Journal of Combinatorial Theory, Series A 119 (2012) 1194–1199

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Random runners are very lonely

Sebastian Czerwiński

Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, 65-546 Zielona Góra, Poland

ARTICLE INFO

Article history:
Received 8 March 2011
Available online 28 February 2012

Keywords:
Lonely runner problem
Diophantine approximation
Integer distance graphs

ABSTRACT

Suppose that k runners having different constant speeds run laps on a circular track of unit length. The Lonely Runner Conjecture states that, sooner or later, any given runner will be at distance at least $1/k$ from all the other runners. We prove that, with probability tending to one, a much stronger statement holds for random sets in which the bound $1/k$ is replaced by $1/2 - \varepsilon$. The proof uses Fourier analytic methods. We also point out some consequences of our result for colouring of random integer distance graphs.

1. Introduction

Suppose that k runners run laps on a unit-length circular track. They all start together from the same point and run in the same direction with pairwise different constant speeds $d_1, d_2, \dots, d_k$. At a given time t, a runner is said to be *lonely* if no other runner is within a distance of $1/k$, both in front and rear. The Lonely Runner Conjecture states that for every runner there is a time at which he is lonely. For instance if $k = 2$, one can imagine easily that at some time or other, the two runners will find themselves on antipodal points of the circle, both becoming lonely at that moment.

To give a precise statement, let $\mathbb{T} = [0, 1)$ denote the *circle* (the one-dimensional torus). For a real number x, let $\{x\}$ be the fractional part of x (the position of x on the circle), and let $\|x\|$ denote the distance of x to the nearest integer (the circular distance from $\{x\}$ to zero). Notice that $\|x - y\|$ is just the length of the shortest circular arc determined by the points $\{x\}$ and $\{y\}$ on the circle. It is not difficult to see that the following statement is equivalent to the Lonely Runner Conjecture.

E-mail address: s.czerwinski@wmie.uz.zgora.pl.

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2012.02.002

Conjecture 1. *For every integer $k \geqslant 1$ and for every set of positive integers $\{d_1, d_2, \dots, d_k\}$ there exists a real number t such that*

$$\|td_i\| \geqslant \frac{1}{k+1}$$

for all $i = 1, 2, \dots, k$.

The above bound is sharp as is seen for the sets $\{1, 2, \dots, k\}$. The paper of Goddyn and Wong [11] contains items of interesting exemplars of such extremal sets. The problem was posed for the first time by Wills [16] in connection to Diophantine approximation. Cusick [8] raised the same question independently, as a view obstruction problem in discrete geometry (cf. [5]). Together with Pomerance [9], he confirmed the validity of the conjecture for $k \leqslant 4$. Bienia et al. [3] gave a simpler proof for $k = 4$ and found interesting application to flows in graphs and matroids. Next the conjecture was proved for $k = 5$ by Bohman et al. [4]. A simpler proof for that case was provided by Renault [13]. Recently the case $k = 6$ was established by Barajas and Serra [2], using a new promising idea.

Let $D = \{d_1, d_2, \dots, d_k\}$ be a set of k positive integers. Consider the quantity

$$\kappa(D) = \sup_{x \in \mathbb{T}} \min_{d_i \in D} \|xd_i\|$$

and the related function $\kappa(k) = \inf \kappa(D)$, where the infimum is taken over all k-element sets of positive integers. So, the Lonely Runner Conjecture states that $\kappa(k) \geqslant \frac{1}{k+1}$. The trivial bound is $\kappa(k) \geqslant \frac{1}{2k}$, as the sets $\{x \in \mathbb{T}\colon \|xd_i\| < \frac{1}{2k}\}$ simply cannot cover the whole circle (since each of them is a union of d_i open arcs of length $\frac{1}{kd_i}$ each). Surprisingly, nothing much better was proved so far. Currently the best general bound is

$$\kappa(k) \geqslant \frac{1}{2k - 1 + \frac{1}{2k-3}}$$

for every $k \geqslant 5$ [6]. A slightly improved inequality $\kappa(k) \geqslant \frac{1}{2k-3}$ holds when $k \geqslant 4$ and $2k - 3$ is prime [7]. Using the probabilistic argument we proved in [10] that every set D contains an element d such that

$$\kappa\big(D \setminus \{d\}\big) \geqslant \frac{1}{k}.$$

In this paper we prove another general result supporting the Lonely Runner Conjecture.

Theorem 1. *Let k be a fixed positive integer and let $\varepsilon > 0$ be fixed real number. Let $D \subseteq \{1, 2, \dots, n\}$ be a k-element subset chosen uniformly at random. Then the probability that $\kappa(D) \geqslant \frac{1}{2} - \varepsilon$ tends to 1 with $n \to \infty$.*

The proof uses elementary Fourier analytic technique for subsets of $\mathbb{Z}_p$. We give it in the next section. In the last section we point to a striking consequence of our result for colouring of integer distance graphs.

2. Proof of the main result

Let k be a fixed positive integer and let $p \geqslant k$ be a prime number. For $a \in \mathbb{Z}_p$, let $\|a\|_p = \min\{a, p - a\}$ be the circular distance from a to zero in $\mathbb{Z}_p$. We will need the following notion introduced by [14]. Let L be a fixed positive integer. A set $D = \{d_1, \dots, d_k\} \subseteq \mathbb{Z}_p$ is called *L-independent* in $\mathbb{Z}_p$ if equation

$$d_1x_1 + d_2x_2 + \dots + d_kx_k = 0$$

has no solutions satisfying

$$0 < \sum_{i=1}^{k} \|x_i\|_p \leqslant L.$$

We will show that for appropriately chosen L, any L-independent set can be pushed away arbitrarily far from zero. Then we will demonstrate that for such L, almost every set in $\mathbb{Z}_p$ is L-independent.

Let $f : \mathbb{Z}_p \to \mathbb{C}$ be any function and let $\hat{f} : \mathbb{Z}_p \to \mathbb{C}$ denote its Fourier transform, that is

$$\hat{f}(r) = \sum_{x \in \mathbb{Z}_p} f(x)\omega^{rx},$$

where $\omega = e^{\frac{2\pi}{p}i}$. For a set $A \subseteq \mathbb{Z}_p$, by $A(x)$ we denote its characteristic function. We will make use of the following basic properties of the Fourier transform:

(F1) $|\hat{f}(r)| = |\hat{f}(-r)|$ for every $r \in \mathbb{Z}_p$.
(F2) $f(x) = \frac{1}{p}\sum_{r \in \mathbb{Z}_p} \hat{f}(r)\omega^{-rx}$ for every $x \in \mathbb{Z}_p$.
(F3) $\hat{A}(0) = |A|$ for every subset of $\mathbb{Z}_p$.

In the lemma below we give a bound for the Fourier coefficient $\hat{A}(r)$ for the sets of the form

$$A = \{s, s+1, \ldots, l\}, \tag{$*$}$$

where l and s are elements of $\mathbb{Z}_p$, such that $s < l$. This bound does not depend on l and s. The following lemma can be easily proved, as for instance in [12, p. 39]. We proved this for the reader convenience.

Lemma 1. *If* $0 < r < \frac{p}{2}$, *then*

$$\left|\hat{A}(r)\right| \leqslant \frac{p}{2r}.$$

Proof. By simple calculations we have

$$\left|\hat{A}(r)\right| = \left|\sum_{x=s}^{l} \omega^{rx}\right| = \left|\frac{\omega^{r(l+1)} - \omega^{rs}}{\omega^r - 1}\right|$$
$$= \left|\frac{\omega^{\frac{r(l+s+1)}{2}}}{\omega^{\frac{r}{2}}} \cdot \frac{\omega^{\frac{r(l+1-s)}{2}} - \omega^{\frac{-r(l+1-s)}{2}}}{\omega^{\frac{r}{2}} - \omega^{\frac{-r}{2}}}\right| = \left|\frac{\sin(\frac{\pi r}{p})}{\sin(\frac{\pi r}{p})}\right|.$$

Using inequality $\sin(x) \geqslant \frac{2x}{\pi}$ for $x < \frac{\pi}{2}$, we get

$$\left|\hat{A}(r)\right| \leqslant \frac{p}{2r}. \qquad \square$$

Now, we state and prove the aforementioned property of L-independent sets.

Theorem 2. *Let* $0 < \varepsilon < \frac{1}{2}$ *be a fixed real number. Let* D *be a* k*-element,* L*-independent set in* $\mathbb{Z}_p$*, where*

$$L > \sqrt{\frac{k^3 3^{k-1}}{2^{k+1}\varepsilon^{2k}}}.$$

Then

$$\kappa(D) \geqslant 1/2 - \varepsilon.$$

Proof. Let

$$C = \left\{ x \in \mathbb{Z}_p \colon \left(\frac{1}{4} - \frac{\varepsilon}{2}\right)p < x < \left(\frac{1}{4} + \frac{\varepsilon}{2}\right)p \right\}$$

and let $C(x)$ be the characteristic function of the set C. Define convolution of two functions f and g by

$$(f * h)(x) = \sum_{y \in \mathbb{Z}_p} f(y) \cdot g(x - y).$$

Denote by $B(x) = (C * C)(x)$ convolution of function C with itself. It is easy to see that $\hat{B}(r) = \hat{C}(r) \cdot \hat{C}(r)$ for all $r \in \mathbb{Z}_p$.

So, if we find $t \in \mathbb{Z}_p$ such that $tD \subseteq \operatorname{supp} B$, where $\operatorname{supp} B = \{x \in \mathbb{Z}_p \colon\ B(x) \neq 0\}$, then at the same time we push the set D away into the small arc $(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)$ on the torus $\mathbb{T}$.

Then the expression

$$I = \sum_{t \in \mathbb{Z}_p} B(td_1) B(td_2) \cdots B(td_k)$$

counts those numbers t which push the set D away to a distance $\frac{1}{2} - \varepsilon$ from zero. We will show that $I \neq 0$. From properties of the Fourier transform it results that

$$I = \sum_{t \in \mathbb{Z}_p} \left(\frac{1}{p} \sum_{r_1 \in \mathbb{Z}_p} \hat{B}(r_1) \omega^{-td_1 r_1} \right) \cdots \left(\frac{1}{p} \sum_{r_k \in \mathbb{Z}_p} \hat{B}(r_k) \omega^{-td_k r_k} \right).$$

Denoting $\vec{r} = (r_1, r_2, \ldots, r_k)$, we get

$$p^k I = \sum_{\vec{r} \in \mathbb{Z}_p^k} \hat{B}(r_1) \cdots \hat{B}(r_k) \sum_{t \in \mathbb{Z}_p} \omega^{-t(d_1 r_1 + \cdots + d_k t_k)}.$$

The expression $\sum_t \omega^{-t(d_1 r_1 + \cdots + d_k t_k)}$ is equal to p when

$$d_1 r_1 + \cdots + d_k r_k \equiv 0 \pmod{p}, \tag{$**$}$$

and is equal to zero in the contrary case. As a consequence we may write

$$p^{k-1} I = \sum_{\vec{r} \in \mathbb{Z}_p^k} \hat{B}(r_1) \cdots \hat{B}(r_k) R(\vec{r}),$$

where $R(\vec{r}) = 1$ for $r_1, \ldots, r_k$ satisfying Eq. $(**)$, and $R(\vec{r}) = 0$ in the opposite situation. Since D is L-independent, the identity $R(\vec{r}) = 1$ holds only for those $r_1, \ldots, r_k$ satisfying condition $\sum_{i=1}^k \|r_i\|_p > L$, or $r_1 = r_2 = \cdots = r_k = 0$. Hence,

$$p^{k-1} I - |C|^{2k} = \sum_{\vec{r} \in \mathbb{Z}_p^k,\ \sum \|r_i\|_p > L} \hat{B}(r_1) \cdots \hat{B}(\vec{r}),$$

as for $r_i = 0$ the Fourier coefficient $\hat{B}(r_i)$ is equal to square of the size of C. So, by showing that

$$|C|^{2k} > \sum_{\sum \|r_i\|_p > L} \left|\hat{B}(r_1)\right| \cdots \left|\hat{B}(r_k)\right| R(\vec{r}),$$

we will confirm that $I \neq 0$.

The property of L-independence of the set D implies that in any nontrivial solution of $(**)$ there is some r_i satisfying $\|r_i\|_p > \frac{L}{k}$. The estimates for those r_i

$$\left|\hat{B}(r_i)\right| = \left|\hat{C}(r_i)\right|^2 \leqslant \left(\frac{p}{2r_i}\right)^2 \leqslant \left(\frac{kp}{2L}\right)^2$$

result from Lemma 1.

Denote by $\overrightarrow{r_j} = (r_1, \dots, r_{j-1}, r_{j+1}, \dots, r_k)$, the vector $\overrightarrow{r}$ with jth coordinate missing. Substituting this to the previous sum we obtain

$$\sum_{\sum \|r_i\|_p > L} \left|\hat{B}(r_1)\right| \cdots \left|\hat{B}(r_k)\right| R(r_1, \dots, r_k)$$

$$\leqslant \left(\frac{kp}{2L}\right)^2 \sum_{j=1}^{k} \sum_{\overrightarrow{r_j} \in \mathbb{Z}_p^{k-1}} \left|\hat{B}(r_1)\right| \cdots \left|\hat{B}(r_{j-1})\right| \left|\hat{B}(r_{j+1})\right| \dots \left|\hat{B}(r_k)\right|$$

$$\leqslant k\left(\frac{kp}{2L}\right)^2 \sum_{\overrightarrow{r_k} \in \mathbb{Z}_p^{k-1}} \left|\hat{B}(r_1)\right| \cdots \left|\hat{B}(r_{k-1})\right|.$$

The last sum may be estimated further. Let $S_p = \{0, 1, \dots, \frac{p-1}{2}\}$ and we get

$$\sum_{\overrightarrow{r_k} \in \mathbb{Z}_p^{k-1}} \left|\hat{B}(r_1)\right| \cdots \left|\hat{B}(r_{k-1})\right| \leqslant 2^{k-1} \sum_{\overrightarrow{r_k} \in S_p^{k-1}} \left|\hat{B}(r_1)\right| \cdots \left|\hat{B}(r_{k-1})\right|.$$

Thus, applying Lemma 1 again we get

$$\sum_{\sum \|r_i\|_p > L} \left|\hat{B}(r_1)\right| \cdots \left|\hat{B}(r_k)\right| R(\overrightarrow{r})$$

$$\leqslant k\left(\frac{kp}{2L}\right)^2 \cdot 2^{k-1} \cdot \left(\frac{p^{k-1}}{2^{k-1}}\right)^2 \cdot \left(1 + \sum_{r \in S_p} \frac{1}{r^2}\right)^{k-1}$$

$$\leqslant k\left(\frac{kp}{2L}\right)^2 \cdot 2^{k-1} \cdot \left(\frac{p^{k-1}}{2^{k-1}}\right)^2 \cdot \left(1 + \frac{\pi^2}{2}\right)^{k-1}$$

since $1 + \frac{\pi^2}{2} \leqslant 3$, we obtain

$$\sum_{\sum \|r_i\| p > L} \left|\hat{B}(r_1)\right| \cdots \left|\hat{B}(r_k)\right| R(\overrightarrow{r}) \leqslant \frac{k^3 p^{2k} 3^{k-1}}{2^{k+1} L^2}.$$

So, by the assumption on L we obtain

$$\sum_{\sum \|r_i\|_p > L} \left|\hat{B}(r_1)\right| \cdots \left|\hat{B}(r_k)\right| R(\overrightarrow{r}) < (\varepsilon p)^{2k} \leqslant |C|^{2k}.$$

This completes the proof. □

Proof of Theorem 1. Let L be a number satisfying inequalities

$$\sqrt{\frac{k^3 3^{k-1}}{2^{k+1} \varepsilon^{2k}}} < L < \sqrt[k+1]{p}.$$

Such numbers L exist provided that p is sufficiently large. By Theorem 2, $\kappa(D) \geqslant \frac{1}{2} - \varepsilon$ for every L-independent set D. We show that the second inequality implies that almost every set in $\mathbb{Z}_p^*$ is L-independent. Indeed, the number of sets that are not L-independent is at most

$$(2L+1)^k \binom{p-1}{k-1}.$$

So, the fraction of those sets in $\mathbb{Z}_p^*$ is equal

$$\frac{(2L+1)^k\binom{p-1}{k-1}}{\binom{p-1}{k}} = \frac{(2L+1)^k k}{p-k} < \frac{(2\sqrt[k+1]{p}+1)^k k}{p-k}.$$

The last expression tends to zero with p tending to infinity. This completes the proof, as the ratios of two consecutive primes tend to one. □

3. Integer distance graphs

We conclude the paper with a remark concerning *integer distance graphs*. For a given set D, consider a graph $G(D)$ whose vertices are positive integers, with two vertices a and b joined by an edge if and only if $|a-b| \in D$. Let $\chi(D)$ denote the chromatic number of this graph. It is not hard to see that $\chi(D) \leqslant |D| + 1$.

To see a connection to parameter $\kappa(D)$, put $N = \lceil \kappa(D)^{-1} \rceil$ and split the circle into N intervals $I_i = [(i-1)/N, i/N)$, $i = 1, 2, \dots, N$ (cf. [15]). Let t be a real number such that $\min_{d \in D} \|dt\| = \kappa(D)$. Then define a colouring $c : \mathbb{N} \to \{1, 2, \dots, N\}$ by $c(a) = i$ if and only if $\{ta\} \in I_i$. If $c(a) = c(b)$ then $\{ta\}$ and $\{tb\}$ are in the same interval I_i. Hence $\|ta - tb\| < 1/N \leqslant \kappa(D)$, and therefore $|a-b|$ is not in D. This means that c is a proper colouring of a graph $G(D)$. So, we have a relation

$$\chi(D) \leqslant \left\lceil \frac{1}{\kappa(D)} \right\rceil.$$

Now, by Theorem 1 we get that $\chi(D) \leqslant 3$ for almost every graph $G(D)$.

A different proof of a stronger version of this result has been recently found by Alon [1]. He also extended the theorem for arbitrary Abelian groups, and posed many intriguing questions for general groups.

Acknowledgments

I would like to thank Tomasz Schoen for an inspiring idea of using independent sets, and to Jarek Grytczuk for stimulating discussions and help in preparation of the manuscript. I thank the anonymous referees for valuable suggestions concerning the merit of the paper. I also acknowledge a support from Polish Ministry of Science and Higher Education (MNiSW) (N N201 271335).

References

[1] N. Alon, The chromatic number of random Cayley graphs, European J. Combin., in press.
[2] J. Barajas, O. Serra, The lonely runner with seven runners, Electron. J. Combin. 15 (1) (2008), Paper R48.
[3] W. Bienia, L. Goddyn, P. Gvozdjak, A. Sebö, M. Tarsi, Flows, view obstructions, and the lonely runner, J. Combin. Theory Ser. B 72 (1) (1998) 1–9.
[4] T. Bohman, R. Holzman, D. Kleitman, Six lonely runners. In honor of Aviezri Fraenkel on the occasion of his 70th birthday, Electron. J. Combin. 8 (2) (2001), Research paper 3, 49 pp. (electronic).
[5] P. Brass, W. Moser, J. Pach, Research Problems in Discrete Geometry, Springer, New York, 2005.
[6] Y.G. Chen, View-obstruction problems and a generalization in E^n, Acta Math. Sinica 37 (1994) 551–562.
[7] Y.G. Chen, T.W. Cusick, The view-obstruction problem for n-dimensional cubes, J. Number Theory 74 (1999) 126–133.
[8] T.W. Cusick, View-obstruction problems, Aequationes Math. 9 (1973) 165–170.
[9] T.W. Cusick, C. Pomerance, View-obstruction problems. III, J. Number Theory 19 (2) (1984) 131–139.
[10] S. Czerwiński, J. Grytczuk, Invisible runners in finite fields, Inform. Process. Lett. 108 (2008) 64–67.
[11] L. Goddyn, E. Wong, Tight instances of the lonely runner, Integers 6 (2006), #A38, 14 pp.
[12] H.L. Montgomery, Ten Lectures on the Interface between Analytic Number Theory and Harmonic Analysis, Amer. Math. Soc., 1994.
[13] J. Renault, View-obstruction: a shorter proof for 6 lonely runners, Discrete Math. 287 (1–3) (2004) 93–101.
[14] I. Ruzsa, Arithmetic progressions in sumset, Acta Arith. 60 (2) (1991) 191–202.
[15] I. Ruzsa, Z. Tuza, M. Voigt, Distance graphs with finite chromatic number, J. Combin. Theory Ser. B 85 (2002) 181–187.
[16] J.M. Wills, Zwei Sätze über inhomogene diophantische Approximation von Irrationalzahlen, Monatsch. Math. 71 (1967) 263–269.