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Sums over Positive Integers with Few Prime Factors*

D. G. HAZLEWOOD

Department of Mathematics, Southwest Texas State University, San Marcos, Texas 78666

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The author has presented estimates for sums of multiplicative functions, satisfying certain conditions, extended over positive integers n such that n is less than or equal to x^t, the greatest prime factor of n is at most equal to x, and n is relatively prime to a natural number k. These estimates are uniform in t, x, and k.

1. Introduction

Let $p(n)$ denote the largest prime factor of a positive integer n, let $x \geqslant 1$, $t \geqslant 1$ be real numbers, let $g(n)$ be a multiplicative function, and let k be a natural number. We define $\psi_k(x^t, x; g)$ by the following:

$$\psi_k(x^t, x; g) = \sum_{\substack{n \leqslant x^t \\ p(n) \leqslant x \\ (n,k)=1}} g(n). \tag{1.1}$$

Levin and Fainleib [1] have given a systematic discussion of estimates of such sums subject to g satisfying certain conditions and also subject to $k < (\log x)^D$ for some absolute constant D. In this paper, we make the observation that the restriction on k can be removed in three of their major theorems. In particular, if we define $S_k(x^t; g)$ by

$$S_k(x^t; g) = \sum_{\substack{n \leqslant x^t \\ (n,k)=1}} g(n), \tag{1.2}$$

then, by use of Theorems 1 and 2 below, we shall obtain good asymptotic estimates for $\psi_k(x^t, x; g)$ if we have good asymptotic estimates for $S_k(x^t; g)$

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(the restriction on k arises from the term $(B_1/\log x)\, m_f(x^t)$ in Lemma 3.1.1 of [1]).

As an application of Theorem 2, we shall prove Theorem 4, which gives an asymptotic estimate for $\psi_k(x^t, x)$, the number of positive integers n such that $n \leqslant x^t$, $p(n) \leqslant x$, and $(n, k) = 1$, that is uniform in t, x, and k.

Throughout the discussion, the O-constants will be absolute unless otherwise indicated and $\epsilon(n)$ will denote the function given by

$$\epsilon(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \tag{1.3}$$

2. Statement of the Main Theorems

We begin with some basic definitions after the manner of Levin and Fainleib [1]. In association with the multiplicative function $g(n)$, define the function $\lambda_g(n)$ by the relation

$$g(n) \log n = \sum_{d \mid n} g(d)\, \lambda_g\left(\frac{n}{d}\right).$$

In particular, if $g(n) = 1$ for each n, then $\lambda_g(n)$ is von Mangoldt's function.

The function $\lambda_g(n)$ can be characterized explicitly in the following way using Lemma 1.1.2 of Levin and Fainleib [1]:

$$\lambda_g(n) = \begin{cases} \log p^r \displaystyle\sum_{m=1}^{r} \frac{(-1)^{m-1}}{m} \sum_{k_1+k_2+\cdots+k_m=r} g(p^{k_1})\, g(p^{k_2}) \cdots g(p^{k_m}) & \text{if } n = p^r, \\ 0 \quad \text{if } n \neq p^r, \end{cases} \tag{2.1}$$

where the k_i's, $1 \leqslant i \leqslant m$, are natural numbers. If $g(n)$ is 0 for all integers that are not square-free, then

$$\lambda_g(p^r) = (-1)^{r-1} \log p (g(p))^r. \tag{2.2}$$

Next we define

$$L_g(x, y) = \sum_{\substack{p^r \leqslant x \\ p \leqslant y}} \lambda_g(p^r), \tag{2.3}$$

and

$$\Pi_g(x) = \prod_{p \leqslant x} \left(1 + \sum_{r=1}^{\infty} | g(p^r)|\right). \tag{2.4}$$

We begin with the class of multiplicative functions $g(n)$ satisfying the following two conditions:

$$L_g(x, y) = \tau \log \min(x, y) + B + h(\min(x, y)), \tag{2.5}$$

where τ is a fixed complex number, B is an absolute constant, and $h(x) = O(\rho(x))$ with $\rho(x)$ a nonincreasing, nonnegative function; and

$$\Pi_g(x) = O(\log^A x), \tag{2.6}$$

where A is an absolute constant.

Let $Z(t)$ be the function satisfying the differential–difference equation

$$t\, Z'(t) = -\tau\, Z(t-1) \tag{2.7}$$

with initial condition $Z(t) = 1$ for $0 \leqslant t \leqslant 1$. We assume $Z(t)$ is continuous at $t = 1$. From the results of Section 3 of Chapter 1 of Levin and Fainleib [1], the behavior of $Z(t)$ as $t \to \infty$ is as follows:

$$Z(t) = \begin{cases} P_{-\tau}(t) + O(e^{-\alpha(t)}) & \text{if } -\tau \geqslant 0 \text{ is an integer,} \\ O(e^{-\alpha(t)}) & \text{if } -\tau < 0 \text{ is an integer,} \\ \sim t^{\tau}\left(A_0 + \dfrac{A_1}{t} + \cdots\right) & \text{if } \tau \text{ is not an integer,} \end{cases}$$

where $P_{-\tau}(t)$ is a polynomial in t of degree $-\tau$ with leading coefficient

$$\frac{e^{\gamma\tau}}{\Gamma(-\tau+1)},$$

where γ is Euler's constant and $\Gamma(s)$ is the gamma function,

$$\alpha(t) = t \log t + O(t),$$

and A_0, A_1, A_2,..., are absolute constants. The function $Z(t)$ may also be expressed as the function satisfying the integral–difference equation

$$t\, Z(t) - \int_0^t Z(v)\, dv + \tau \int_0^{t-1} Z(v)\, dv = 0 \tag{2.8}$$

with initial condition $Z(t) = 1$ for $0 \leqslant t \leqslant 1$, which is the form needed in Lemma 3 of Section 3.

We now state our first theorem, which is a special case[1] of Theorem 3.2.1 of Levin and Fainleib [1].

[1] It is free of their restriction on the size of k. The same remark applies to Theorem 2 below.

THEOREM 1. *Let $g(n)$ be a multiplicative function satisfying conditions* (2.5) *and* (2.6). *Let k be a natural number and let $x \geqslant 1$, $t \geqslant 1$ be real numbers. Then*

$$\psi_k(x^t, x; g) = S_k(x^t; g) + \int_0^t Z'(t-v)\, S_k(x^v; g)\, dv + O(t^{A_1} \rho(x; k)(\log x)^{A_2}), \tag{2.9}$$

uniformly in t, x, and k, where A_1 and A_2 are absolute constants, $Z(t)$ satisfies (2.7), *and*

$$\rho(x; k) = (\nu(k) + 1)\, \rho(x), \tag{2.10}$$

where $\nu(k)$ is the number of distinct prime factors of k.

Given a multiplicative function $g(n)$, we also define two functions similar to (2.3) and (2.4):

$$L_g^*(x, y) = \sum_{\substack{p^r \leqslant x \\ p \leqslant y}} \frac{\lambda_g(p^r)}{p^r}, \tag{2.11}$$

and

$$\Pi_g^*(x) = \prod_{p \leqslant x} \left(1 + \sum_{r=1}^{\infty} \frac{|g(p^r)|}{p^r}\right). \tag{2.12}$$

Now we introduce another class of multiplicative functions $g(n)$, namely those satisfying the following conditions:

$$L_g^*(x, y) = \tau \log \min(x, y) + B + h(\min(x, y)), \tag{2.13}$$

where τ is a fixed complex number, B is an absolute constant, and $h(x) = O(\rho(x))$ with $\rho(x)$ a nonincreasing, nonnegative function; and

$$\Pi_g^*(x) = O(\log^A x), \tag{2.14}$$

where A is an absolute constant.

Using Abel summation and Theorem 1, we prove the following theorem which is a special case of Theorem 3.2.2 of Levin and Fainleib [1].

THEOREM 2. *Let $g(n)$ be a multiplicative function satisfying* (2.13) *and* (2.14). *Let k be a natural number and let $x \geqslant 1$, $t \geqslant 1$ be real numbers. Then*

$$\psi_k(x^t, x; g) = S_k(x^t; g) + \int_0^t x^{t-v} Z'(t-v)\, S_k(x^v; g)\, dv + O(x^t t^{A_3} \rho(x; k)(\log x)^{A_4}) \tag{2.15}$$

uniformly in t, x, and k, where A_3 and A_4 are absolute constants, $Z(t)$ satisfies (2.7), *and $\rho(x; k)$ is defined by* (2.10).

In applications, Theorems 1 and 2 give genuine asymptotic estimates only for values of t bounded with respect to x; for example, in Theorem 4, $1 \leqslant t \leqslant (\log x)^{3/5-\delta}$ for some $\delta > 0$. The next theorem gives an upper bound for certain sums extended over positive integers with small prime factors that is valid for $t \leqslant x/(e \log x)$; it extends Theorem 3.3.1 of Levin and Fainleib [1].

THEOREM 3. *Let $g(n)$ be a nonnegative multiplicative function satisfying the following conditions*:

$$L_g(x, y) = \tau \log \min(x, y) + B + h(\min(x, y)) + \rho(\min(x, y)), \tag{2.16}$$

where

$$h(v) = O\left(\frac{1}{\log v}\right) \quad \text{and} \quad \int_2^\infty \frac{|\rho(\min(v, y))|}{v^\delta \log v}\, dv < +\infty \tag{2.17}$$

for every $\delta > 0$; and for every prime p and $s > -1$,

$$\sum_{r=1}^{\infty} \frac{g(p^r)}{p^{rs}} < +\infty. \tag{2.18}$$

Then for every t such that $\tau e < t < \tau x/(e \log x)$ and for every natural number k,

$$\sum_{\substack{n \leqslant x^t \\ p(n) \leqslant x \\ (n,k)=1}} n\, g(n) \leqslant \prod_{\substack{p \leqslant x \\ p \mid k}} \left(1 + \sum_{r=1}^{\infty} g(p^r)\right)^{-1} x^t$$
$$\times \exp\{-t \log t - t \log \log t + \eta(t, x)\}, \tag{2.19}$$

where

$$\eta(t, x) = t\left\{1 + \log \tau - \frac{\log \log t}{\log t} + O\left(\frac{1}{\log t}\right)\right\}$$
$$+ O(\log \log x) + O\left(\frac{(t \log t)^2}{x \log x}\right). \tag{2.20}$$

As an application of Theorem 2 with $g(n) = 1$ for every n, we shall prove the following asymptotic estimate for the function $\psi_k(x^t, x)$, defined in the Introduction.

THEOREM 4. *If k is a natural number and $1 \leqslant t \leqslant (\log x)^{3/5-\delta}$ where $\delta > 0$ is an arbitrary real number, then*

$$\psi_k(x^t, x) = x^t \left\{k^{-1}\varphi(k)\, Z(t) - \sum_{m=0}^{N-1} \frac{(-1)^m}{m!} \frac{Z^{(m+1)}(t)}{(\log x)^{m+1}}\, \xi_0(k, m)\right\}$$
$$+ O_{N,\epsilon}\left(x^t \left\{t^{A_1}\rho(x; k)(\log x)^{A_2} + 2^{\nu(k)}x^{-\epsilon}(1 + Z(t)) + \xi_1(k, N)\frac{t\, Z^{(N)}(t)}{(\log x)^{N+1}}\right\}\right) \tag{2.21}$$

for every even natural number N, where $Z(t)$ satisfies (2.7) with $\tau = 1$;

$$\xi_r(k, m) = \sum_{d|k} \mu_r(d) \int_1^\infty \left(\frac{v}{d} - \left[\frac{v}{d}\right]\right) \frac{(\log v)^m}{v^2}\, dv \tag{2.22}$$

for $0 \leqslant m \leqslant N$ and $r = 0, 1$ where

$$\mu_r(d) = \begin{cases} \mu(d) & \text{for } r = 0, \\ \mu^2(d) & \text{for } r = 1, \end{cases} \tag{2.23}$$

with $\mu(d)$ representing the Möbius function and $[v]$ denoting the greatest integer $\leqslant v$;

$$\rho(x; k) = (\nu(k) + 1)\exp(-\tfrac{1}{2}(\log x)^{3/5-\delta}), \tag{2.24}$$

where *$\nu(k)$ denotes the number of distinct prime factors of k; $\varphi(k)$ is Euler's totient function; and A_1 and A_2 are absolute constants. The estimate is uniform in t, x, and k outside the intervals $(\gamma, \gamma + \epsilon)$, where $\gamma = 1, 2, 3, \ldots, N + 1$ are the discontinuities of $Z^{(N+1)}(t)$ and ϵ is an arbitrary positive real number. In particular, if $t > N + 1$, then ϵ may be chosen such that*

$$1 \leqslant \epsilon < t - N.$$

To give a clearer impression of Theorem 4, we point out the following corollary.

COROLLARY. *If k is a natural number and $1 \leqslant t \leqslant (\log x)^{3/5-\delta}$ for $\delta > 0$, then*

$$\psi_k(x^t, x) = k^{-1}\varphi(k)\, Z(t)\, x^t + O_\epsilon(x^t\{t^{A_1}\rho(x; k)(\log x)^{A_2} + 2^{\nu(k)}x^{-\epsilon}(1 + Z(t)) + \xi_1(k, 0)\, |\, Z'(t - \epsilon)|\, (\log x)^{-1}\}) \tag{2.25}$$

uniformly in t, x, and k outside the interval $(1, 1+\epsilon)$, *where* $\epsilon > 0$ *is arbitrary.*

In particular, if $2 \leqslant t \leqslant (\log x)^{3/5-\delta}$, then

$$\psi_k(x^t, x) = k^{-1}\varphi(k)\, Z(t)\, x^t + O_\epsilon\Big(x^t\Big\{t^{A_1}\rho(x; k)(\log x)^{A_2} + 2^{\nu(k)}x^{-\epsilon}(1 + Z(t)) + \xi_1(k, 0)\,\frac{Z(t-2)}{\log x}\Big\}\Big) \tag{2.26}$$

uniformly in t, x, and k with $1 \leqslant \epsilon < t - 1$.

The proof of the corollary follows directly from Theorem 4 with $N = 1$ (if N is an odd natural number, there is no change in (2.21) with the exception of the last term of the O-term; see (7.13)).

We define

$$\lambda_n = 1 - \sum_{m=0}^{n} \frac{c_m}{m!} \tag{2.27}$$

for $n = 0, 1,\ldots,$ with

$$c_m = \lim_{r\to\infty}\left(\sum_{t=1}^{r} \frac{(\log t)^m}{t} - \frac{(\log r)^{m+1}}{m+1}\right) \tag{2.28}$$

(c_0 = Euler's constant), and

$$f_r(s) = \sum_{d\mid k} \mu_r(d)\, d^{-s} \tag{2.29}$$

for any complex number s and $r = 0, 1$. Then Lemma 1 gives some information about $\xi_r(k, m)$.

LEMMA 1. *If k is a natural number, M is a nonnegative integer and $r = 0, 1$, then*

$$\xi_r(k, M) = \frac{1}{M+1}\sum_{d\mid k}\frac{\mu_r(d)}{d}(\log d)^{M+1} + \sum_{m=0}^{M}\frac{M!}{(M-m)!}\lambda_m\sum_{d\mid k}\frac{\mu_r(d)}{d}(\log d)^{M-m} = O_M(f_r(1)(\log\log 3k)^{M+1}). \tag{2.30}$$

Levin and Fainleib, combining their Theorem 3.4.1 and their argument on page 199 of [1], gave essentially the same asymptotic estimate for $\psi_k(x^t, x)$ as (2.21), but with the restriction $k < (\log x)^D$ for some absolute constant D.

Norton in Theorem 5.21 of [2] removed the restriction on k. For comparison, we state Norton's result in the form of Theorem 5.48 of [2] with $2 < t \leqslant (\log x)^{1/2}$:

$$\psi_k(x^t, x) = k^{-1}\varphi(k)\, Z(t)\, x^t + O\left(x^t \left\{\frac{k}{\varphi(k)} (\log x)^{-1} + 2^{\nu(k)}x^{-1} + \xi_1(k) \frac{Z(t-2)}{\log x}\right\}\right), \tag{2.31}$$

where

$$\xi_1(k) = O((\log\log 3k)^2).$$

If we assume that $1 < \epsilon < t - 1$ with t fixed and k to be any of the infinitely many integers for which $\nu(k) > \frac{1}{2}(\log k/\log\log k)$, then setting

$$x = 2^{\nu(k)}(\log\log k)^{1/\epsilon},$$

it is easy to see that the error term of (2.26) is

$$O_t\left(\frac{x^t}{(\log\log k)^M}\right)$$

for every positive number M, while the leading term is

$$k^{-1}\varphi(k)\, Z(t)\, x^t > \alpha_1(t) \frac{x^t}{\log\log k}\,.$$

Thus, (2.26) is a genuine asymptotic estimate for $\psi_k(x^t, x)$. On the other hand, the error in Norton's estimate (2.31) is at least

$$2^{\nu(k)}x^{t-1} = \frac{x^t}{(\log\log k)^{1/\epsilon}},$$

which is larger than the leading term.

Theorem 4 gives a genuine asymptotic estimate for $\psi_k(x^t, x)$ only for $1 \leqslant t \leqslant (\log x)^{3/5-\delta}$. However, using Theorem 3 with $g(n) = n^{-1}$ for every n, we can write

$$\psi_k(x^t, x) \leqslant k^{-1}\varphi(k)\, x^t \exp\{-t\log t - t\log\log t + \eta(t, x)\} \tag{2.32}$$

for $k \leqslant x$ and $e < t < x/(e\log x)$, where $\eta(t, x)$ is defined by (2.20) with $\tau = 1$.

Now (2.32) compares favorably with the following upper estimate of Norton [2, (3.29)]:

$$\psi_k(x, x^{1/t}) < c_1 k^{-1} \varphi(k)\, Z(t) x$$
$$< c_2 k^{-1} \varphi(k) x \exp\left\{-t\left(\log t + \log\log t - 1 - \frac{1}{\log t}\right)\right\} \tag{2.33}$$

for $k \leqslant x$, $x > e^e$, and $e \leqslant t \leqslant \log\log x/\log\log\log x$, with c_1 and c_2 as absolute constants.

The proofs of Theorems 1 and 2, Lemma 1, and Theorem 4 are given in Sections 4–7, respectively. The next section, Section 3, contains some preliminary groundwork.

3. Preliminary Results

Let $g(n)$ be a multiplicative function and let k be a natural number. We define a multiplicative function $f(n)$ by

$$f(n) = \begin{cases} g(n) & \text{if } (n, k) = 1, \\ 0 & \text{if } (n, k) > 1. \end{cases} \tag{3.1}$$

Then we have the following form of (2.5).

Lemma 2. *Let $g(n)$ be a multiplicative function satisfying* (2.5) *with* $\rho(x) = \exp(-A(\log x)^a)$, $A > 0$, *and* $a > 0$. *Let k be a natural number and define $f(n)$ by* (3.1). *Then*

$$L_f(x, y) = \sum_{\substack{p^r \leqslant x \\ p \leqslant y \\ p \nmid k}} \lambda_g(p^r) = \tau \log \min(x, y) + B(k) + h(\min(x, y); k), \tag{3.2}$$

where

$$B(k) = B - \sum_{p \mid k} \sum_{r=1}^{\infty} \lambda_g(p^r), \tag{3.3}$$

$$h(\min(x, y); k) = h(\min(x, y)) + \sum_{\substack{p \mid k \\ p^r > x}} \lambda_g(p^r) + \sum_{\substack{p \mid k \\ p > y}} \lambda_g(p^r) - \sum_{\substack{p \mid k \\ p^r > x \\ p > y}} \lambda_g(p^r),$$

with

$$h(\min(x, y); k) = O((\nu(k) + 1)\exp\{-(A/2)(\log \min(x, y))^a\}) \quad (3.4)$$

uniformly in x, y and k.

Lemma 2 is Lemma 4.1.3 of Levin and Fainleib [1].

We also extend (2.13)

LEMMA 3. *Let $g(n)$ be a multiplicative function satisfying* (2.13) *with $\rho(x) = \exp(-A(\log x)^a)$, $A > 0$, and $a > 0$. Let k be a natural number and define $f(n)$ by* (3.1). *Then*

$$L_f^*(x, y) = \sum_{\substack{p^r \leqslant x \\ p \leqslant y \\ p \nmid k}} \frac{\lambda_g(p^r)}{p^r} = \tau \log \min(x, y) + B(k) + h(\min(x, y); k), \quad (3.5)$$

where

$$B(k) = B - \sum_{p \mid k} \sum_{r=1}^{\infty} \frac{\lambda_g(p^r)}{p^r},$$

$$h(\min(x, y); k) = h(\min(x, y)) + \sum_{\substack{p \mid k \\ p^r > x}} \frac{\lambda_g(p^r)}{p^r} + \sum_{\substack{p \mid k \\ p > y}} \frac{\lambda_g(p^r)}{p^r} - \sum_{\substack{p \mid k \\ p^r > x \\ p > y}} \frac{\lambda_g(p^r)}{p^r},$$

with

$$h(\min(x, y); k) = O((\nu(k) + 1)\exp\{-(A/2)(\log \min(x, y))^a\}) \quad (3.6)$$

uniformly in x, y, and k.

The proof of Lemma 3 is essentially the same as the proof of Lemma 2.

Now we prove the following lemma which is a special case of Theorem 3.1.2 of Levin and Fainleib [1].

LEMMA 4. *Let $g(n)$ be a multiplicative function satisfying* (2.5) *and* (2.6). *Let k be a natural number and $x \geqslant 1$, $t \geqslant 1$. Then*

$$\sum_{\substack{n \leqslant x^t \\ (n, P(x)) = 1 \\ (n, k) = 1}} g(n) = Z(t) + O(t^{A_1} \rho(x; k)(\log x)^{A_2}), \quad (3.7)$$

where $P(x) = \prod_{p \leqslant x} p$ and $Z(t)$ satisfies Eq. (2.8) *with τ replaced by $-\tau$.*

Proof. In the notation of Chapter 3 of Levin and Fainleib [1], let $0 = \beta_0 < \beta_1 = 1 < \beta_2 = +\infty$. Let $n \in M_\nu$, $\nu = 1, 2$, denote that either

$n = 1$ or that all the prime factors of n belong to the interval $(x^{\beta_{\nu-1}}, x^{\beta_\nu}]$. Then any positive integer n can be uniquely expressed in the form $n = n_1 n_2$, where $n_\nu \in M_\nu$, $\nu = 1, 2$. Further, let $f_1(n)$ and $f_2(n)$ be multiplicative functions. Let $f(n) = f_1(n_1) f_2(n_2)$ and

$$m_f(x^t) = \sum_{n \leqslant x^t} f(n) = \sum_{n \leqslant x^t} f_1(n_1) f_2(n_2). \tag{3.8}$$

In particular, if $f_1(n) = \epsilon(n)$ and $f_2(n)$ is defined by (3.1),

$$m_f(x^t) = \sum_{\substack{n \leqslant x^t \\ (n, P(x)) = 1 \\ (n,k) = 1}} g(n). \tag{3.9}$$

Now $L_{f_1}(x, y) = 0$ and, by Lemma 2,

$$L_{f_2}(x, y) = \tau \log \min(x, y) + B(k) + h(\min(x, y); k). \tag{3.10}$$

It is easy to see that the conditions of Lemma 3.1.1. of Levin and Fainleib [1] are satisfied so that

$$\begin{aligned} t\, m_f(x^t) &- \int_0^t m_f(x^v)\, dv \\ &= \tau \int_0^{t-1} m_f(x^v)\, dv \\ &\quad + \frac{1}{\log x} \sum_{n \leqslant x^{t-1}} f(n) \left\{ h\left(\frac{x^t}{n}; k\right) - h\left(\min\left(\frac{x^t}{n}, x\right); k\right) \right\}, \end{aligned}$$

since $\tau_1 = 0$, $B_1 = 0$, $\tau_2 = \tau$ and $B_2 = B(k)$.

Therefore, since $\sum_{n \leqslant x^t} |f(n)| = O(t^A \log^A x)$ from (2.6), we have

$$t\, m_f(x^t) - \int_0^t m_f(x^v)\, dv - \tau \int_0^{t-1} m_f(x^v)\, dv = O(t^A \rho(x; k)(\log x)^{A-1}). \tag{3.11}$$

Letting

$$m_f(x^t) = Z(t) + R_k(t, x)\, \rho(x; k)(\log x)^{A-1}$$

and substituting into (3.11), with (2.8), with τ replaced by $-\tau$, and Lemma 1.2.1 of Levin and Fainleib [1], we obtain (3.7), which completes the proof of Lemma 4.

Although we do not need it in this paper, we can derive a lemma similar to Lemma 4, but for $g(n)$ satisfying (2.13) and (2.14), by using Abel summation on (3.7).

In order to prove Theorem 3, we need Lemma 3.3.1 of Levin and Fainleib [1]. The statement of their lemma is incorrect and should read as follows.

LEMMA 5. *Let $g(n)$ be a nonnegative multiplicative function satisfying conditions* (2.16)–(2.18). *Define*

$$P(s, y) = \prod_{p \leqslant y} \left(1 + \sum_{r=1}^{\infty} \frac{g(p^r)}{p^{rs}}\right) \tag{3.12}$$

for $y \geqslant 2$ and s a complex variable. If $0 < \delta = \delta(y) \leqslant 1 - 1/\log y$, then

$$\log P(\delta - 1, y) = \frac{\tau\, y^{1-\delta}}{(1-\delta)\log y} + \tau \log\left(\frac{1}{1-\delta}\right) + O\left(\frac{y^{1-\delta}}{(1-\delta)^2 \log y}\right)$$
$$+ O\left(\int_2^{\infty} \frac{|\rho(v, y)|}{v^{\delta} \log v}\, dv\right). \tag{3.13}$$

The proof of Lemma 5 is the same as the proof of Lemma 3.3.1 of Levin and Fainleib [1].

4. PROOF OF THEOREM 1

Let $f_1(n)$ be defined by (3.1) and let $f_2(n) = \epsilon(n)$. Then

$$m_f(x^t) = \psi_k(x^t, x; g) \tag{4.1}$$

and

$$m_{f_1}(x^t) = S_k(x^t; g). \tag{4.2}$$

We now define multiplicative functions $\hat{f}_\nu(n)$ by the relations

$$\sum_{d \mid n} \hat{f}_\nu(d) f_\nu\left(\frac{n}{d}\right) = f_1(n), \qquad \nu = 1, 2.$$

Then $\hat{f}_1(n) = \epsilon(n)$ and $\hat{f}_2(n) = f_1(n)$.

Thus

$$m_{\hat{f}}(x^t) = \sum_{\substack{n \leqslant x^t \\ (n, P(x)) = 1 \\ (n,k)=1}} g(n) = \hat{Z}(t) + O(t^{A_1} \rho(x; k)(\log x)^{A_2}), \tag{4.3}$$

where $\hat{Z}(t)$ satisfies the equation

$$t\, \hat{Z}'(t) = \tau\, \hat{Z}(t-1)$$

with initial condition $\hat{Z}(t) = 1$ for $0 \leqslant t \leqslant 1$, by Lemma 4.

Now

$$m_{f_1}(x^t) = \sum_{n \leqslant x^t} f_1(n) = \sum_{n \leqslant x^t} f_1(n_1) f_1(n_2)$$

$$= \sum_{nm \leqslant x^t} f_1(n_1) f_2(n_2) \hat{f}_1(m_1) \hat{f}_2(m_2)$$

$$= \sum_{n \leqslant x^t} f_1(n_1) f_2(n_2)\, m_{\hat{f}}(x^t/n).$$

Thus, using (4.3), we get

$$m_{f_1}(x^t) = \sum_{n \leqslant x^t} f_1(n_1) f_2(n_2)\, \hat{Z}\left(t - \frac{\log n}{\log x}\right)$$

$$+ O\left(t^{A_1} \rho(x; k)(\log x)^{A_2} \sum_{n \leqslant x^t} |f_1(n_1) f_2(n_2)|\right)$$

$$= m_f(x^t) + \sum_{n \leqslant x^t} f_1(n_1) f_2(n_2) \int_0^{t - \log n/\log x} \hat{Z}'(v)\, dv$$

$$+ O(t^{A_3} \rho(x; k)(\log x)^{A_4}).$$

Hence,

$$m_{f_1}(x^t) = m_f(x^t) + \int_0^t \hat{Z}'(t - v)\, m_f(x^v)\, dv + O(t^{A_3} \rho(x; k)(\log x)^{A_4}). \tag{4.4}$$

Now (4.4) is an integral equation with respect to $m_f(x^t)$. To solve it, we proceed as in Levin and Fainleib [1]. Let $Z(t)$ satisfy (2.7). Then using the Levin and Fainleib argument from (3.2.9)–(3.2.11) of their Chapter 3, we have

$$\int_0^t Z'(t - v)\, \hat{Z}'(v)\, dv + Z'(t) + \hat{Z}'(t) = 0. \tag{4.5}$$

From (4.4), we get

$$\int_0^t Z'(t - v)\, m_{f_1}(x^v)\, dv = \int_0^t Z'(t - v)\, m_f(x^v)\, dv$$

$$+ \int_0^t m_f(x^u) \int_0^{t-u} Z'(t - u - v)\, \hat{Z}'(v)\, dv\, du$$

$$+ O(t^{A_3} \rho(x; k)(\log x)^{A_4}),$$

which, using (4.5) and (4.4) again, is the same as

$$\int_0^t Z'(t-v)\, m_f(x^v)\, dv = -\int_0^t \hat{Z}'(t-v)\, m_f(x^v)\, dv + O(t^{A_3}\rho(x;k)(\log x)^{A_4})$$

$$= m_f(x^t) - m_{f_1}(x^t) + O(t^{A_3}\rho(x;k)(\log x)^{A_4}).$$

Hence, using (4.1) and (4.2), we get (2.9) to prove Theorem 1.

5. Proof of Theorem 3

Let $0 < \delta < 1 - 1/\log x$; then

$$\sum_{\substack{n \leqslant x^t \\ p(n) \leqslant x \\ (n,k)=1}} n\, g(n) \leqslant x^{t\delta} \sum_{\substack{n \leqslant x^t \\ p(n) \leqslant x \\ (n,k)=1}} n^{1-\delta} g(n)$$

$$\leqslant x^{t\delta} \prod_{\substack{p \leqslant x \\ p \nmid k}} \left(1 + \sum_{r=1}^{\infty} \frac{g(p^r)}{p^{(\delta-1)r}}\right)$$

$$\leqslant x^{t\delta} \prod_{\substack{p \leqslant x \\ p \mid k}} \left(1 + \sum_{r=1}^{\infty} \frac{g(p^r)}{p^{(\delta-1)r}}\right)^{-1} P(\delta - 1, x)$$

$$\leqslant x^{t\delta} \prod_{\substack{p \leqslant x \\ p \mid k}} \left(1 + \sum_{r=1}^{\infty} g(p^r)\right)^{-1} P(\delta - 1, x).$$

Thus, using Lemma 5 and also the argument used by Levin and Fainleib [1] to prove their Theorem 3.3.1, we have the proof of Theorem 3.

6. Proof of Lemma 1

To prove Lemma 1, we need three steps. The argument generalizes work by Norton [3].

Lemma 6. *Let N and k be natural numbers; then*

$$\sum_{p \mid k} \frac{\log^N p}{p} = O_N((\log\log 3k)^N). \tag{6.1}$$

Proof. We have

$$\sum_{p|k} \frac{\log^N p}{p} = \sum_{\substack{p|k \\ p \leqslant (\log 3k)^N}} \frac{\log^N p}{p} + \sum_{\substack{p|k \\ p > (\log 3k)^N}} \frac{\log^N p}{p}$$
$$= O\Bigl(\sum_{p \leqslant (\log 3k)^N} \frac{\log^N p}{p}\Bigr) + O\Bigl((\log 3k)^{-N} \sum_{p|k} \log^N p\Bigr)$$
$$= O((N \log\log 3k)^N),$$

which proves the lemma.

LEMMA 7. *Define the function*

$$g_r(s) = \sum_{p|k} (-\log p) \frac{\mu_r(p)}{(p^s + \mu_r(p))} \tag{6.2}$$

for any complex number s *and* $r = 0, 1$ *with* $\mu_r(n)$ *defined by* (2.23). *Then, for any natural number* N, *there exist integers* $a(N, j)$, $1 \leqslant j \leqslant N + 1$, *with* $a(N, 1) = 1$, *such that*

$$g_r^{(N)}(s) = \sum_{p|k} (-\log p)^{N+1} \mu_r(p) \sum_{j=1}^{N+1} \frac{a(N, j)}{(p^s + \mu_r(p))^j}. \tag{6.3}$$

Proof. The result follows by a straightforward argument using induction on N.

LEMMA 8. *If* k *and* N *are natural numbers, then*

$$\sum_{d|k} \frac{\mu_r(d)}{d} \log^N d = \sum_{n=0}^{N-1} \binom{N-1}{n} \Bigl(\sum_{d|k} \frac{\mu_r(d)}{d} \log^n d\Bigr) ((-1)^{N-n} g_r^{(N-n-1)}(1))$$
$$= O_N (f_r(1)(\log\log 3k)^N) \tag{6.4}$$

for $r = 0, 1$; $f_r(s)$ *is defined by* (2.29) *and* $g_r(s)$ *is defined by* (6.2).

Proof. Taking the logarithmic derivative of (2.29), we get

$$f_r'(s) = f_r(s)\, g_r(s) \tag{6.5}$$

with

$$f_r'(s) = \sum_{d|k} \mu_r(d)\, d^{-s}(-\log d).$$

Differentiating $N - 1$ times with Leibnitz's rule, we get the first equality of (6.4).

The second equality of (6.4) follows from the first using induction on N with the fact that

$$g_r^{(m)}(1) = O_N\Big(\sum_{p|k} \frac{\log^{m+1} p}{p}\Big)$$
$$= O_N((\log\log 3k)^{m+1})$$

by Lemma 6.

To prove Lemma 1, we write

$$\xi_r(k, N) = \sum_{d|k} \frac{\mu_r(d)}{d} \int_{1/d}^{\infty} \frac{(v - [v])}{v^2} (\log v + \log d)^N \, dv$$

$$= \sum_{d|k} \frac{\mu_r(d)}{d} \sum_{n=0}^{N} \binom{N}{n} (\log d)^{N-n} \int_{1/d}^{\infty} \frac{(v - [v])}{v^2} (\log v)^n \, dv. \tag{6.6}$$

Breaking the integral on the right-hand side of (6.6) into two parts, we get

$$\int_{1/d}^{\infty} = \int_{1/d}^{1} \frac{(\log v)^n}{v} \, dv + \int_{1}^{\infty} \frac{(v - [v])}{v^2} (\log v)^n \, dv,$$

so that

$$\int_{1/d}^{1} = \frac{(-1)^n}{n+1} (\log d)^{n+1} \tag{6.7}$$

and

$$\int_{1}^{\infty} = n! \, \lambda_n \tag{6.8}$$

with λ_n defined by (2.27) (see Norton [2, Lemma 3.14]). Putting (6.7) and (6.8) in (6.6), we get the first equality of (2.30). Using Lemma 8, we get the second to complete the proof of Lemma 1.

7. Proof of Theorem 4

Let $g(n) = 1$ for every positive integer n.

Then

$$L_g^*(x, y) = \log \min(x, y) + B + h(\min(x, y)) \tag{7.1}$$

with $h(x) = O(\exp(-(\log x)^{3/5-\delta}))$, $\delta > 0$,

$$\Pi_g^*(x) = O(\log x), \tag{7.2}$$

where $L_g{}^*(x, y)$ and $\Pi_g{}^*(x)$ are (2.11) and (2.12), respectively. Then by Theorem 2

$$\psi_k(x^t, x) = S_k(x^t) + \int_0^t x^{t-v} Z'(t-v)\, S_k(x^v)\, dv + O(x^t t^{A_1} \rho(x; k)(\log x)^{A_2}) \tag{7.3}$$

uniformly in t, x, and k, where A_1 and A_2 are absolute constants $Z(t)$ satisfies (2.7) with $\tau = 1$, $\rho(x; k)$ is defined by (2.24), and

$$S_k(x^t) = \sum_{\substack{n \leqslant x^t \\ (n,k)=1}} 1. \tag{7.4}$$

Now

$$S_k(x^t) = k^{-1}\varphi(k)\, x^t + x^t R(x^t, k), \tag{7.5}$$

where

$$R(x^t, k) = -x^{-t} \sum_{d \mid k} \mu(d) \left(\frac{x^t}{d} - \left[\frac{x^t}{d} \right] \right). \tag{7.6}$$

Putting (7.5) in (7.3), we get

$$\psi_k(x^t, x) = x^t \Big\{ k^{-1}\varphi(k) \left(1 + \int_0^t Z'(t-v)\, dv \right) + R(x^t, k) + \int_0^t Z'(t-v)\, R(x^v, k)\, dv + O(t^{A_1} \rho(x, k)(\log x)^{A_2}) \Big\}. \tag{7.7}$$

First, we note that

$$Z(t) = 1 + \int_0^t Z'(t-v)\, dv. \tag{7.8}$$

Now we let $\epsilon > 0$ be so small that the interval $(t - \epsilon, t)$ does not contain any discontinuities of $Z^{(N+1)}(v)$. The discontinuities of $Z^{(N+1)}(v)$ are the points where $Z^{(m+1)}(v)$, $0 \leqslant m \leqslant N - 1$, might not be differentiable, so that $Z'(v)$ is N times differentiable on $(t - \epsilon, t)$. Also, since $Z^{(N+1)}(v)$ has only right discontinuities, ϵ must be small enough that $t - \epsilon$ is not a discontinuity of $Z^{(N+1)}(v)$. Thus $Z^{(N+1)}(v)$ is continuous on $[t - \epsilon, t]$. Hence, for $0 < v < \epsilon$, we can apply Taylor's theorem to get

$$Z'(t-v) = \sum_{m=0}^{N-1} \frac{(-1)^m}{m!} Z^{(m+1)}(t)\, v^m + O(v^N \mid Z^{(N+1)}(t - \epsilon_1)|)$$

for some ϵ_1 such that $0 < \epsilon_1 < \epsilon$.

Thus,

$$\int_0^\epsilon Z'(t-v)\, R(x^v, k)\, dv = \sum_{m=0}^{N-1} \frac{(-1)^m}{m!} Z^{(m+1)}(t) \int_0^\epsilon v^m R(x^v, k)\, dv$$
$$+ O\left(| Z^{(N+1)}(t-\epsilon_1)| \int_0^\epsilon v^N \,| R(x^v, k)|\, dv\right). \tag{7.9}$$

The sum on the right-hand side of (7.9) is equal to

$$\sum_{m=0}^{N-1} \frac{(-1)^m}{m!} \frac{Z^{(m+1)}(t)}{(\log x)^{m+1}} \xi_0(k, m)$$
$$- \sum_{m=0}^{N-1} \frac{(-1)^m}{m!} \frac{Z^{(m+1)}(t)}{(\log x)^{m+1}} \int_{x^\epsilon}^\infty \frac{R(v, k)(\log v)^m}{v}\, dv, \tag{7.10}$$

and the second sum of (7.10) is

$$O_N\left(\frac{| Z'(t)|}{\log x} \int_{x^\epsilon}^\infty \frac{| R(v, k)|}{v}\, dv\right) = O_N\left(\frac{2^{\nu(k)} x^{-\epsilon}\, | Z'(t)|}{\log x}\right), \tag{7.11}$$

since

$$| R(v, k)| = O(2^{\nu(k)} v^{-1}).$$

For the O-term of (7.9), we see that

$$O\left(| Z^{(N+1)}(t-\epsilon_1)| \int_0^\epsilon v^N \,| R(x^v, k)|\, dv\right)$$
$$= O\left(\frac{| Z^{(N+1)}(t-\epsilon_1)|}{(\log x)^{N+1}} \xi_1(k, N)\right). \tag{7.12}$$

By the same argument as used by Levin and Fainleib on page 187 of [1],

$$Z^{(N+1)}(t-\epsilon_1) = O(t\, | Z^{(N)}(t)|). \tag{7.13}$$

Further,

$$\int_\epsilon^t Z'(t-v)\, R(x^v, k)\, dv = O\left(2^{\nu(k)} x^{-\epsilon} \int_\epsilon^t | Z'(t-v)|\, dv\right)$$
$$= O(2^{\nu(k)} x^{-\epsilon}(1 + Z(t))). \tag{7.14}$$

Putting (7.13) in (7.12) and (7.11) in (7.10), then (7.12) and (7.10) in (7.9), and finally (7.9), (7.14), and (7.8) in (7.7), we get (2.21) to complete the proof of Theorem 4.

References

1. B. V. Levin and A. S. Fainleib, Application of some integral equations to problems in number theory, *Usp. Mat. Nauk* **22** (1967), 119–197; *Russ. Math. Surveys* **22** (1967), 119–204.
2. K. K. Norton, Numbers with small prime factors, and the least k-th power non-residue, *Mem. Amer. Math. Soc.* **106** (1971), 106 pp.