Upper bounds on the paired-domination number

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Received 8 October 2007; accepted 8 October 2007

Abstract

A set \( S \) of vertices in a graph \( G \) is a \textit{paired-dominating set} of \( G \) if every vertex of \( G \) is adjacent to some vertex in \( S \) and the subgraph induced by \( S \) contains a perfect matching. The minimum cardinality of a paired-dominating set of \( G \) is the \textit{paired-domination number} of \( G \), denoted by \( \gamma_{pr}(G) \). In this work, we present several upper bounds on the paired-domination number in terms of the maximum degree, minimum degree, girth and order.

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Keywords: Paired-domination number; Maximum degree; Minimum degree; Girth

1. Introduction

Domination and its variations in graphs are now well studied. The literature on this subject has been surveyed in detail in the two books by Haynes, Hedetniemi, and Slater [1,2]. In this work we investigate some upper bounds on paired-domination number in terms of the maximum degree, minimum degree, girth and order.

A \textit{matching} in a graph \( G \) is a set of independent edges in \( G \). A vertex \( v \) of \( G \) is \textit{saturated} by a matching \( M \) if it is the endpoint of an edge of \( M \). A \textit{perfect matching} \( M \) in \( G \) is a matching such that every vertex of \( G \) is saturated by \( M \). For a bipartite graph with partite sets \( X \) and \( Y \), a matching \( M \) is a \textit{complete matching} of \( X \) into \( Y \) if every vertex of \( X \) is saturated by \( M \).

Paired-domination was introduced by Haynes and Slater [3]. A \textit{paired-dominating set} of a graph \( G \) is a set \( S \) of vertices of \( G \) such that every vertex is adjacent to some vertex in \( S \) and the subgraph induced by \( S \) contains a perfect matching. The \textit{paired-domination number} of \( G \), denoted by \( \gamma_{pr}(G) \), is the minimum cardinality of a paired-dominating set of \( G \). A set \( S \) of vertices of \( G \) is called an \textit{independent set} if no two vertices in \( S \) are adjacent. The maximum cardinality of a maximal independent set is called the \textit{independent number}, denoted by \( \alpha(G) \).

For notation and graph theory terminology we in general follow [1]. Specifically, let \( G = (V, E) \) be a simple graph of order \( n \). The degree, neighborhood and closed neighborhood of a vertex \( v \) in the graph \( G \) are denoted by \( d(v), N(v) \) and \( N[v] = N(v) \cup \{v\} \), respectively. For a subset \( S \) of \( V \), \( N(S) = \bigcup_{v \in S} N(v) \) and \( N[S] = N(S) \cup S \). The graph induced by \( S \subseteq V \) is denoted by \( G[S] \). The minimum degree and maximum degree of the graph \( G \) are denoted by

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0893-9659/$ - see front matter © 2008 Published by Elsevier Ltd
doi:10.1016/j.aml.2007.10.029
\[ \delta(G) \text{ and } \Delta(G), \] respectively. A cycle on \( n \) vertices is denoted by \( C_n \) and a path on \( n \) vertices by \( P_n \). The girth \( g(G) \) of \( G \) is the length of a shortest cycle in \( G \). A vertex of degree 1 is called a leaf. A vertex \( v \) of \( G \) is called a support if it is adjacent to a leaf. If \( T \) is a tree, \( L(T) \) and \( S(T) \) denote the set of leaves and supports, respectively. A star is the tree \( K_{1,n-1} \) of order \( n \geq 2 \). A subdivided star is a star where each edge is subdivided exactly once.

2. Known results

The decision problem of determining the paired-domination number of a graph is known to be NP-complete. Hence it is of interest to determine upper bounds on the paired-domination number of a graph. Haynes and Slater obtained the following upper bounds on the paired-domination number of a connected graph in terms of the order of the graph.

**Theorem 1** (Haynes and Slater [3]). If \( G \) is a connected graph of order \( n \geq 3 \), then \( \gamma_{pr}(G) \leq n - 1 \) with equality if and only if \( G \) is \( C_3, C_5 \) or a subdivided star.

If we restrict the minimum degree to being at least 2 and the order to being at least 6, then the upper bound in **Theorem 1** can be improved from one less than its order to two-thirds of its order.

**Theorem 2** (Haynes and Slater [3]). If \( G \) is a connected graph of order \( n \geq 6 \) with \( \delta(G) \geq 2 \), then \( \gamma_{pr}(G) \leq \frac{2n}{3} \).

Henning [4] showed that there are exactly ten graphs that achieve equality in the bound of **Theorem 2**. Furthermore, he gave the following result.

**Theorem 3** (Henning [4]). If \( G \) is a connected graph of order \( n \geq 10 \) with \( \delta(G) \geq 2 \), then \( \gamma_{pr}(G) \leq \frac{2(n-1)}{3} \).

Favaron and Henning in [5] gave the following results.

**Theorem 4** (Favaron and Henning [5]). If \( G \) is claw-free and diamond-free, then \( \gamma_{pr}(G) \leq \frac{2n}{5} \).

**Theorem 5** (Favaron and Henning [5]). If \( G \) is claw-free, then \( \gamma_{pr}(G) \leq \frac{n}{2} \).

3. Main results

First, we present an upper bound on the paired-domination number of a graph with girth at least 6 in terms of its minimum degree.

**Lemma 1** (Hall [6]). Let \( G \) be a bipartite graph with partite sets \( X \) and \( Y \). Then \( G \) has a complete matching of \( X \) into \( Y \) if and only if \( |N(S)| \geq |S| \) for all subsets \( S \) of \( X \).

**Theorem 6**. If \( G \) is a connected graph of order \( n \) with minimum degree \( \delta \geq 2 \) and girth \( g(G) \geq 6 \), then \( \gamma_{pr}(G) \leq \frac{2}{3}(n - \frac{1}{2}(\delta - 2) - \frac{1}{2} \delta) \).

**Proof**. If \( \delta = 2 \), then the result follows from **Theorem 2**. Hence we may assume that \( \delta \geq 3 \). Since \( g(G) \geq 6 \), the open neighborhood of every vertex is an independent set of vertices. So \( \alpha(G) \geq \Delta \). Let \( S \) be an independent set of \( \delta - 2 \) vertices. Since \( g(G) \geq 6 \), it follows that \( G \) is triangle-free and \( C_4 \)-free. So every vertex in \( N(S) \) is adjacent to at least one vertex of \( V(G) - N[S] \). Let \( H = G[V(G) - N[S)] \).

For each vertex \( v \in V(H) \), let \( N_v = N(v) \cap N[S] \). Since \( G \) is \( C_4 \)-free, it follows that \(|N_v| \leq \delta - 2 \) for any \( v \in H \). So \( \delta(H) \geq 2 \).

Let \( S^* \) be the set of vertices in \( N(S) \) such that each is adjacent to two or more vertices of \( S \). Since \( S \) is an independent set and \( G \) is \( C_4 \)-free, every pair of vertices in \( S \) has at most one common neighbor. Thus, \(|S^*| \leq \frac{1}{2}(\delta - 2) \). It follows that

\[
|N[S]| = |S| + |N(S)| \\
\geq |S| + \left( \sum_{x \in S} d_G(x) \right) - |S^*| \\
\geq (\delta - 2) + \delta(\delta - 2) - \left( \frac{1}{2}(\delta - 2) \right) \\
= \frac{(\delta - 2)(\delta + 5)}{2}.
\]
Now, we consider the bipartite graph with partite sets $S$ and $N(S)$. It is obvious that $|N(S')| \geq |S'|$ for any subset $S' \subseteq S$. By Lemma 1, it follows that there exists a complete matching $M$ of $S$ into $N(S)$. Without loss of generality, we can assume that for each $x \in S$, $x'$ is a neighbor of $x$ in $G$ such that $xx' \in M$. Let $S'' = \bigcup \{x'\}$ where the union is taken over all vertices $x \in S$.

Suppose that $H$ contain components $H_1, \ldots, H_t$, where $t \geq 1$. Let $|V(H_i)| = n_i$ for $i = 1, \ldots, t$. It is obvious that $n_1 + \cdots + n_t = |V(H)|$. Since $\delta(H) \geq 2$, it follows that $H_i$ contains a cycle for $i = 1, \ldots, t$. Since $g(G) \geq 6$, $|V(H_i)| \geq 6$. By Theorem 2, $\gamma_{pr}(H_i) \leq \frac{2n_i}{3}$ for $i = 1, \ldots, t$. So, $\gamma_{pr}(H) = \gamma_{pr}(H_1) + \cdots + \gamma_{pr}(H_t) \leq \frac{2n_1}{3} + \cdots + \frac{2n_t}{3} = \frac{2|V(H)|}{3}.

Since every $\gamma_{pr}(H)$-set can be extended to a paired-dominating set of $G$ by adding to it the vertices in the set $S \cup S''$, it follows that

$$
\gamma_{pr}(G) \leq |S \cup S''| + \gamma_{pr}(H)
= 2|S| + \gamma_{pr}(H)
\leq 2|S| + \frac{2|V(H)|}{3}
= 2|S| + \frac{2(n - |N(S)|)}{3}
= 2(\delta - 2) + \frac{2n}{3} - \frac{2}{3}|N(S)|
\leq 2(\delta - 2) + \frac{2n}{3} - \frac{2}{3}(\delta - 2)(\delta + 5)
= \frac{2}{3} \left(n - \frac{\delta - 1}{2}(\delta - 2)\right).
$$

\begin{remark}
The condition $g(G) \geq 6$ is necessary. For example, cycle $C_5$ with $\delta(C_5) = 2$ and $g(C_5) = 5$ satisfies $\gamma_{pr}(C_5) > \frac{2}{3} \left(n - \frac{\delta - 1}{2}(\delta - 2)\right).
\end{remark}

By Henning’s result on characterizing the graphs with the bound of Theorem 2, it is easy to find that each such graph has minimum degree $2$. Now, we consider graph with $\delta \geq 3$ and $g(G) \geq 6$.

\begin{theorem}
If $G$ is a connected graph of order $n$ with minimum degree $\delta \geq 3$ and girth $g(G) \geq 6$, then $\gamma_{pr}(G) \leq \frac{2}{3} \left(n + 1 - \delta\right).
\end{theorem}

\begin{proof}
Let $v$ be a vertex of maximum degree $\Delta(G)$, and let $G_v = G[V - N[v]]$. Since $g(G) \geq 6$, $N(v)$ is an independent set. Furthermore, $G_v \neq \emptyset$. Suppose $G_v$ has $t$ components $G_1, G_2, \ldots, G_t$. Then we have the following claims.

\begin{claim}
$\delta(G_i) \geq 2$ for $i = 1, 2, \ldots, t$.
\end{claim}

Otherwise, suppose that there exists a component $G_i$ such that $\delta(G_i) < 2$. If $\delta(G_i) = 0$, then $G_i$ is an isolated vertex. Assume $G_i = \{u\}$. Since $\delta(G) \geq 3$, it follows that $u$ is adjacent to at least three vertices of $N(v)$ and a 4-cycle is formed, which is a contradiction.

If $\delta(G_i) = 1$, then $G_i$ has a leaf $u$. Since $\delta(G) \geq 3$, it follows that $u$ is adjacent to at least two vertices of $N(v)$ and a 4-cycle is formed, which is a contradiction.

\begin{claim}
No two adjacent vertices of $G_i$ both have degree 2 for $i = 1, 2, \ldots, t$.
\end{claim}

Otherwise, suppose that $G_i$ contain two adjacent vertices $u$ and $w$ both having degree 2 in $G_i$. Since $\delta(G) \geq 3$, it follows that each of $u$ and $w$ is adjacent to at least one vertex of $N(v)$. So there is a cycle of length at most 5 containing both $u$ and $w$, which is a contradiction.

\begin{claim}
$|V(G_i)| \geq 10$ for $i = 1, 2, \ldots, t$.
\end{claim}

By Claim 1, $G_i$ contains a cycle $C_i$. Suppose that $C_i = v_1v_2 \cdots v_nv_1$, where $n_i = |V(C_i)|$. Since $g(G) \geq 6$, it follows that $|V(G_i)| \geq 6$ for $i = 1, 2, \ldots, t$.

Suppose that $6 \leq |V(G_i)| \leq 9$. Let $S_i = V(G_i) - V(C_i)$. By Claim 2, $G_i \neq C_i$. That is $S_i \neq \emptyset$ and $|V(G_i)| \geq 7$. Suppose that $|S_i| = 1$. Without loss of generality, we can assume that $S_i = \{u\}$ and $uv \in E(G)$. Since $g(G) \geq 6$,
it follows that $uv_2, uv_3 \notin E(G)$. Then $v_2$ and $v_3$ are two adjacent vertices of $G_i$ both having degree 2, which is a contradiction.

Suppose that $|S_i| = 2$. Assume that $S_i = \{u, v\}$. If $S$ is an independent set, then $u$ is adjacent to at least two vertices of $C_i$, say $v_j$ and $v_k$. Replacing the longer $v_j - v_k$ path in $C_i$ by the path $v_juv_k$ produces a cycle of length less than 6, which is a contradiction. Hence, $uv \in E(G)$. Since $\delta(G_i) \geq 2$, it follows that $u$ is adjacent to exactly one vertex of $C_i$, say $v_1$. Since $g(G) \geq 6$, it follows that $uv_2, uv_3, uv_2, uv_3 \notin E(G)$. Then $v_2$ and $v_3$ are two adjacent vertices of $G_i$ both having degree 2, which is a contradiction.

Suppose that $|S_i| = 3$. Then $|V(G_i)| = 9$ and $|V(C_i)| = 6$. Assume that $S_i = \{u, v, w\}$. Similarly, each vertex of $S_i$ is adjacent to at most one vertex of $C_i$. Hence, $G[S_i]$ is path $P_3$. Say $uv, vw, uv_1 \in E(G)$. Since $g(G) \geq 6$, it follows that $vuv_1, uv_2, uv_3, vv_5, vv_6 \notin E(G_i)$. By Claim 2, $vv_4 \in E(G)$. So $d_{G_i}(w) = 1$, which is a contradiction.

By Theorem 3, it follows that $\gamma_{pr}(G_i) \leq \frac{2}{3}(|G_i| - 1)$ for $i = 1, \ldots, t$. So $\gamma_{pr}(G_v) \leq \frac{2}{3}(|G_1| - 1) + \cdots + \frac{2}{3}(|G_t| - 1) = \frac{2}{3}(\mid G_v\mid - t).

Adding the vertex $v$ along with a neighbor of $v$ to a $\gamma_{pr}(G_v)$-set produces a paired-dominating set of $G$. So

$$\gamma_{pr}(G) \leq 2 + \frac{2}{3}(\mid G_v\mid - t)$$

$$= 2 + \frac{2}{3}(n - \Delta - 1 - t)$$

$$= \frac{2}{3}(n - \Delta + 2 - t)$$

$$\leq \frac{2}{3}(n + 1 - \Delta). \quad \blacksquare$$

**Remark 2.** The condition $g(G) \geq 6$ is necessary. For example, the Petersen graph $G$ with $\delta(G) = 3$ and $g(G) = 5$ satisfies $\gamma_{pr}(G) > \frac{2}{3}(n + 1 - \Delta)$.

Now we establish an upper bound on the paired-domination number of a graph in terms of its order and girth.

**Lemma 2.** For $n \geq 3$, $\gamma_{pr}(C_n) = 2\lceil \frac{n}{4} \rceil$.

**Theorem 8.** If $G$ is a connected graph of order $n$ with minimum degree $\delta \geq 3$, then $\gamma_{pr}(G) \leq \frac{2n}{3} - \frac{g(G)}{6} + \frac{5}{6}$.

**Proof.** If $g(G) = 3, 4, 5$ and $n \geq 6$, by Theorem 2, the result holds. If $g(G) = 3$ and $n = 4, 5$, it is obvious that the result holds. It is easy to prove that there is no graph with $\delta \geq 3$, $g(G) = 4$ and $n = 5$. If $g(G) = 6$, then by Theorem 7, $\gamma_{pr}(G) \leq \frac{2}{3}(n + 1 - \Delta) \leq \frac{2n}{3} - \frac{g(G)}{6} + \frac{5}{6}$. So we can assume that $g(G) \geq 7$. Let $C$ be a $g$-cycle of $G$, and let $H = G[V - V(C)]$. Suppose that $H$ has $t$ components $H_1, H_2, \ldots, H_t$. Since $g \geq 7$, it follows that every vertex of $H$ is adjacent to at most one vertex on the cycle $C$. Since $\delta \geq 3$, it follows that $\delta(H) \geq 2$. Hence, each component $H_i$ of $H$ contains a cycle $C_i$ and $|V(H_i)| \geq g(G)$ for $i = 1, 2, \ldots, t$. Suppose that $H_i$ contain two adjacent vertices $u$ and $w$ both having degree 2 in $H_i$. Without loss of generality, assume that $uv_i, vw_j \in E(G)$, where $v_i, v_j \in V(C)$.

Replacing the longer $v_i - v_j$ path in $C$ by the path $v_iuv_j$ produces a cycle of length less than $\frac{g(G)}{2} + 3$. Then $g(G) \leq \frac{g(G)}{2} + 3$. So $g(G) \leq 6$, which is a contradiction. Hence no two adjacent vertices of $H_i$ both have degree 2. Obviously, $H_i$ is not a cycle. Using a similar approach to the proof of Claim 3 of Theorem 7, it is easy to prove that $|V(H_i)| \geq 10$.

By Theorem 3, $\gamma_{pr}(H_i) \leq \frac{2(|V(H_i)| - 1)}{3}$ for $i = 1, 2, \ldots, t$. So $\gamma_{pr}(H) = \gamma_{pr}(H_1) + \cdots + \gamma_{pr}(H_t) \leq \frac{2(|V(H)| - 1)}{3} + \cdots + \frac{2(|V(H)| - 1)}{3} = \frac{2(|V(H)| - t)}{3}.$

Let $S$ be a minimum paired-dominating set of $C$. Then $\gamma_{pr}(G) \leq |S| + \frac{2(|V(H)| - t)}{3} = 2\lceil \frac{g(G)}{4} \rceil + \frac{2(|V(H)| - t)}{3} = 2\lceil \frac{g(G)}{4} \rceil + \frac{2(n - g(G) - t)}{3} \leq \frac{2n}{3} - \frac{g(G)}{6} + \frac{5}{6}. \quad \blacksquare$

**Acknowledgements**

This work was partially supported by the CERG Research Grant Council of Hong Kong and the Faculty Research Grant of Hong Kong Baptist University. It was supported by a Doctoral Research Grant of North China Electric Power University (200722026).
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