose labels differ by 1 in exactly one position, that is, vertex $(u_1, u_2, \dots, u_k, \dots, u_d)$ is connected to exactly those vertices $(u_1, u_2, \dots, u_k + 1, \dots, u_d)$ and $(u_1, u_2, \dots, u_k - 1, \dots, u_d)$ where $1 \leqslant k \leqslant d$. A finite d-dimensional grid, denoted $G_{n_1, n_2, \dots, n_d}$, is the induced subgraph of $G_{(d)}$ on vertices $(u_1, u_2, \dots, u_d)$ with $0 \leqslant u_k \leqslant n_k - 1$ for $1 \leqslant k \leqslant d$. To avoid confusion, we will insist that $n_k \geqslant 2$ for each k.

2. A simple lower bound

Theorem 2.1. *The delay of any maximum degree 3 or 4 spanner of $G_{(d)}$ or any $G_{n_1, n_2, \dots, n_d}$ is $\Omega(d)$.*

Proof. Consider a vertex v in such a grid G. If G is a hypercube, at least half of the vertices of G are within distance $d/2$ of v; that is, at least 2^{d-1} vertices are within distance $d/2$ of v. If G is any other d-dimensional grid, the number of vertices within distance $d/2$ of v is at least as large.

Let S be any spanner of G with maximum degree Δ and delay b. Let T_v be the depth $d/2 + b$ breadth-first search tree in S rooted at v. This tree must contain all vertices of G within distance $d/2$ of v in G. The tree T_v, being of maximum degree Δ and depth $d/2 + b$, contains at most $\Delta((\Delta - 1)^{d/2+b} - 1)/(\Delta - 2) + 1$ vertices.

Thus,

$$\Delta\left(\frac{(\Delta - 1)^{d/2+b} - 1}{\Delta - 2}\right) + 1 \geqslant 2^{d-1},$$

$$\Delta\left(\frac{(\Delta - 1)^{d/2+b} - 1}{\Delta - 2}\right) \geqslant 2^{d-2},$$

$$(\Delta - 1)^{d/2+b} - 1 \geqslant \frac{\Delta - 2}{\Delta} 2^{d-2},$$

$$(\Delta - 1)^{d/2+b} \geqslant \frac{\Delta - 2}{\Delta} 2^{d-2},$$

$$\left(\frac{d}{2}+b\right)\log(\Delta-1) \geqslant \log\left(\frac{\Delta-2}{\Delta}\right)+(d-2),$$

$$\frac{d}{2}+b \geqslant \frac{\log\left(\frac{\Delta-2}{\Delta}\right)-2}{\log(\Delta-1)}+\frac{d}{\log(\Delta-1)},$$

$$b \geqslant \frac{\log\left(\frac{\Delta-2}{\Delta}\right)-2}{\log(\Delta-1)}+\frac{2d}{2\log(\Delta-1)}-\frac{d\log(\Delta-1)}{2\log(\Delta-1)},$$

$$b \geqslant \frac{\log\left(\frac{\Delta-2}{\Delta}\right)-2}{\log(\Delta-1)}+\frac{(2-\log(\Delta-1))d}{2\log(\Delta-1)}.$$

When $\log(\Delta-1) < 2$ (or, equivalently, $\Delta < 5$), the delay b is in $\Omega(d)$. □

3. Construction of grid spanners with maximum degree 4

In this section, we describe four different maximum degree 4 spanners of $G_{(d)}$ and modifications which yield maximum degree 4 spanners of finite 4-dimensional grids. The first three constructions yield different delays and are each best for some particular range of d. The fourth construction has a lower delay than the others for most values of d, but requires more complicated routing.

All four of the spanners constructed here are based on the idea of highways and interchanges as introduced in [4]. The idea is that there are highways (uninterrupted long paths) in each dimension which are connected by interchanges. To route between a pair of vertices, it is necessary to travel along the highways of the dimensions in which the addresses differ. Switching between highways is done by traveling on interchanges.

To construct a maximum degree 4 *path interchange* spanner S of an infinite d-dimensional grid, $G_{(d)}$, we connect every vertex $(u_1, u_2, \ldots, u_d)$ to $(u_1+1, u_2, \ldots, u_d)$ and also to $(u_1, u_2, \ldots, u_{k-1}, u_k+1, u_{k+1}, \ldots, u_d)$ where $(k-2) \equiv u_1 \bmod (d-1)$. For any integer j, the jth layer denotes the collection of vertices of the grid having first coordinate equal to j. In the path interchange spanner, each layer contains all of its dimension k edges for some $2 \leqslant k \leqslant d$. Fig. 1 shows a 2-dimensional slice (dimensions 1 and 2) of a path interchange spanner.

Theorem 3.1. *The path interchange spanner S is a $(2d-4+x)$-spanner of $G_{(d)}$ with $\Delta_S = 4$ and $\delta_S = 4$.*

Proof. Given two vertices $u = (u_1, u_2, \ldots, u_d)$ and $v = (v_1, v_2, \ldots, v_d)$, $d_G(u, v) = \sum_{i=1}^{d} |u_i - v_i|$. We wish to bound the delay between u and v in S. Since S contains all

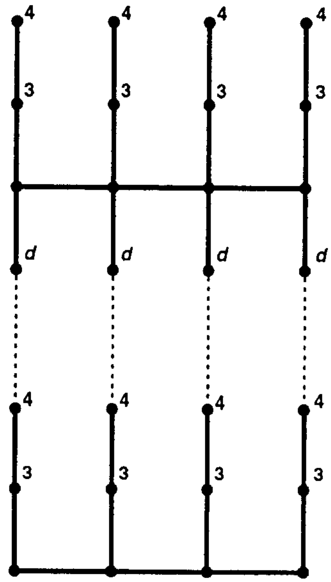


Fig. 1. A slice of a path interchange spanner showing dimension 1 vertically and dimension 2 horizontally. Label i on a vertex indicates that the vertex is connected to its neighbors in dimension i.

edges of dimension 1, there is a path from u to v consisting of a path in dimension 1 from u to $u' = (v_1, u_2, u_3, \ldots, u_d)$, followed by a path from u' to v. Thus, $d_S(u, v) \leqslant d_S(u', v) + |u_1 - v_1|$. Since $d_G(u, v) = d_G(u', v) + |u_1 - v_1|$, $d'_S(u, v) \leqslant d'_S(u', v)$.

From vertex $u' = (v_1, u_2, u_3, \ldots, u_d)$, we can move in dimension $i = v_1 \bmod (d-1) + 2$ without delay. In particular, we can move from $u' = (v_1, u_2, \ldots, u_i, \ldots, u_d)$ to $(v_1, u_2, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_d)$ using exactly $|u_i - v_i|$ edges. From this vertex, we move to $(v_1 + 1, u_2, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_d)$ using 1 edge. From this vertex, we can move in dimension $j = (v_1 + 1) \bmod (d-1) + 2$ without delay. Note that j will be either $i + 1$ or 2. (The latter occurs when $i = d$.)

As before, we move until the coordinate in dimension j is v_j. By performing $(d-1)$ iterations of this process of stepping positively one edge in dimension 1 and then $|u_\alpha - v_\alpha|$ edges in dimension α, we eventually arrive at $v' = (v_1 + (d-2), v_2, \ldots, v_d)$. From v', we follow $(d-2)$ edges in dimension 1 to arrive at v.

The length of this path from u' to v is $\sum_{k=2}^{d} |u_k - v_k| + 2(d-2) = 2(d-2) + d_G(u', v)$. Thus, $d'_S(u', v) \leqslant 2d - 4$, $d'_S(u, v) \leqslant 2d - 4$, and thus, for any x, $f'(x) \leqslant 2d - 4$. □

We can easily construct a finite version of this spanner for any finite d-dimensional grid.

Corollary 3.1a *For any finite d-dimensional grid* $G' = G_{n_1, n_2, \ldots, n_d}$ *with* $n_i \geqslant d - 1$ *for some* $1 \leqslant i \leqslant d$, *there exists a* $(2d - 4 + x)$ *-spanner of* G' *with* $\Delta = 4$ *and* $\delta \leqslant 4$.

Proof. Let $G' = G_{n_1,n_2,\dots,n_d}$ be a finite d-dimensional grid and S' be the subspanner of the d-dimensional path interchange spanner S induced by the vertices of G'. If $n_1 \geqslant 2d-3$, then the analysis in the above proof holds for S', except when $v_1 > (n_1-1)-(d-1)$. In this case, one cannot step positively $(d-2)$ times in dimension 1 as required by the description of the path. However, by substituting negative steps in dimension 1 for positive ones (and vice versa), we stay within the grid and the same analysis holds.

We can view the constructed path as a cycle of length $2d-2$ in dimension 1 interspersed with one single-dimensional path P_k in each other dimension k. It is not necessary that all of the P_k's are grouped at the beginning of the cycle. In fact, any cycle in dimension 1 that encounters $(d-1)$ distinct dimension 1 coordinates can be used in the construction, inserting the P_k appropriately. In particular, we can use any cycle that consists of l dimension 1 edges in the positive direction followed by $(d-1)$ dimension 1 edges in the negative direction followed by $(d-1-l)$ dimension 1 edges in the positive direction. Thus, we can relax our requirement of $n_1 \geqslant 2d-3$ in our analysis of the delay of S' to $n_1 \geqslant d-1$. In fact, as long as $n_i \geqslant d-1$ for some $1 \leqslant i \leqslant d$, we can construct a path interchange spanner with the same delay bound by exchanging dimensions i and 1. □

To construct a maximum degree 4 *cycle interchange* spanner S of the infinite d-dimensional grid, $G_{(d)}$, we begin by connecting every vertex $(u_1, u_2, \dots, u_d)$ to $(u_1+1, u_2, \dots, u_d)$. When u_2 is even, we connect $(u_1, u_2, \dots, u_d)$ to $(u_1, u_2, \dots, u_{k-1}, u_k+1, u_{k+1}, \dots, u_d)$ where $k = u_1 (\mathrm{mod} \lceil d/2 \rceil) + 2$. When u_2 is odd, we consider three cases: If $u_1 \equiv 0 (\mathrm{mod} \lceil d/2 \rceil)$, then we connect $(u_1, u_2, u_3, \dots, u_d)$ to $(u_1, u_2+1, u_3, \dots, u_d)$. If $u_1 \equiv \lceil d/2 \rceil - 1 (\mathrm{mod} \lceil d/2 \rceil)$ and d is odd, we add no further connections to $(u_1, u_2, \dots, u_d)$. Otherwise, we connect $(u_1, u_2, \dots, u_d)$ to $(u_1, u_2, \dots, u_{k-1}, u_k+1, u_{k+1}, \dots, u_d)$ where $k = u_1 (\mathrm{mod} \lceil d/2 \rceil) + \lceil d/2 \rceil + 1$. Fig. 2 shows a 2-dimensional slice (dimensions 1 and 2) of a cycle interchange spanner.

Theorem 3.2. *The cycle interchange spanner S is a $(2\lceil d/2 \rceil + 2\lceil (d-2)/4 \rceil + 2 + x)$-spanner of $G_{(d)}$ with $\Delta_S = 4$ and $\delta_S = 4$.*

Proof. Given two vertices $u = (u_1, u_2, \dots, u_d)$ and $v = (v_1, v_2, \dots, v_d)$, $d_G(u,v) = \sum_{i=1}^{d} |u_i - v_i|$. We wish to bound the delay between u and v in S. Since S contains all edges of dimension 1, there is a path from u to v consisting of a path in dimension 1 from u to $u' = (v_1, u_2, u_3, \dots, u_d)$, followed by a path from u' to v. Thus, $d_S(u,v) \leqslant d_S(u',v) + |u_1 - v_1|$. Since $d_G(u,v) = d_G(u',v) + |u_1 - v_1|$, $d'_S(u,v) \leqslant d'_S(u',v)$.

From vertex u', we move to $u'' = (v_1, v_2, u_3, \dots, u_d)$ by taking $b \leqslant \lceil d/2 \rceil / 2 = \lceil (d-2)/4 \rceil$ dimension 1 edges to the nearest vertex with neighbors in dimension 2, followed by a path of $|u_2 - v_2|$ edges in dimension 2 and b more edges to u''.

We will construct a path from u'' to v by interspersing single-dimensional paths in dimensions 3 through d into a short cycle. The cycle consists of 2 paths of $\lceil d/2 \rceil$ edges

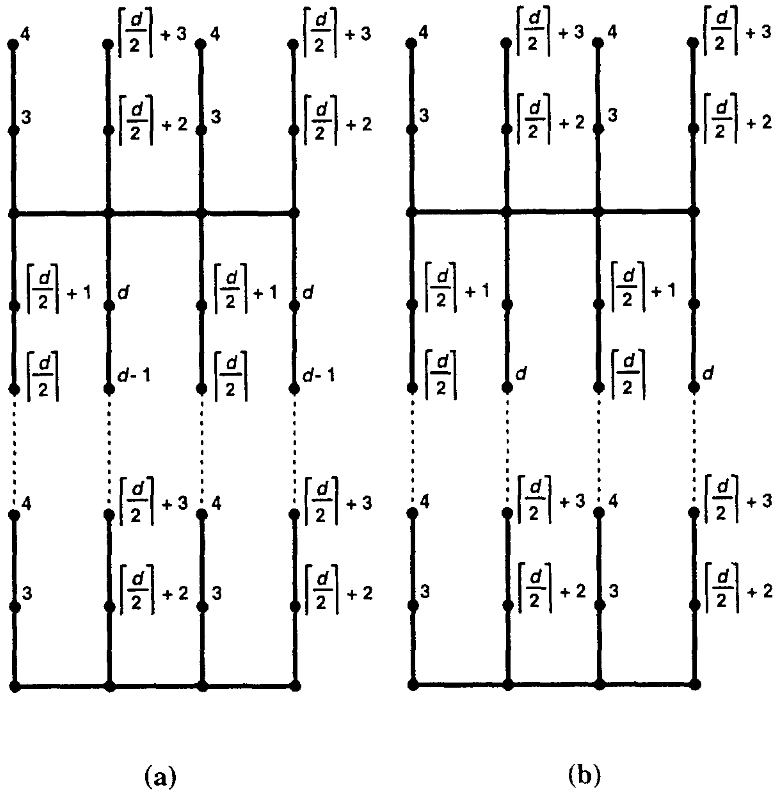


Fig. 2. A slice of a cycle interchange spanner showing dimension 1 vertically and dimension 2 horizontally. Label i on a vertex indicates that the vertex is connected to its neighbors in dimension i. In (a) d is even and in (b) d is odd.

in dimension 1 joined on both ends by a single dimension 2 edge. Note that such a cycle will encounter a vertex with edges in dimension k for all $3 \leqslant k \leqslant d$. At each such vertex, we include the appropriate path of length $|u_k - v_k|$ in dimension k.

The total path length from u to v is $|u_1 - v_1| + (2b + |u_2 - v_2|) + 2\lceil d/2 \rceil + 2 + \sum_{k=3}^{d} |u_k - v_k| = d_G(u, v) + 2b + 2\lceil d/2 \rceil + 2 \leqslant d_G(u, v) + 2\lceil (d-2)/4 \rceil + 2\lceil d/2 \rceil + 2$. $\square$

Corollary 3.2a. *For any finite d-dimensional grid $G' = G_{n_1, n_2, \ldots, n_d}$ with $n_i \geqslant \lceil d/2 \rceil + 1$ for some $1 \leqslant i \leqslant d$, there exists a $(2\lceil d/2 \rceil + 2\lceil (d-2)/4 \rceil + 2 + x)$-spanner of G' with $\Delta = 4$ and $\delta \leqslant 4$.*

Proof. We assume that the dimensions have been permuted so that $n_1 \geqslant \lceil d/2 \rceil + 1$. We could construct a finite cycle interchange spanner by simply taking an induced subspanner of the infinite cycle interchange spanner. However, if $n_1 (\text{mod} \lceil d/2 \rceil)$ is large, then vertices with the largest coordinate in dimension 1 would incur a large delay because they are far from all of the short cycles used in the paths from u'' to v in the proof of Theorem 3.2. Consequently, to construct a cycle interchange spanner for $G_{n_1,n_2,\ldots,n_d}$, we first shift the infinite cycle interchange spanner $\lfloor n_1(\bmod\lceil d/2\rceil)/2\rfloor$ units positively in dimension 1, and then take the induced subspanner. The analysis is then the same as in Theorem 3.2 except that u'' may not be on a short cycle. If this is the case, then the path described from u' to u'' (even if $u = u''$) contains a vertex with edges in both dimensions 1 and 2. This vertex is on a short cycle, and thus the short cycle (interspersed with single dimensional paths) can be spliced into the path at this point (rather than occuring after it). Thus, we arrive at the same delay bound of $2\lceil d/2\rceil + 2\lceil(d-2)/4\rceil + 2$. □

There are several possible generalizations of the path interchange and cycle interchange spanners.

Note that paths in the path interchange spanner were described in terms of a cycle of dimension 1 edges interspersed with single dimensional paths in higher dimensions. Paths in the cycle interchange spanner were described in terms of a cycle of two paths of roughly $d/2$ dimension 1 edges joined by two dimension 2 edges. An obvious generalization is to construct a cycle of four paths of roughly $d/4$ dimension 1 edges joined by two dimension 2 edges and two dimension 3 edges. This generalization leads to a spanner that we call a folded cycle interchange spanner.

To construct a maximum degree 4 *folded cycle interchange* spanner S of an infinite d-dimensional grid, $G_{(d)}$, we begin by connecting every vertex $(u_1, u_2, \ldots, u_d)$ to $(u_1 + 1, u_2, \ldots, u_d)$. When $u_1 \equiv 0(\bmod\lceil(d+5)/4\rceil)$, we connect $(u_1, u_2, \ldots, u_d)$ to $(u_1, u_2 + 1, u_3, \ldots, u_d)$. When $u_1 \equiv (\lceil(d+5)/4\rceil - 1)$ $(\bmod\lceil(d+5)/4\rceil)$ we connect $(u_1, u_2, \ldots, u_d)$ to $(u_1, u_2, u_3 + 1, u_4, \ldots, u_d)$. Otherwise. we let $k = (2(u_3 \bmod 2) + (u_2 \bmod 2))$ $\lceil(d-3)/4\rceil + u_1 \bmod\lceil(d+5)/4\rceil + 2$, and connect each vertex $(u_1, u_2, \ldots, u_k, \ldots, u_d)$ to $(u_1, u_2, \ldots, u_k + 1, \ldots, u_d)$, if $k < d$. A crucial point in the analysis of this generalized spanner is to note that vertices with edges in dimension 2 on one short cycle are adjacent by dimension 1 edges to vertices with edges in dimension 3 on another short cycle. This helps us to obtain a short path from $u = (u_1, u_2, u_3, \ldots, u_d)$ to $u'' = (v_1, v_2, v_3, u_4, \ldots, u_d)$.

Theorem 3.3. *The folded cycle interchange spanner S is a $(2\lceil(d-6)/8\rceil + 4\lceil(d+1)/4\rceil + 6 + x)$-spanner of $G_{(d)}$ with $\Delta_S = 4$ and $\delta_S \leqslant 4$.*

Proof. Given two vertices $u = (u_1, u_2, \ldots, u_d)$ and $v = (v_1, v_2, \ldots, v_d)$, $d_G(u, v) = \sum_{i=1}^{d} |u_i - v_i|$. We wish to bound the delay between u and v in S. Since S contains all edges of dimension 1, there is a path from u to v consisting of a path in dimension 1 from u to $u' = (v_1, u_2, u_3, \ldots, u_d)$, followed by a path from u' to v. Thus, $d_S(u, v) \leqslant d_S(u', v) + |u_1 - v_1|$. Since $d_G(u, v) = d_G(u', v) + |u_1 - v_1|$, $d'_S(u, v) \leqslant d'_S(u', v)$.

From vertex u', we begin to move to $u'' = (v_1, v_2, v_3, u_4, \ldots, u_d)$ by taking $b \leqslant \lfloor(\lceil(d-3)/4\rceil + 1)/2\rfloor = \lceil(d-6)/8\rceil$ dimension 1 edges to the nearest vertex with neighbors in either dimension 2 or dimension 3. In the former case, we move $|u_2 - v_2|$ steps in dimension 2 followed by one edge negatively in dimension 1, arriving at a vertex with neighbors in dimension 3. From here, we continue with $|u_3 - v_3|$ steps in dimension 3, followed by $b + 1$ edges positively in dimension 1, arriving at u''. In the latter case we move $|u_3 - v_3|$ steps in dimension 3 followed by one edge positively in dimension 1, arriving at a vertex with neighbors in dimension 2. From here, we continue with $|u_2 - v_2|$ steps in dimension 2, followed by $b + 1$ edges negatively in dimension 1, arriving at u''. In either case, the length of the path from u' to u'' is at most $2\lceil (d-6)/8 \rceil + 2 + |u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|$.

We will construct a path from u'' to v by interspersing single-dimensional paths in dimensions 4 through d into a short cycle. The cycle consists of 4 paths of $\lceil (d+1)/4 \rceil$ edges in dimension 1 joined alternately by dimension 2 and dimension 3 edges. Note that such a cycle will encounter a vertex with edges in dimension k for all $4 \leqslant k \leqslant d$. At each such vertex, we include the appropriate path of length $|u_k - v_k|$ in dimension k.

The total path length from u to v is $(2\lceil (d-6)/8 \rceil + 2 + \sum_{k=1}^{3} |u_k - v_k|) + (4\lceil (d+1)/4 \rceil + 4 + \sum_{k=4}^{d} |u_k - v_k|) = 2\lceil (d-6)/8 \rceil + 4\lceil (d+1)/4 \rceil + 6 + \sum_{k=1}^{d} |u_k - v_k| = 2\lfloor (d-6)/8 \rfloor + 4\lceil (d+1)/4 \rceil + 6 + d_G(u,v)$. □

Corollary 3.3a. *For any finite d-dimensional grid* $G' = G_{n_1,n_2,\ldots,n_d}$ *with* $n_i \geqslant \lceil d/2 \rceil + 1$ *for some* $1 \leqslant i \leqslant d$, *there exists a* $(2\lceil (d-2)/8 \rceil + 4\lceil (d+1)/4 \rceil + 6 + x)$*-spanner of* G' *with* $\Delta = 4$ *and* $\delta \leqslant 4$.

Proof. We assume that the dimensions have been permuted so that $n_1 \geqslant \lceil d/2 \rceil + 1$. In the proof of Corollary 3.2a, we shifted the construction to ensure that no vertex was at distance more than (roughly) $d/4$ from a vertex with neighbors in dimension 2. Similarly, we shift this construction to ensure that no vertex is at distance more than (roughly) $d/8$ from a pair of adjacent vertices with neighbors in dimensions 2 and 3. Let $m = \lceil (d+5)/4 \rceil$. (This is the dimension 1 length of an interchange.) To construct a folded cycle interchange spanner for $G_{n_1,n_2,\ldots,n_d}$, we first shift the infinite folded cycle interchange spanner s units positively in dimension 1 and then take the induced subspanner, where $s = \lceil (m + (n_1 \bmod m))/2 \rceil$ if $0 \leqslant n_1 \bmod m \leqslant 1$ and $s = \lceil (2 + (n_1 \bmod m))/2 \rceil$ if $2 \leqslant n_1 \bmod m \leqslant m-1$.

The path from u to v is constructed in the same manner as in Theorem 3.3 except when u'' is not on a short cycle. If this is the case, the path described from u' to u'' (even if $u = u''$) contains both a vertex with neighbors in dimension 2 and a vertex with neighbors in dimension 3. One of these vertices must be on a short cycle and thus the short cycle (interspersed with single-dimensional paths) can be spliced into the path at this point (rather than occurring after it). The delay analysis is essentially the same as in the proof of Theorem 3.3 except that, due to shifting, $b \leqslant \lceil (d-2)/8 \rceil$ (rather than $\lceil (d-6)/8 \rceil$) and, thus, the delay bound is $2\lceil (d-2)/8 \rceil + 4\lceil (d+1)/4 \rceil + 6$. □

To describe our next generalization, we consider the layers of a cycle interchange spanner. Starting from a layer with dimension 2 edges, by increasing the dimension 1 coordinate we encounter $\lceil (d-2)/2 \rceil$ layers that each contain edges in two different dimensions. This pattern of a dimension 2 layer followed by $\lceil (d-2)/2 \rceil$ other layers is cyclically repeated across the entirety of dimension 1. When d is large, one may improve the delay of the spanner by interleaving l additional dimension 2 layers equally spaced in each set of $\lceil (d-2)/2 \rceil$ layers. This increases the length of the cycle by $2l$ but decreases the delay from u' to u'' in our analysis. We call such a spanner an l shortcut interchange spanner.

To construct a maximum degree 4 l *shortcut interchange* spanner S of an infinite d-dimensional grid, $G_{(d)}$, for $1 \leqslant l \leqslant \lceil d/2 \rceil - 2$, we begin by connecting every vertex $(u_1, u_2, \ldots, u_d)$ to $(u_1 + 1, u_2, \ldots, u_d)$. Let $\lambda = (\lceil d/2 \rceil + l)/(l+1)$. When $u_1 \equiv \lfloor i\lambda \rfloor$ $(\mathrm{mod} \lceil d/2 \rceil + l)$ for $0 \leqslant i \leqslant l$, then we connect $(u_1, u_2, \ldots, u_d)$ to $(u_1, u_2 + 1, u_3, \ldots, u_d)$. Note that this adds l dimension 2 layers which are roughly equally spaced as described above. When u_2 is even and $u_1 \not\equiv \lfloor i\lambda \rfloor (\mathrm{mod} \lceil d/2 \rceil + l)$ for any $0 \leqslant i \leqslant l$, we connect $(u_1, u_2, \ldots, u_k, \ldots, u_d)$ to $(u_1, u_2, \ldots, u_k + 1, \ldots, u_d)$, where $k \neq u_1 (\mathrm{mod} \lceil d/2 \rceil + l) + 2 - \lfloor u_1 (\mathrm{mod} \lceil d/2 \rceil + l)/\lambda \rceil$. When u_2 is odd and $u_1 \not\equiv \lfloor i\lambda \rfloor (\mathrm{mod} \lceil d/2 \rceil + l)$ for any $0 \leqslant i \leqslant l$, we consider two cases: If $u_1 \equiv \lceil d/2 \rceil + l - 1$ $(\mathrm{mod} \lceil d/2 \rceil + l)$ and d is odd, we add no further connections to $(u_1, u_2, \ldots, u_d)$. Otherwise, we connect $(u_1, u_2, \ldots, u_d)$ to $(u_1, u_2, \ldots, u_k + 1, \ldots, u_d)$, where $k = u_1 (\mathrm{mod} \lceil d/2 \rceil + l) + \lceil d/2 \rceil + 1 - \lfloor u_1 (\mathrm{mod} \lceil d/2 \rceil + l)/\lambda \rfloor$. Fig. 3 shows a 2-dimensional slice (dimensions 1 and 2) of a 3 shortcut interchange spanner of $G_{(14)}$.

Theorem 3.4. *The l shortcut interchange spanner S (for $1 \leqslant l \leqslant \lceil d/2 \rceil - 2$) is a $(\lceil (\lceil d/2 \rceil + l)/(l+1) \rceil + 2\lceil d/2 \rceil + 2l + 2 + x)$-spanner of $G_{(d)}$ with $\Delta_S = 4$ and $\delta_S \leqslant 4$.*

Proof. We proceed as in the proof of Theorem 3.2. Given two vertices $u = (u_1, u_2, \ldots, u_d)$ and $v = (v_1, v_2, \ldots, v_d)$, $d_G(u, v) = \sum_{i=1}^{d} |u_i - v_i|$. We wish to bound the delay between u and v in S. Since S contains all edges of dimension 1, there is a path from u to v consisting of a path in dimension 1 from u to $u' = (v_1, u_2, u_3, \ldots, u_d)$, followed by a path from u' to v. Thus, $d_S(u, v) \leqslant d_S(u', v) + |u_1 - v_1|$. Since $d_G(u, v) = d_G(u', v) + |u_1 - v_1|$, $d'_S(u, v) \leqslant d'_S(u', v)$.

From vertex u', we move to $u'' = (v_1, v_2, u_3, \ldots, u_d)$ by taking $b \leqslant \lfloor \lceil \lambda \rceil / 2 \rfloor$ dimension 1 edges to the nearest vertex with neighbors in dimension 2, followed by a path of $|u_2 - v_2|$ edges in dimension 2 and b more edges to u''.

We will construct a path from u'' to v by interspersing single dimensional paths in dimensions 3 through d into a short cycle. The cycle consists of 2 paths of $\lceil d/2 \rceil + l$ edges in dimension 1 joined on both ends by a single dimension 2 edge. Note that such a cycle will encounter a vertex with edges in dimension k for all $3 \leqslant k \leqslant d$. At each such vertex, we include the appropriate path of length $|u_k - v_k|$ in dimension k.

The total path length from u to v is $|u_1 - v_1| + (2b + |u_2 - v_2|) + 2(\lceil d/2 \rceil + l) + 2 + \sum_{k=3}^{d} |u_k - v_k| = d_G(u, v) + 2b + 2(\lceil d/2 \rceil + l) + 2 \leqslant d_G(u, v) + 2\lfloor \lceil \lambda \rceil / 2 \rfloor + 2(\lceil d/2 \rceil + l) + 2 \leqslant d_G(u, v) + \lceil \lambda \rceil + 2(\lceil d/2 \rceil + l) + 2 = d_G(u, v) + \lceil (\lceil d/2 \rceil + l)/(l+1) \rceil + 2\lceil d/2 \rceil + 2l + 2$. □

Corollary 3.4a. *For any finite d-dimensional grid $G' = G_{n_1, n_2, \ldots, n_d}$ with $n_i \geqslant \lceil d/2 \rceil + 1$ for some $1 \leqslant i \leqslant d$, there exists a $(\lceil (\lceil d/2 \rceil + l)/(l+1) \rceil + 2\lceil d/2 \rceil + 2l + 2 + x)$-spanner of G' for each $1 \leqslant l \leqslant \lceil d/2 \rceil - 2$ with $\Delta = 4$ and $\delta \leqslant 4$.*

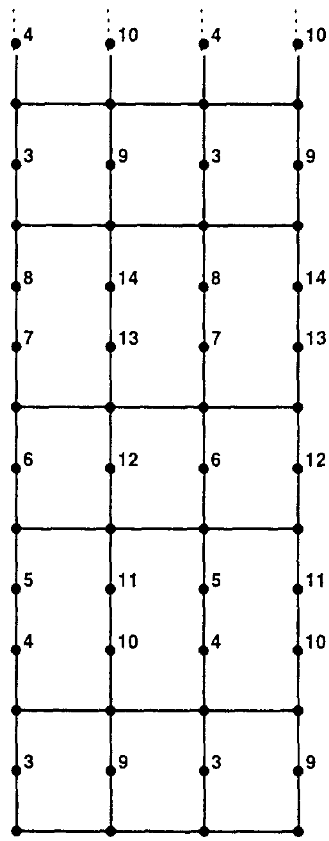


Fig. 3. A slice of a 3 shortcut interchange spanner of $G_{(14)}$ showing dimension 1 vertically and dimension 2 horizontally. Label i on a vertex indicates that the vertex is connected to its neighbors in dimension i.

Proof. We proceed by modifying the proof of Theorem 3.4 exactly as the proof of Corollary 3.2a followed from the proof of Theorem 3.2 except that we use the following procedure to determine how much to shift the infinite construction before inducing the subspanner on the finite grid. Suppose that we induce the subspanner without shifting. Let j be the distance from a vertex with maximum x_1-coordinate to a vertex with an edge in dimension 2. Shift the infinite spanner $\lceil j/2 \rceil$ units positively in dimension 1. This ensures that every vertex is within distance $\lfloor \lceil \lambda \rceil / 2 \rfloor$ of a vertex with neighbors in dimension 2. □

Among the first three constructions (path interchange, cycle interchange, and folded cycle interchange), the path interchange spanner yields the lowest delay for $d < 14$, the cycle interchange yields the lowest delay for $14 \leqslant d < 25$, and the folded cycle interchange yields the lowest delay for $d \geqslant 25$. The 1 shortcut interchange spanner has lower delay than the first three constructions for $d \geqslant 15$. The 2 shortcut interchange spanner has lower delay than the 1 shortcut interchange spanner for $d \geqslant 31$.

Generally, the l shortcut interchange spanners will outperform the other types for sufficiently large d when $1 \leqslant l < d/8$. However, routing in the l shortcut interchange spanners is more complicated than in the first three types.

We can decrease the delay on these four constructions by allowing the maximum degree to increase. For instance, we may create a spanner like the path interchange spanner but where each layer contains all of the edges in k dimensions instead of just a single dimension.

4. Construction of grid spanners with maximum degree 3

In this section we construct a good spanner of $G_{(d)}$ with maximum degree 3. We begin by discussing the general method of construction and then present a more detailed specific construction.

We first decompose $G_{(d)}$ into d-dimensional hypercubes as follows: Each vertex $(u_1, u_2, \ldots u_d)$ belongs to a hypercube $H_{\alpha_1, \alpha_2, \ldots, \alpha_d}$ where $\alpha_i = \lfloor u_i/2 \rfloor$ for $1 \leqslant i \leqslant d$. The edges of $H_{\alpha_1, \alpha_2, \ldots, \alpha_d}$ are those induced by the vertices of $H_{\alpha_1, \alpha_2, \ldots, \alpha_d}$ in $G_{(d)}$. The remaining edges of $G_{(d)}$ connect adjacent hypercubes.

A *highway* is an infinite line of edges in some dimension i, that is, the subgraph induced in $G_{(d)}$ by the set of vertices $(u_1, \ldots u_{i-1}, x, u_{i+1}, \ldots u_d)$ where the u's are constant and the x varies over the integers. Given that $G_{(d)}$ is decomposed into hypercubes as described above, any highway consists of a sequence of edges which are alternately edges of a hypercube and edges connecting adjacent hypercubes. The highway is said to *intersect* those hypercubes with which it shares an edge.

A d-dimensional *general bypass spanner* is a spanner of $G_{(d)}$ such that every hypercube $H_{\alpha_1, \alpha_2, \ldots, \alpha_d}$ is intersected by at least one highway in each dimension, and each hypercube $H_{\alpha_1, \alpha_2, \ldots, \alpha_d}$ induces a connected subgraph of the spanner.

Consider the two vertices $u = (u_1, u_2, \ldots u_d)$ in $H_{\alpha_1, \alpha_2, \ldots, \alpha_d}$ and $v = (v_1, v_2, \ldots, v_d)$ in $H_{\beta_1, \beta_2, \ldots, \beta_d}$. To find a path from u to v in a general bypass spanner S of the grid $G = G_{(d)}$, we examine the indices $\alpha_1, \alpha_2, \ldots \alpha_d$ and $\beta_1, \beta_2, \ldots \beta_d$. Suppose that $\alpha_i \neq \beta_i$ for some $1 \leqslant i \leqslant d$. Then, by moving from u to a vertex in $H_{\alpha_1, \alpha_2, \ldots, \alpha_d}$ from which a highway in dimension i exits the hypercube in the direction of $H_{\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \beta_i, \alpha_{i+1}, \ldots, \alpha_d}$ and following the dimension i highway from $H_{\alpha_1, \alpha_2, \ldots, \alpha_d}$ to the first vertex u' of $H_{\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \beta_i, \alpha_{i+1}, \ldots, \alpha_d}$ encountered, we have reduced the number of indices in which our source and destination hypercubes differ. By repeating this process, we eventually arrive at a vertex v' of $H_{\beta_1, \beta_2, \ldots, \beta_d}$ (which could be u if $\alpha_i = \beta_i$ for $1 \leqslant i \leqslant d$). To get from v' to v, we follow any path between v' and v in $H_{\beta_1, \beta_2, \ldots, \beta_d}$. Because each hypercube of the construction is connected, any such path between two vertices of the same hypercube is of length no more than $2^d - 1$.

Let d' denote the number of dimensions i such that $\alpha_i \neq \beta_i$. The constructed path from u to v consists of $(d' + 1)$ subpaths each in a single hypercube alternating with d' sections of highway edges. Thus, the total path length is at most $(d' + 1)\,(2^d - 1)$ plus the length of the highway sections. As the highway sections used in the path construction are shortest paths between the hypercubes involved, the length of the section of dimension i highway is $\leqslant |u_i - v_i|$. Thus, the total length of all highway sections is $\leqslant \sum_{i=1}^{d} |u_i - v_i| = d_G(u, v)$. This implies that the total path length in S is at most $(d' + 1)(2^d - 1) + d_G(u, v)$ and, thus, the delay is at most $(d' + 1)(2^d - 1)$. As $d' \leqslant d$, this gives delay at most $(d + 1)(2^d - 1)$. Thus, any general bypass spanner of the infinite d-dimensional grid is a $(c + x)$-spanner for some constant c.

If a general bypass spanner is such that the highways are "regularly spaced" and each hypercube contains the same substructure, we call it a *regular bypass spanner*. More formally, we can describe a regular bypass spanner S by indicating which edges of $H_{0,0,\ldots,0}$ are included in S and designating at least one such edge in each dimension as a highway edge. The edges included in $H_{0,0,\ldots,0}$ are included in each hypercube $H_{\alpha_1,\alpha_2,\ldots,\alpha_d}$; that is, if the edge from $(u_1, \ldots u_{i-1}, 0, u_{i+1}, \ldots u_d)$ to $(u_1, \ldots u_{i-1}, 1, u_{i+1}, \ldots u_d)$ is included in S, then all edges $(u_1 + 2\alpha_1, \ldots, u_{i-1} + 2\alpha_{i-1}, 2\alpha_i, u_{i+1} + 2\alpha_{i+1}, \ldots, u_d + 2\alpha_d)$ to $(u_1 + 2\alpha_1, \ldots, u_{i-1} + 2\alpha_{i-1}, 2\alpha_i + 1, u_{i+1} + 2\alpha_{i+1}, \ldots u_d + 2\alpha_d)$ are also included in S. The endpoints of a dimension i highway edge also have edges leading to their other neighbor in dimension i. Consider two adjacent hypercubes $H_{\alpha_1,\alpha_2,\ldots,\alpha_d}$ and $H_{\alpha_1,\ldots,\alpha_{i-1},\alpha_i+1,\ldots,\alpha_d}$. These two hypercubes each contain a designated dimension i highway edge. Furthermore, there is a dimension i edge connecting these two designated highway edges. To complete S, we include all such connecting highway edges.

We now construct a maximum degree 3 regular bypass spanner S_d of $G_{(d)}$ for $d \geqslant 3$. The constant delay of this spanner is $O(d)$ which is significantly better than the $O(d2^d)$ delay of a general bypass spanner.

We label the nodes of a d-dimensional hypercube with d-bit binary strings. We follow the convention that the leftmost bit corresponds to the vertex's coordinate in dimension 1, and the rightmost bit corresponds to the vertex's coordinate in dimension d.

A *double-rooted binary tree* is a full binary tree in which the root node has been split into two nodes (one with the left subtree hanging off of it, and the other with the right subtree) that are connected by an edge. Each of the vertices created by the split is known as a *root* of the double-rooted binary tree, and the edge between them is called the *root edge*. A double-rooted binary tree is shown in Fig. 4.

In [6], it is shown that the double-rooted binary tree is a spanning subgraph of a d-cube. We give a detailed description of an embedding of the double-rooted binary tree in the d-cube in order to establish properties of this embedding which will be used in the spanner construction that follows. We denote this embedded double-rooted binary tree as *DRB*(d). First, we let *DRB*(1), *DRB*(2), and *DRB*(3) be the trees shown in Figs. 5(a), (b), and (c) respectively.

Note that for *DRB*(3), the root edge of the tree is in dimension d, and is incident on the origin 000. The two edges adjacent to the root edge are in dimension $d - 1$ (from 000) and in dimension $d - 2$ (from 001).

We wish for our general construction to have the above properties; i.e. the root edge is in dimension d from $00\ldots00$, and the two edges adjacent to the root are in

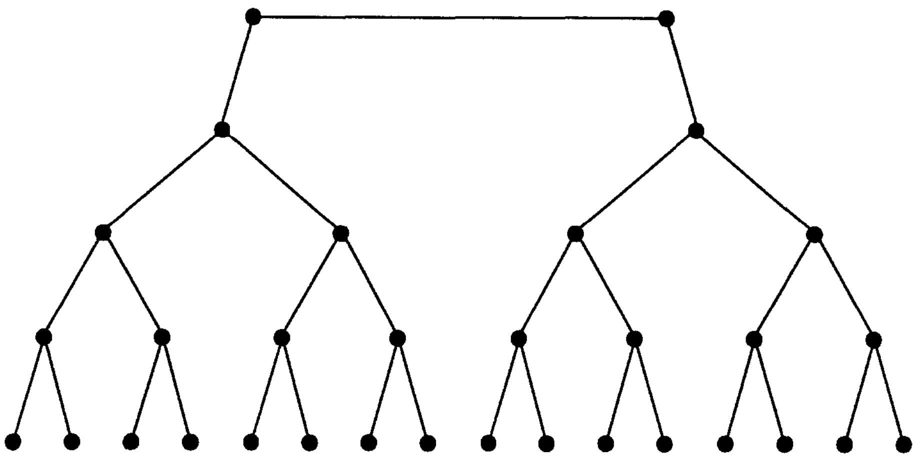


Fig. 4. The double-rooted binary tree $DRB(5)$.

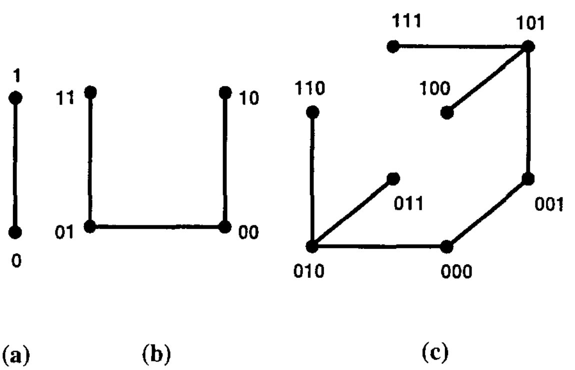


Fig. 5. The embedded double-rooted binary trees $DRB(1)$, $DRB(2)$, and $DRB(3)$.

dimension $d-1$ from $00\ldots00$ and in dimension $d-2$ from $00\ldots01$. When constructing $DRB(d+1)$, we assume that this is true for all of the trees $DRB(d')$ where $1 \leqslant d' \leqslant d$.

We now construct another embedding of the double-rooted binary tree, symmetric to and derived from $DRB(d)$. We call this embedding $TWIST(d)$. Each vertex $X = (x_1, x_2, \ldots x_{d-3}, x_{d-2}, x_{d-1}, x_d)$ of $DRB(d)$ is mapped to a vertex $\phi(X) = (x_1, x_2, \ldots x_{d-3}, x'_{d-2}, x'_{d-1}, x'_d)$ of $TWIST(d)$, where

$$x'_{d-2} = x_{d-1}, \quad x'_{d-1} = \overline{x_d}, \quad \text{and} \quad x'_d = x_{d-2}.$$

Each edge (X, Y) of $DRB(d)$ gives rise to an edge $(\phi(X), \phi(Y))$ of $TWIST(d)$. In $TWIST(d)$, the root edge is in dimension $d-1$ from $00\ldots00$, and the two edges

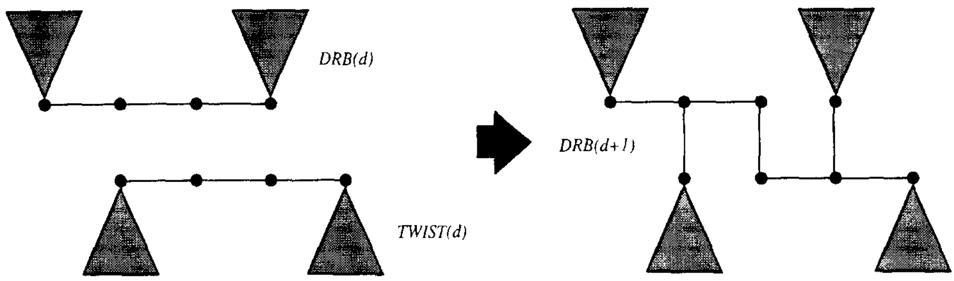


Fig. 6. Making a double-rooted binary tree from two smaller ones.

adjacent to the root are in dimension d from $00\dots00$ and in dimension $d-2$ from $00\dots010$.

To construct $DRB(d+1)$, we start by placing $DRB(d)$ in the d-cube with $(d+1)$-coordinate 0, and placing $TWIST(d)$ in the d-cube with $(d+1)$-coordinate 1. We then add three edges in dimension $(d+1)$, from the vertices $00\dots0000$, $00\dots0010$, and $00\dots0100$, to the vertices $00\dots0001$, $00\dots0011$, and $00\dots0101$, respectively. We then remove the edge from $00\dots0000$ to $00\dots0100$, and the edge from $00\dots0001$ to $00\dots0011$. Conceptually, we have spliced together two double-rooted binary trees to obtain a larger one as in Fig. 6. The resulting tree is $DRB(d+1)$, and indeed it has the root edge in dimension $d+1$ from $00\dots00$, and the two edges adjacent to the root edge in dimension d from $00\dots00$ and in dimension $d-1$ from $00\dots01$. $DRB(4)$ is shown in Fig. 7.

Lemma 4.1. *For $d \geqslant 4$, a vertex is a leaf of $DRB(d)$ iff it is a leaf of either the $DRB(d-1)$ or the $TWIST(d-1)$ used in the construction of $DRB(d)$.*

Proof. For $d \geqslant 4$, $DRB(d)$ was constructed from the $DRB(d-1)$ and $TWIST(d-1)$ by adding and deleting some edges incident on the roots of these trees or incident on vertices adjacent to such roots. As $d \geqslant 4$, the depths of the underlying double-rooted binary trees for both $DRB(d-1)$ and $TWIST(d-1)$ are at least two. Thus, no leaf is a root and no leaf is adjacent to a root. Therefore, the degree of any leaf in either the $DRB(d-1)$ or $TWIST(d-1)$ is unchanged when the two copies are put together. Furthermore, no new leaves are created. □

Corollary 4.1a. *For $d \geqslant 3$, the vertices $1000\dots0$, $1100\dots0$, and $1110\dots0$ are leaves of $DRB(d)$.*

We now describe which edges of $H_{0,0,\dots,0}$ are included in S_d. The structure in $H_{0,0,\dots,0}$ is composed of a short cycle containing highway edges in each dimension, and a large tree connected to the cycle at a single vertex. The large tree is essentially composed of several instances of $DRB(i)$ of different sizes i, all hanging off of a short path.

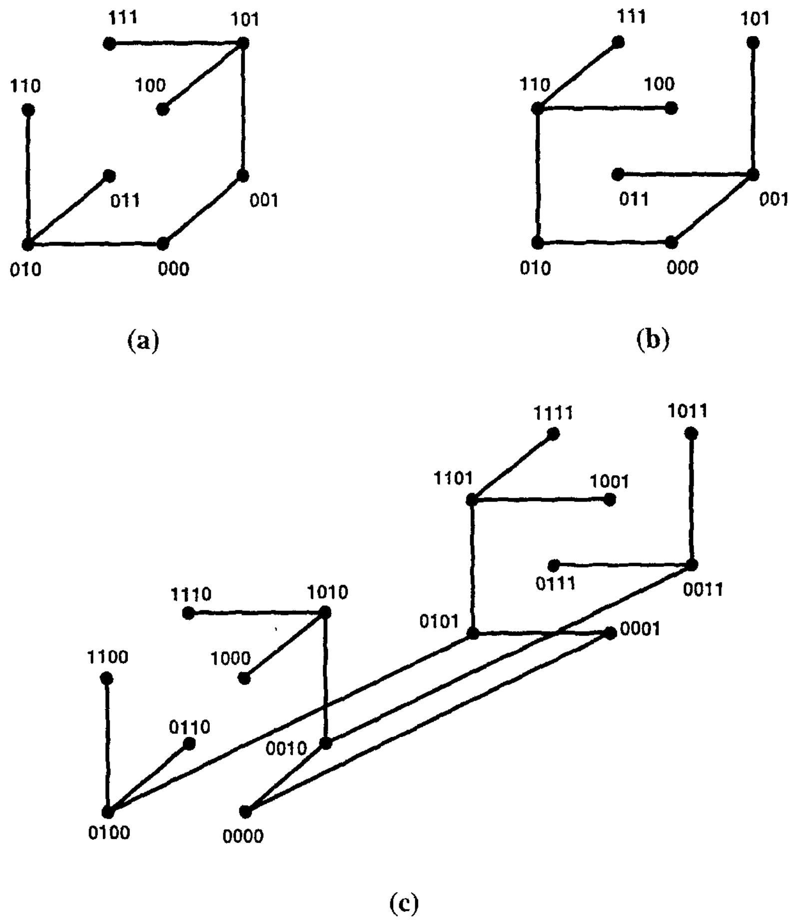


Fig. 7. (a) The tree $DRB(3)$, (b) the tree $TWIST(3)$, (c) the tree $DRB(4)$.

For $d \geqslant 4$, the vertices of the d-cube can be decomposed into the vertices of a $(d-1)$-cube Q_{d-1} with $x_d = 0$, a $(d-2)$ -cube Q_{d-2} with $x_{d-1} = 0$ and $x_d = 1$, a $(d-3)$-cube Q_{d-3} with $x_{d-2} = 0$ and $x_{d-1} = x_d = 1$, and so on until we have one 3-cube Q_3 with $x_4 = 0$ and $x_5 = x_6 = \cdots = x_d = 1$, and another 3-cube $\ddot{Q}$ with $x_4 = x_5 = \cdots = x_d = 1$. For $3 \leqslant i \leqslant d-1$, we form a tree $T(i)$ in Q_i by first placing $DRB(i)$ in Q_i and then deleting the three vertices $v_1(i)$, $v_2(i)$, and $v_3(i)$ that have the first i coordinates $1000\ldots0$, $1100\ldots0$, and $1110\ldots0$, respectively. (These are coordinates in Q_i, not in $H_{0,0,\ldots,0}$.) Note that these vertices were leaves of $DRB(i)$, by Corollary 4.1a, and thus $T(i)$ is a tree.

The cycle of the spanner of $H_{0,0,\ldots,0}$ will contain the vertices of $\ddot{Q}$ and all of the vertices $v_2(i)$ and $v_3(i)$. The vertices $v_1(i)$ will be used to connect together the trees $T(i)$ to form a large tree that will be connected to the cycle in Q_{d-1}.

The order of vertices on the cycle is $v_3(d-1)$, $v_3(d-2)$, $\ldots$, $v_3(3)$, $1111\ldots1$, $0111\ldots1, 0011\ldots1, 1011\ldots1, 1001\ldots1, 0001\ldots1, 0101\ldots1, 1101\ldots1, v_2(3), v_2(4), \ldots,$

$v_2(d-1)$. The high-dimensional part of this cycle is shown in Fig. 8. The highway edges are in dimension d between $v_3(d-1)$ and $v_3(d-2)$, in dimension $d-1$ between $v_2(d-2)$ and $v_2(d-3)$, in dimension $d-2$ between $v_3(d-3)$ and $v_3(d-4)$, and so on, alternating between v_2 vertices and v_3 vertices.

If d is odd, this alternation ends at a dimension 5 edge between $v_3(4)$ and $v_3(3)$. The remaining highway edges are in dimension 4 between $v_2(3)$ and $1101\ldots1$, in dimension 3 between $1001\ldots1$ and $1011\ldots1$, in dimension 2 between $0101\ldots1$ and $0001\ldots1$, and in dimension 1 between $1111\ldots1$ and $0111\ldots1$ (see Fig. 9(a)). If d is even, the alternation ends at a dimension 5 edge between $v_2(4)$ and $v_2(3)$. The remaining highway edges are in dimension 4 between $v_3(3)$ and $1111\ldots1$, in dimension 3 between $1001\ldots1$ and $1011\ldots1$, in dimension 2 between $0111\ldots1$ and $0011\ldots1$, and in dimension 1 between $1101\ldots1$ and $0101\ldots1$ (see Fig. 9(b)).

Recall that vertices on highway edges are connected outside of the hypercube in the direction of the highway edge. Thus, each highway edge effectively contributes two to the degree of its endpoints in the final bypass spanner construction. Note that, in the cycle as constructed, no vertex has more than one highway edge incident on it, and

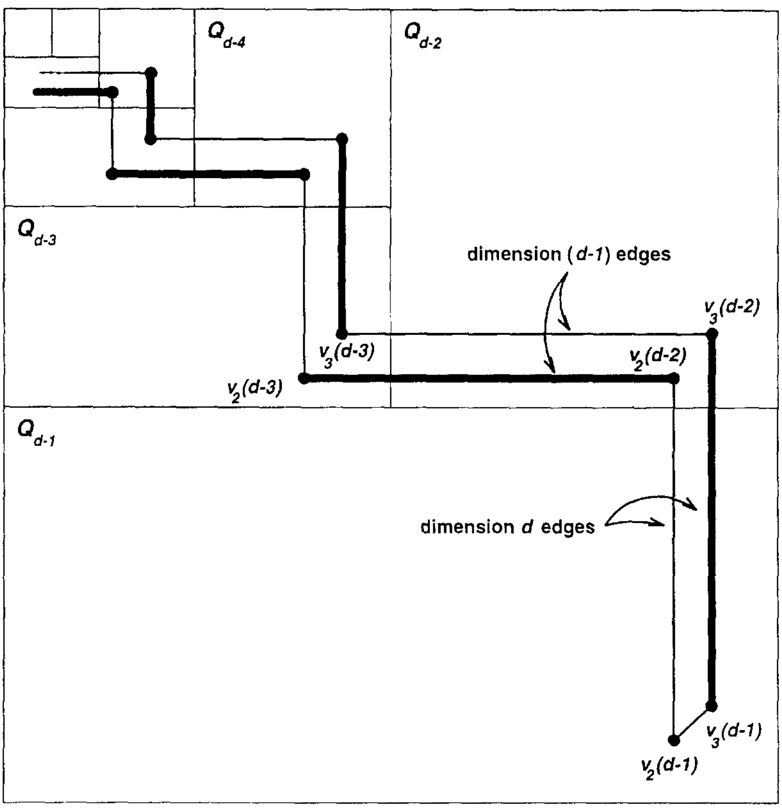
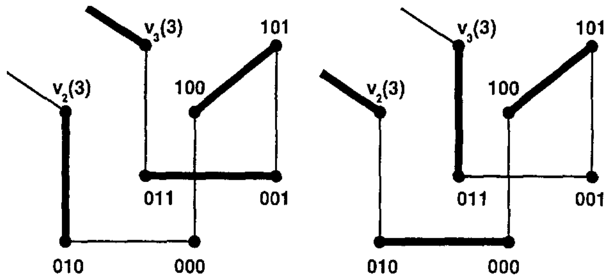


Fig. 8. Upper part of cycle with highway edges shown as thick edges.



(a) (b)

Fig. 9. Lower part of cycle with highway edges shown as thick edges: (a) d is even and (b) d is odd.

thus these vertices will have degree at most three in the bypass spanner. Furthermore, the vertex $v_2(d-1)$ has no highway edges incident on it, and thus has only degree two so far.

To complete our construction (forming the large tree referred to above), we extend the path $v_2(d-1), v_1(d-1), v_1(d-2), \ldots, v_1(3)$ from the vertex $v_2(d-1)$ on the cycle. This makes $v_2(d-1)$ degree 3, $v_1(3)$ degree 1, and the remaining vertices $v_1(i)$ degree 2. In each Q_i we attach $T(i)$ to this path by the edge in dimension 1 between the vertex $r(i)$ with the first i coordinates $000\ldots0$ and $v_1(i)$, which has the first i coordinates $100\ldots0$.

Adding these trees makes the degree of each of the vertices $v_1(i)$ be at most 3. Note that the vertex $r(i)$, being the origin of Q_i, is a root of $DRB(i)$, and hence has degree two in $T(i)$. Adding the edge to $v_1(i)$ makes each $r(i)$ have degree 3.

This completes the construction of the substructure of $H_{0,0,\ldots,0}$, and, together with the inclusion of the appropriate highway edges between adjacent hypercubes as described above, it also completes the construction of the bypass spanner S_d of $G_{(d)}$, which has maximum degree three.

Since the construction of the bypass spanner S_d partitions the d-grid into d-cubes, and then replicates the same structure on each d-cube, the average degree for the spanner is the same as the average degree for a single hypercube. In any such d-cube, there are 2^d vertices. Recall that the structure of S_d in a d-cube is a cycle with an attached tree and, therefore, has 2^d edges. Each of these edges contributes 2 to the total degree in the d-cube and each of the $2d$ highway edges leading to other d-cubes contributes 1 to the total degree. Thus, the average degree in S_d is

$$\frac{2\cdot 2^d + 2d}{2^d} = 2 + \frac{d}{2^{d-1}}.$$

Given two vertices $u = (u_1, u_2, \ldots u_d)$ in $H_{\alpha_1,\alpha_2,\ldots,\alpha_d}$ and $v = (v_1, v_2, \ldots v_d)$ in $H_{\beta_1,\beta_2,\ldots,\beta_d}$, we construct a path from u to v in S_d consisting of three sections: section A from u to vertex $v_2(d-1)$ of $H_{\alpha_1,\alpha_2,\ldots,\alpha_d}$, section B from vertex $v_2(d-1)$ of

$H_{\alpha_1,\alpha_2,\ldots,\alpha_d}$ to vertex $v_2(d-1)$ of $H_{\beta_1,\beta_2,\ldots,\beta_d}$, and section C from vertex $v_2(d-1)$ of $H_{\beta_1,\beta_2,\ldots,\beta_d}$ to v. If u is a vertex on the cycle in $H_{\alpha_1,\alpha_2,\ldots,\alpha_d}$, then section A is the shortest path on the cycle from u to $v_2(d-1)$ consisting of at most $d+1$ edges. Otherwise, section A is the path from u to $v_2(d-1)$ in the large tree of $H_{\alpha_1,\alpha_2,\ldots,\alpha_d}$ which is of length at most $d+1$ edges. Thus, in either case, section A contains at most $d+1$ edges. Similarly, section C contains at most $d+1$ edges. Section B can be thought of as one traversal of the cycle from $v_2(d-1)$ to itself interspersed with single dimensional paths in each dimension i in which $\alpha_i \neq \beta_i$. The length of this section is $2d + 2 + \sum_{i=1}^{d} 2|\alpha_i - \beta_i|$ edges. By the way the grid has been subdivided into hypercubes, we have that $|u_i - v_i| \geqslant 2|\alpha_i - \beta_i| - 1$ for all i. The number of edges in the path from u to v may be rewritten as $4d + 4 + \sum_{i=1}^{d}|u_i - v_i| + \sum_{i=1}^{d} 1 \leqslant 4d + 4 + \sum_{i=1}^{d}|u_i - v_i| + d \leqslant 5d + 4 + d_{G_{(d)}}(u,v)$. Thus, the delay in S_d is at most $5d+4$, giving:

Theorem 4.2. *The bypass spanner S_d is a $(5d+4+x)$-spanner of $G_{(d)}$ with $\Delta = 3$ and $\delta = 2 + d/2^{d-1}$.*

We may also obtain good spanners of finite grids by taking induced subspanners of of S_d.

Corollary 4.2a. *For any finite d-dimensional grid $G' = G_{n_1,n_2,\ldots,n_d}$ with n_i even for all $1 \leqslant i \leqslant d$, there exists a $(5d+4+x)$-spanner of G' with $\Delta = 3$ and $\delta = 2 + d/2^{d-1}$.*

Proof. Consider the induced subspanner of the bypass spanner S_d. The delay bound follows directly from the proof of Theorem 4.2. □

5. Construction of some specific spanners

We have constructed other bypass spanners that improve the above delay bound for 3 and 4 dimensions.

We now construct a maximum degree 3 spanner S of the infinite 3-dimensional grid. This spanner is much like a regular bypass spanner except that each hypercube does not induce a connected subgraph of the spanner. We begin by creating the highways. We create the dimension 1 highways by connecting (u_1, u_2, u_3) to (u_1+1, u_2, u_3) when u_2 is even and u_3 is odd. We then create the dimension 2 and dimension 3 highways by connecting (u_1, u_2, u_3) to (u_1, u_2+1, u_3) when u_3 is even and u_1 is odd, and (u_1, u_2, u_3) to (u_1, u_2, u_3+1) when u_1 is even and u_2 is odd. To complete the spanner, we connect all vertices (u_1, u_2, u_3) that are not on any highways (which we shall refer to as connector vertices) to (u_1+1, u_2, u_3), (u_1, u_2+1, u_3), and (u_1, u_2, u_3+1).

Theorem 5.1. *The spanner S described above is a $(6+x)$-spanner of $G_{(3)}$ with $\Delta_S = 3$ and $\delta_S = 3$.*

Proof. Given two vertices $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, $d_G(u, v) = \sum_{i=1}^{3} |u_i - v_i|$. We wish to bound the delay between u and v in S.

If one of the vertices, say u, is a vertex on a highway of dimension i, let u' be the vertex on that highway with ith coordinate equal to v_i. One path from u to v in S consists of a section along the ith-dimensional highway from u to u' followed by a path from u' to v. The delay along this path is equal to the delay from u' to v, and $d'_S(u, v) \leqslant d'_S(u', v)$.

Otherwise, both u and v are connector vertices. If u and v differ in two or fewer coordinates, let $u' = u$; clearly $d'_S(u, v) = d'_S(u', v)$. If u and v differ in all three coordinates, then one of them, say u, has lower first coordinate. From this vertex, we can step one edge upward in the first dimension to a highway in one of the other dimensions i. We follow this highway in the appropriate direction until we reach a vertex u' with ith coordinate equal to v_i. As the first step and the highway section are both in the direction of v in G, $d'_S(u, v) \leqslant d'_S(u', v)$.

In any case, u' differs from v in at most two coordinates, u' is either a connector vertex or is on a highway orthogonal to a plane containing u' and v, and $d'_S(u, v) \leqslant d'_S(u', v)$.

Consider the plane containing u' and v; if there is more than one such plane, choose one orthogonal to the highway used to get from u to u', if any. By symmetry, we can orient this plane to look like Fig. 10 where H denotes vertices on highways orthogonal to the plane and C denotes connector vertices. Also, by symmetry, we can assume that the connector vertices are each incident on an edge orthogonal to this plane that connects to a vertex on a horizontal highway. For convenience, we will call the horizontal direction dimension 1, the vertical direction dimension 2, and the orthogonal direction dimension 3. Note that u' is labeled either C or H.

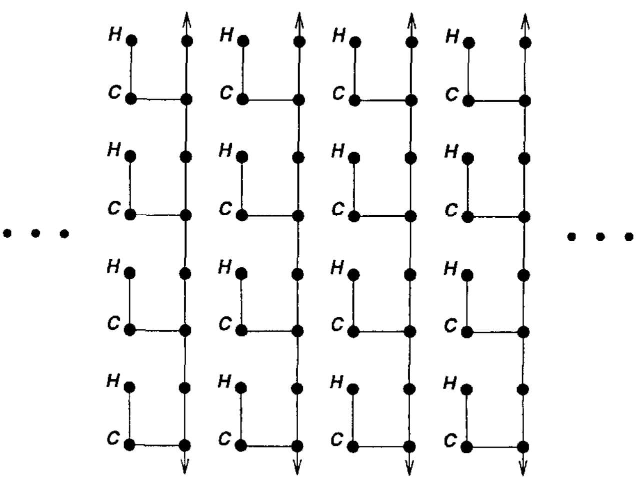


Fig. 10. A two-dimensional slice of spanner S.

If v is on a highway in dimension 2, then let v' be the vertex on that highway with the same dimension 2 coordinate as u'. One path from u' to v in S goes from u' to v' and then along the highway to v, thus $d'_S(u', v) \leqslant d'_S(u', v')$. If u' is a vertex of type H, then a path from u' to v' can be formed by stepping down in dimension 2, orthogonally in dimension 3 to a highway in dimension 1, following the highway, stepping from the highway to a connector vertex via a dimension 3 edge, moving right on a dimension 1 edge, and finally stepping up on a dimension 2 edge. The two possible shapes of this path are shown in Fig. 11. As we can see from the figure, this path has delay at most 6. If u' is a type C vertex, then a similar path from u' to v' can be found by omitting the first and last edges of the path described above. Such a path has delay at most 4.

It remains to consider the case where v is either a type C or a type H vertex. Let v' be v if v is a type C vertex or the type C vertex adjacent to v by a dimension 2 edge, otherwise. Similarly, let u'' be u' if u' is a type C vertex or the type C vertex adjacent to u' by a dimension 2 edge, otherwise. Note that $d'_S(u', v) \leqslant d'_S(u'', v') + 2$ since either $d_G(u', v) = d_G(u'', v')$ and $d_S(u', v) \leqslant d_S(u'', v') + 2$, or $d_G(u', v) \geqslant d_G(u'', v') - 1$ and $d_S(u', v) \leqslant d_S(u'', v') + 1$. Since u'' and v' are both type C vertices, we can construct a path from u'' to v' by stepping orthogonally in dimension 3 to a highway in dimension 1, following the highway, stepping from the highway to a connector vertex via a dimension 3 edge, stepping along a dimension 1 edge to a highway of dimension 2, following the highway, and then stepping from the highway to v' along a dimension 1 edge as shown in Fig. 12. As the distance traveled along the highway sections is the distance between u'' and v', this section has delay 4 and, hence, $d'_S(u', v) \leqslant 6$.

Thus, in all cases we have shown that $d'_S(u', v) \leqslant 6$ and, therefore, $d'_S(u, v) \leqslant 6$. □

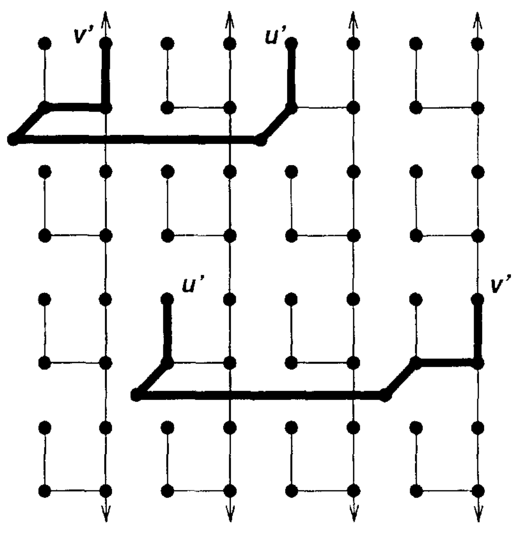


Fig. 11. Possible path shapes.

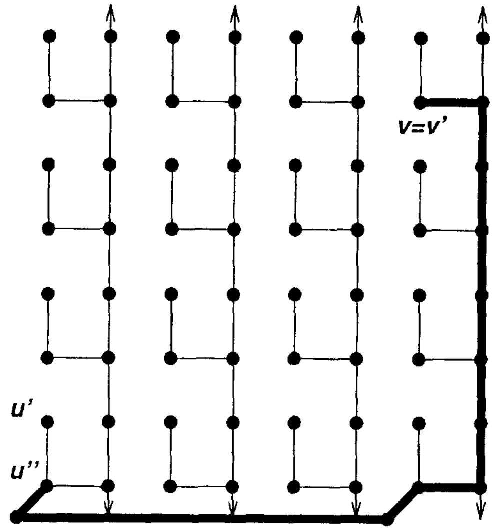


Fig. 12. Path from u'' to u'.

We can modify the above construction to obtain a maximum degree 3 spanner S of a finite 3-dimensional grid G_{n_1,n_2,n_3}. We begin by taking the subgraph of the infinite spanner induced by the vertices of G_{n_1,n_2,n_3}.

Consider the face corresponding to $x_3 = n_3 - 1$. The structure on this face is a set of parallel highways of dimension i ($i = 1$ or 2) alternating with lines containing connector vertices labeled C and vertices on highways in dimension 3 (orthogonal to the face) labeled H, as shown in Fig. 10 (except that the face is finite). In the infinite spanner, each vertex labeled C or H was connected to its neighbor with $x_3 = n_3$. Thus, in the induced subgraph these vertices have degree at most 2. We can therefore add edges so that all edges of dimension i are included in this face without increasing the degree of any vertex beyond 3. The resulting structure on this face is shown in Fig. 13. To continue the construction of S, we modify the faces corresponding to $x_1 = n_1 - 1$ and $x_2 = n_2 - 1$ similarly. Note that some vertices are on more than one face, but any edge added to a vertex corresponds to an edge deleted from the vertex when taking the induced subgraph, so no vertex has degree greater than 3.

The case analysis to establish the delay of this finite spanner would be lengthy and would lend no special insight into the structure. Rather than constructing and verifying such an analysis, we note that due to the regular structure of the spanner, we can calculate the delay on any such spanner by computing the delay on a particular small spanner with the same parity in each dimension. Thus, a small number of computations can establish a general bound. To this end, we constructed a program that calculates the delay of such a spanner and ran it on various sizes of grids with all possible parity combinations to establish the following.

Corollary 5.1a. *The spanner S described above for G_{n_1,n_2,n_3} has $\Delta_S = 3$ and $\delta_S = 3$. When all n_i are even or all n_i are odd, S has delay 6; otherwise, S has delay 8.*

We now construct a maximum degree 3 regular bypass spanner S of the infinite 4-dimensional grid that has two highways of each dimension passing through each hypercube. The edges we include in $H_{0,0,0,0}$ form the Hamiltonian cycle 0000, 1000, 1010, 1110, 0110, 0100, 1100, 1101, 1111, 0111, 0101, 0001, 1001, 1011, 0011, 0010. The highway edges within $H_{0,0,0,0}$ are alternate edges of this cycle starting with the edge between 0000 and 1000. This structure is shown in Fig. 14 with thick edges indicating highway edges.

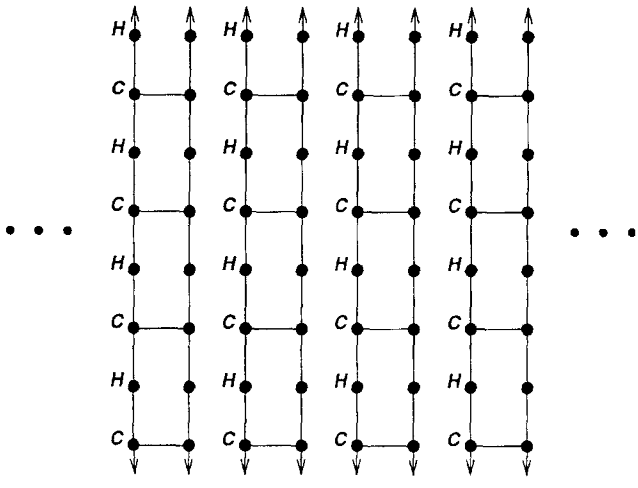


Fig. 13. Structure of face of spanner S corresponding to $x_3 = n_3 - 1$.

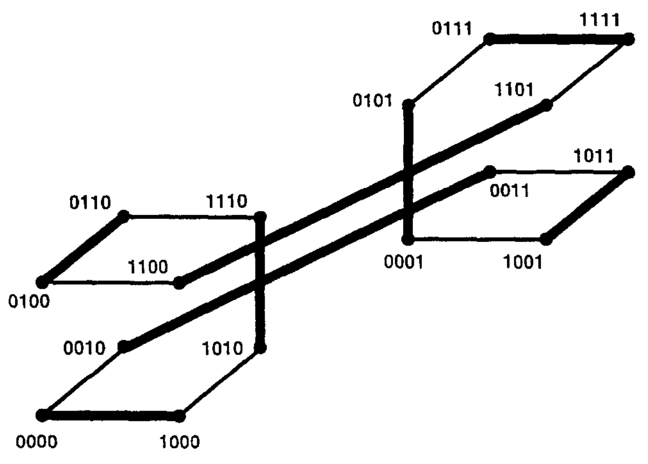


Fig. 14. Cycle in $H_{0,0,0,0}$.

Theorem 5.2. *The spanner S described above is a* $(14 + x)$*-spanner of* $G_{(4)}$ *with* $\Delta_S = 3$ *and* $\delta_S = 3$.

Proof. Consider any pair of vertices, u in $H_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ and v in $H_{\beta_1,\beta_2,\beta_3,\beta_4}$. If $\alpha_i = \beta_i$, $1 \leqslant i \leqslant 4$ then both u and v are on the cycle of $H_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ and there exists a path along that cycle between the two vertices of length at most 8 and therefore $d'_S(u, v) \leqslant 8$.

At this point, we could proceed with an analysis similar to that in our previous proof using a path consisting of a full cycle traversal interspersed with single dimensional paths. However, since our cycle includes two highway edges in each dimension, we can improve on this obvious bound.

Note that all vertices of $H_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ are symmetric under a cyclic permutation of dimensions and possibly a reflection, and we may thus, without loss of generality, assume that u is the vertex of $H_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ that has all coordinates even. The vertex u is therefore on a highway in dimension 1 and incident on a non-highway edge to a vertex on a highway in dimension 4 as shown in Fig. 15.

We consider two paths from u to v in S. Let u' be the vertex in $H_{\beta_1,\beta_2,\beta_3,\beta_4}$ with dimension 4 coordinate odd and the other coordinates even. One path from u to v consists of a path from u to u' corresponding to a walk along the cycle (counterclockwise in Fig. 15) to u', interspersed with single-dimensional paths, followed by the shortest path along the cycle from u' to v in $H_{\beta_1,\beta_2,\beta_3,\beta_4}$. Similarly, let u'' be the vertex in $H_{\beta_1,\beta_2,\beta_3,\beta_4}$ with dimension 1 and 2 coordinates odd and the other coordinates even. Another path from u to v consists of a path from u to u'' corresponding to a walk along the cycle (clockwise in Fig. 15) to u'', interspersed with single-dimensional paths, followed by the shortest path along the cycle from u'' to v in $H_{\beta_1,\beta_2,\beta_3,\beta_4}$.

Consider a highway section in dimension i contained in either of these paths, not including the highway edge (if any) of the conceptual cycle that we are traversing. This section goes between two vertices whose ith coordinates have the same parity and whose other coordinates are equal. This section is of length $2|\alpha_i - \beta_i|$, which is at most $|u_i - v_i| + 1$. Thus, in the path from u to u', we have at most $d_G(u, v) + 9$ edges with

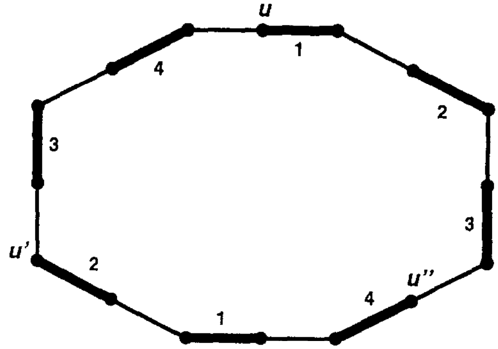


Fig. 15. Redrawn cycle in $H_{0,0,0,0}$.

four of the extra edges coming from the four highway sections and the other five coming from the traversal of our conceptual cycle. Similarly, the number of edges in the path from u to u'' is at most $d_G(u, v) + 10$. Regardless of the location of v on the cycle, the shorter of the two resulting paths from u to v is of length at most $d_G(u, v) + 15$. In fact, the length is at most $d_G(u, v) + 14$ unless v has first coordinate odd and all other coordinates even (the vertex clockwise from u in the figure). In this latter case, we will use the path through u'' with the modification that we replace the highway section and cycle edge in dimension 1 with a highway section in dimension 1 consisting of exactly $|u_1 - v_1|$ edges. This gives a path length of at most $d_G(u, v) + 14$. □

Corollary 5.2a. *For any finite 4-dimensional grid $G' = G_{n_1,n_2,n_3,n_4}$ with n_i even for all $1 \leqslant i \leqslant 4$, there exists a $(14 + x)$ -spanner of G' with $\Delta_S = 3$ and $\delta_S = 3$.*

Proof. Consider the induced subspanner of the above spanner S. The delay bound follows directly from the proof of Theorem 5.2. □

6. Summary

We proved an $\Omega(d)$ lower bound for the delay of maximum degree 3 or 4 spanners of finite or infinite d-dimensional grids. We then constructed spanners with maximum degrees 3 and 4 of both infinite and finite d-dimensional grids with delay $O(d)$. In particular, we constructed four different spanners with maximum degree 4 for infinite d-dimensional grids with delays $2d - 4$, $2\lceil d/2 \rceil + 2\lceil (d-2)/4 \rceil + 2$, $2\lceil (d-6)/8 \rceil + 4\lceil (d+1)/4 \rceil + 6$, and $\lceil (\lceil d/2 \rceil + l)/(l+1) \rceil + 2\lceil d/2 \rceil + 2l + 2$. We showed how to modify these constructions to produce spanners of finite d-dimensional grids with essentially the same delay. We constructed a $(5d + 4 + x)$ -spanner with maximum degree 3 for the infinite d-dimensional grid and showed that an induced subgraph of this spanner is a $(5d + 4 + x)$ -spanner with maximum degree 3 for finite d-dimensional grids where all dimensions are even. These general constructions may be improved upon for specific dimensions. In particular, we constructed a $(6 + x)$ -spanner of the infinite 3-dimensional grid and showed how it could be modified to obtain a $(6 + x)$ -spanner of a finite 3-dimensional grid where all dimensions are even or where all dimensions are odd and a $(8 + x)$ -spanner of a finite 3-dimensional grid otherwise. Finally, we constructed a $(14 + x)$ -spanner for the infinite 4-dimensional grids and showed that this yields a $(14 + x)$ -spanner of a finite 4-dimensional grids with all dimensions are even.

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