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BGP-Reflection Functors and Lusztig's Symmetries: A Ringel–Hall Algebra Approach to Quantum Groups[1]

Jie Xiao

E-mail: jxiao@math.tsinghua.edu.cn

and

Shilin Yang

Department of Mathematics, Beijing Polytechnic University, 100022 China; and Department of Mathematics, Beijing Normal University, 1000875 China

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According to the canonical isomorphisms between the Ringel–Hall algebras (composition algebras) and the quantum groups, we deduce Lusztig's symmetries $T''_{i,1}$, $i \in I$, by applying the Bernstein–Gelfand–Ponomarev reflection functors to the Drinfeld doubles of Ringel–Hall algebras. The fundamental properties of $T''_{i,1}$ including the following can be obtained conceptually. (1) $T''_{i,1}$, $i \in I$ induce automorphisms of the quantum groups $U_q(\mathfrak{g})$ and on the integrable modules. (2) $T''_{i,1}$, $i \in I$ satisfy the braid group relations. This extends and completes the results of B. Sevenhant and M. Van den Bergh (1999, *J. Algebra* **221,** 135–160).

Key Words: Ringel–Hall algebra; quantum group; BGP-reflection; braid relation.

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CONTENTS

1. INTRODUCTION

1.1. Let Δ be a symmetrizable generalized Cartan matrix, or $\Delta = (I, (\ ,\ ))$ a Cartan datum in the sense of Luzstig, $\mathfrak{g}$ the symmetrizable Kac–Moody algebra. We have the Drinfeld–Jimbo quantized enveloping algebra $U_q(\mathfrak{g})$ attached to the Cartan datum Δ. Its generators are E_i, F_i, and K_α with $\alpha \in \mathbb{Z}[I]$. One of the great contributions of Lusztig to quantum groups was his introduction of the symmetries acting on integrable $U_q(\mathfrak{g})$-modules and then $T''_{i,1} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ (see also [LS]). In fact, Lusztig gives us four families of symmetries as automorphisms of $U_q(\mathfrak{g})$, but since they all can be defined and investigated in a similar way, we only write down one of them as

$$T''_{i,1}(E_i) = -F_i K_i^{\varepsilon_i}, \qquad T''_{i,1}(F_i) = -K_i^{-\varepsilon_i} E_i$$

$$T''_{i,1}(E_j) = \sum_{r+s=-a_{ij}} (-1)^r v^{-r\varepsilon_i} E_i^{(s)} E_j E_i^{(r)} \qquad \text{for } j \neq i \text{ in } I,$$

$$T''_{i,1}(F_j) = \sum_{r+s=-a_{ij}} (-1)^r v^{r\varepsilon_i} F_i^{(r)} F_j F_i^{(s)} \qquad \text{for } j \neq i \text{ in } I.$$

$$T''_{i,1}(K_\beta) = K_{s_i(\beta)},$$

where $E_i^{(r)} = E_i^r/[r]!_{\varepsilon_i}$ and $(\varepsilon_i)_i$ is the minimal symmetrization. The fundamental results about $T''_{i,1}$ include (1) $T''_{i,1}$ acts isomorphically on $U_q(\mathfrak{g})$ and on integrable $U_q(\mathfrak{g})$-modules. (2) $T''_{i,1}$, $i \in I$ satisfy the braid group relations. (3) The symmetries preserve the bilinear form (Killing form) on $U_q(\mathfrak{g})$. However, the proof of them requires long and some unpleasant calculations (see [L1, part VI, Jan, Chaps. 8 and 8A]).

1.2. If we consider the Ringel–Hall algebra $\mathfrak{h}(A)$ of a finite dimensional hereditary algebra A, according to the Ringel–Green Theorem (see [G, R1, R2]), the composition subalgebra $\mathfrak{c}(A)$ of $\mathfrak{h}(A)$ provides a realization of the positive part U^+ of $U_q(\mathfrak{g})$. Because the comultiplication of $\mathfrak{h}(A)$ is given by Green [G], it is natural to provide a Hopf algebra structure of

$\mathfrak{h}(A)$ by adding the torus algebra, and then, to consider the Drinfeld double of the Ringel–Hall algebra. This was done in [X, Ka]. Therefore, the Drinfeld-double of the composition algebra provides a realization of the whole $U_q(\mathfrak{g})$. This realization builds up a bridge between the quantum groups and the representation theory of hereditary algebras (especially of quivers). Connecting to Lusztig's symmetries, it is natural to consider the reflection functors on representations of quivers given by Bernstein *et al.* [BGP]. It is easily seen that the BGP-reflection functor σ_i induces an automorphism of $\mathfrak{h}(A)\langle i\rangle$. In fact, it has been pointed out, by Lusztig [L2] and Ringel [R3], that the actions of Lusztig's symmetries and the operators induced by BGP-reflection functors coincide in $U^+\langle i\rangle$ for the case of finite type, where $U^+\langle i\rangle = \{x \in U^+ \mid r_i'(x) = 0\}$ and the derivations r_i' are defined as in [Jan, 6.15]. Recently Sevenhant and Van den Bergh [SV] applied the BGP-reflection functor to the double of the Ringel–Hall algebra of a quiver and obtained an alternative construction of Lusztig's symmetries.

1.3. In this article, we apply the BGP-reflection functors to the Drinfeld doubles of Ringel–Hall algebras of all finite dimensional hereditary algebras. It gives a precise construction of Lusztig's symmetries in the quantum groups and on the integrable modules in a global way. Our process is logically independent of the method used in quantum groups. Almost of all properties of $T''_{i,1}$, in particular, three fundamental ones we mentioned above, can be obtained in a more conceptual way. Also this approach avoids a lot of difficult calculations.

1.4. In Section 2, we first review some notations and basic facts of representations of finite dimensional hereditary algebras in the language of Dlab and Ringel [DR]. In particular, the BGP-reflection functors at sink or source vertices are introduced in detail. Then, the Ringel–Hall algebra and its composition algebra of a finite dimensional hereditary algebra are defined. According to [X], we restate in Section 3 the Drinfeld-double structure of Ringel–Hall algebras, namely the formulae for the comultiplication, etc., are presented here in useful forms. By using the derivations and some routine technique of Hopf algebras, we give the simpler formulae for the defining relations of the double structure. The aim of Section 4 is to define the BGP-reflection operators on the whole Drinfeld double. We verify that the operators induce the algebraic isomorphisms not only for the double of the composition algebras, but also for the double of the whole Ringel–Hall algebras (a slight extension of the result in [SV]). Because the BGP-reflection operators and the bilinear form are defined globally on the Drinfeld double, it is very clear to see in Section 5 that the actions of the BGP-reflection operators preserve the Ringel paring. Note that the proof of this fact in quantum groups is very difficult (see [L1, Chap. 38; Jan, Chap. 8A]). In Section 6, we show that our BGP-reflection operators coincide with Lusztig's symmetries. In fact, it is equivalent to express the root vectors corresponding to indecomposable projective or injective representations of the generalized Kronecker algebras (rank 2 cases) into the combinations of monomials of the generators (see [R3, CX]). To prove the braid group relations for the BGP-reflection operators, we need to extend the actions of the operators on integrable modules. It can be defined on the integrable highest weight modules in a global sense. Section 7 is used to show that the actions of BGP-reflection operators and Lusztig's symmetries on integrable modules coincide too; our method to deal with this question stems from [Jan, 8.10]. The last section is devoted to proving that BGP-reflection operators satisfy the braid group relations. Our steps are also according to Lusztig [L1, Part VI]: first we prove the braid group relations on the algebras in all rank 2 cases, then on integrable modules in general, and finally back to the algebras in general. However, the Ringel–Hall algebra approach enables us to avoid almost all unpleasant calculations, for example, the so-called quantum Verma identities on highest weight vectors [L1, 39, 3.7] are a direct consequence of the actions.

2. PRELIMINARIES

2.1. Given a Cartan datum Δ in the sense of Lusztig [L1], there is a valued graph (Γ, d) corresponding to it. A valued graph (Γ, d) is a finite set Γ (of vertices) together with non-negative integers d_{ij} for all $i, j \in \Gamma$ such that $d_{ii} = 0$ and there exist positive integers $\{\varepsilon_i\}_{i \in \Gamma}$ satisfying

$$d_{ij}\varepsilon_j = d_{ji}\varepsilon_i \qquad \text{for all } i, j \in \Gamma.$$

An orientation Ω of a valued graph (Γ, d) is given by prescribing for each edge $\{i, j\}$ of (Γ, d) an order (indicated by an arrow $i \to j$). We call (Γ, d, Ω), or simply Ω, a valued quiver. For $i \in \Gamma$, we can define a new orientation $\sigma_i\Omega$ of (Γ, d) by reversing the direction of arrows along all edges containing i.

2.2. Let k be a finite field and (Γ, d, Ω) a valued quiver. We assume that (Γ, d, Ω) is connected and without oriented cycles in an obvious sense. Let $\mathscr{S} = (F_i, {}_iM_j)_{i,j \in \Gamma}$ be a reduced k-species of type Ω, that is, for all i, $j \in \Gamma$, ${}_iM_j$ is an F_i-F_j-bimodule, where F_i and F_j are finite extensions of k in an algebraic closure of k and $\dim({}_iM_j)_{F_j} = d_{ij}$ and $\dim_k F_i = \varepsilon_i$. A k-representation $(V_i, {}_j\varphi_i)$ of $\mathscr{S}$ is given by vector space $(V_i)_{F_i}$ and F_j-linear mapping ${}_j\varphi_i$: $V_i \otimes {}_iM_j \to V_j$ for any $i \to j$. Such a representation is called finite dimensional if $\sum \dim_k V_i < \infty$. We denote by rep-$\mathscr{S}$ the category of finite dimensional representations of $\mathscr{S}$ over k. Note that the category rep-$\mathscr{S}$ is equivalent to the module category of finite dimensional modules over a finite dimensional hereditary k-algebra A. This hereditary k-algebra A is given by the tensor algebra of $\mathscr{S}$. Furthermore, any finite dimensional hereditary k-algebra can be obtained in this way.

2.3. Let $\mathscr{S} = (F_i, {}_iM_j)_{i,j\in\Gamma}$ be a k-species, $\varepsilon_i = \dim_k F_i$, and $d_{ij} = \dim_i M_{j_{F_j}}$. For a representation $V = (V_i, {}_j\varphi_i) \in$ rep-$\mathscr{S}$, we define the dimension vector of V to be $\mathbf{dim}\, V = (\dim_{F_i} V_i)_{i\in\Gamma}$. If $V, W \in$ rep-$\mathscr{S}$, assume that

$$\alpha = \mathbf{dim}\, V = (a_1, \dots, a_n) \qquad \text{and} \qquad \beta = \mathbf{dim}\, W = (b_1, \dots, b_n),$$

and we define

$$\langle \alpha, \beta \rangle = \sum_{i\in\Gamma} \varepsilon_i a_i b_i - \sum_{i\to j} d_{ij}\varepsilon_j a_i b_j.$$

One sees that (cf. [R4, Lemma 2.2])

$$\langle \alpha, \beta \rangle = \dim \operatorname{Hom}_A(V, W) - \dim \operatorname{Ext}^1_A(V, W).$$

Set

$$(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle.$$

It is well known that both $\langle -, - \rangle$ and $(-,-)$ are well defined on $G_0(A)$: the Grothendieck group of rep-$\mathscr{S}$. The bilinear forms $\langle -, - \rangle$ and $(-,-)$ are called the Euler form and symmetric Euler form, respectively. In fact, the Grothendieck group with the symmetric Euler form is a Cartan datum and any Cartan datum can be realized in this way (see [R2]). Let $\varepsilon(\alpha) = \langle \alpha, \alpha \rangle$. We see that $\varepsilon(i) = \varepsilon_i$.

2.4. Denote by $\mathbb{Q}^\Gamma$ the vector space of all $x = (x_i)_{i\in\Gamma}$ over the rational numbers. In particular, for each $i \in \Gamma$, e_i, or $i \in \mathbb{Q}^\Gamma$ denotes the vector with $x_i = 1$ and $x_j = 0$ for $j \neq i$. Also, for each $i \in \Gamma$, we define the linear transformation $s_i : \mathbb{Q}^\Gamma \to \mathbb{Q}^\Gamma$ by $s_i x = y$ where $y_j = x_j$ for $j \neq i$ and

$$y_i = -x_i + \sum_{j\in\Gamma} d_{ji} x_j.$$

The symbol $W = W_\Gamma$ will denote the *Weyl group*, i.e., the group of all linear transformations of $\mathbb{Q}^\Gamma$ generated by the fundamental reflections s_i, $i \in \Gamma$.

2.5. Let (Γ, d, Ω) be a valued quiver (connected and without oriented cycles) and $\mathscr{S} = (F_i, {}_iM_j)_{i,j\in\Gamma}$ a k-species of type Ω. Let p be a sink or source of (Γ, Ω). We define $\sigma_p\mathscr{S}$ to be the k-species obtained from $\mathscr{S}$ by replacing ${}_rM_s$ by its k-dual for $r = p$ or $s = p$; then $\sigma_p\mathscr{S}$ is a reduced k-species of type $\sigma_p\Omega$.

2.6. Now, we review the concepts of the Bernstein–Gelfand–Ponomarev reflection functors $\sigma_p^{\pm}$: rep-$\mathscr{S} \to$ rep-$\sigma_p\mathscr{S}$, which is most important for our discussion (see [BGP, DR]).

First, let p be a sink of Ω, $V = (V_i, {}_j\varphi_i) \in$ rep-$\mathscr{S}$. Define $\sigma_p^+ V = W = (W_i, {}_j\psi_i)$ as

$$W_i = V_i \qquad \text{for all } i \neq p$$

and let W_p be the kernel of

$$\bigoplus_{j \to p} V_j \otimes_j M_p \xrightarrow{({}_p\varphi_j)_j} V_p,$$

that is, we have the exact sequence of vector spaces

$$0 \to W_p \xrightarrow{({}_j\kappa_p)_j} \bigoplus_{j \to p} V_j \otimes_j M_p \xrightarrow{({}_p\varphi_j)_j} V_p$$

and ${}_j\psi_i = {}_j\varphi_i$ for $i \neq p$ and ${}_j\psi_p = {}_j\overline{\kappa}_p : W_p \otimes_p M_j \to W_j$ where ${}_j\overline{\kappa}_p$ corresponds to ${}_j\kappa_p$ under the natural isomorphism

$$\operatorname{Hom}_{F_j}\big(W_p \otimes_p M_j, W_j\big) \cong \operatorname{Hom}_{F_p}\big(W_p, W_j \otimes_j M_p\big).$$

Also if $f = (f_i): V \to V'$ is a morphism in rep-$\mathscr{S}$, then $\sigma_p^+ f = g = (g_i)$ is defined by $g_i = f_i$ for $i \neq p$ and $g_p : W_p \to W_p'$ as the restriction of $\bigoplus_{j \to p}(f_j \otimes 1)$, that is, we have the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & W_p & \xrightarrow{({}_j\kappa_p)_j} & \bigoplus_{j \to p} V_j \otimes_j M_p & \xrightarrow{({}_p\varphi_j)_j} & V_p \\
\downarrow & & \downarrow{\scriptstyle g_p} & & \downarrow{\scriptstyle \oplus_{j \to p}(f_j \otimes 1)} & & \downarrow{\scriptstyle f_p} \\
0 & \longrightarrow & W_p' & \xrightarrow{({}_j\kappa_p')_j} & \bigoplus_{j \to p} V_j' \otimes_j M_p & \xrightarrow{({}_p\varphi_j')_j} & V_p'.
\end{array}$$

Similarly, if p is a source of Ω and $V = (V_i, {}_j\varphi_i) \in$ rep-$\mathscr{S}$, define $\sigma_p^- V = W = (W_i, {}_j\psi_i)$ as

$$W_i = V_i \qquad \text{for all } i \neq p$$

and let W_p be the cokernel in the exact sequence

$$V_p \xrightarrow{({}_i\overline{\varphi}_p)_i} \oplus V_i \otimes_i M_p \xrightarrow{({}_p\pi_i)_i} W_p \to 0,$$

where ${}_i\overline{\varphi}_p$ corresponds to ${}_i\varphi_p$ under the natural isomorphism,

$$\operatorname{Hom}_{F_p}\big(V_p, V_i \otimes_i M_p\big) \cong \operatorname{Hom}_{F_i}\big(V_p \otimes_p M_i, V_i\big),$$

and ${}_j\psi_i = {}_j\varphi_i$ for all $j \neq p$ and

$${}_p\psi_i = {}_p\pi_i : V_i \otimes_i M_p \to W_p.$$

So $\sigma_p^- V \in$ rep-$\sigma_p\mathscr{S}$. If $f = (f_i): V \to V'$ is a morphism in rep-$\mathscr{S}$, then $\sigma_p^- f = g = (g_i)$ where $g_i = f_i$ for $i \neq p$ and g_p is the map induced by $\bigoplus_i f_i \otimes 1$, so we have the diagram

$$\begin{array}{ccccccc} V_p & \xrightarrow{({}_i\bar{\varphi}_p)_i} & \bigoplus V_i \otimes {}_iM_p & \xrightarrow{({}_p\pi_i)_i} & W_p & \longrightarrow & 0 \\ f_p\big\downarrow & & \oplus_i(f_i\otimes 1)\big\downarrow & & g_p\big\downarrow & & \\ V'_p & \xrightarrow{({}_i\bar{\varphi}'_p)_i} & \bigoplus V'_i \otimes {}_iM_p & \xrightarrow{({}_p\pi'_i)_i} & W'_p & \longrightarrow & 0. \end{array}$$

2.7. If i is a vertex of Γ, let rep-$\mathscr{S}\langle i\rangle$ be the subcategory of rep-$\mathscr{S}$ of all representations which do not have V_i as a direct summand, where V_i is the simple representation with $\mathbf{dim}\, V_i = e_i$. If i is a sink or source, then rep-$\mathscr{S}\langle i\rangle$ is closed under direct summands and extensions. Among the many important properties of $\sigma_i^{\pm}$ we point out that if i is a sink, then σ_i^+ : rep-$\mathscr{S}\langle i\rangle \to$ rep-$\sigma_i\mathscr{S}\langle i\rangle$ is an equivalence and it is exact and induces isomorphisms on both Hom and Ext. The assertion for σ_i^- : rep-$\mathscr{S}\langle i\rangle \to$ rep-$\sigma_i\mathscr{S}\langle i\rangle$ is the same if i is a source.

2.8. Let A be a finite dimensional hereditary k-algebra over a finite field k, $\mathscr{P}$ the set of isomorphism classes of finite dimensional A-modules, and $I \subset \mathscr{P}$ the set of isomorphism classes of simple A-modules. We choose a representative $V_\alpha \in \alpha$ for any $\alpha \in \mathscr{P}$. By abuse of notation, we write

$$\langle \alpha, \beta \rangle = \langle \mathbf{dim}\, V_\alpha, \mathbf{dim}\, V_\beta \rangle \qquad \text{and}$$

$$(\alpha, \beta) = (\mathbf{dim}\, V_\alpha, \mathbf{dim}\, V_\beta) \text{ for } \alpha, \beta \in \mathscr{P}.$$

So the Euler form $\langle -, - \rangle$ and its symmetrization $(-, -)$ are defined on $\mathbb{Z}[I]$.

Obviously, the fundamental reflection $s_i: \mathbb{Q}^\Gamma \to \mathbb{Q}^\Gamma$ preserves the Euler form and $s_i(\mathbf{dim}\, V_\alpha) = \mathbf{dim}\, V_{\sigma_i^+\alpha}$ for $V_\alpha \in$ rep-$\mathscr{S}\langle i\rangle$. The following is easily seen

2.8.1. LEMMA. *Let i be a sink and let $V_\alpha \in$ rep-$\mathscr{S}\langle i\rangle$. Then*

$$\langle \alpha, e_i \rangle = -\langle \sigma_i^+ \alpha, e_i \rangle \qquad \textit{and} \qquad (\alpha, e_i) = -(\sigma_i^+ \alpha, e_i).$$

2.8.2. *Remark.* From Lemma 2.8.1, if i is a sink and V_i the simple module with $\mathbf{dim}\, V_i = e_i$, then V_i is simple projective in rep-$\mathscr{S}$ and simple injective in rep-$\sigma_i\mathscr{S}$. Let $V_\alpha \in$ rep-$\mathscr{S}\langle i\rangle$. Then

$$\dim_k \operatorname{Ext}_{\sigma_i A}(V_{\sigma_i^+\alpha}, V_i) = 0 \qquad \text{and} \qquad \operatorname{Hom}_A(V_\alpha, V_i) = 0.$$

Hence we have

$$\dim_k \operatorname{Hom}_{\sigma_i A}(V_{\sigma_i^+\alpha}, V_i) = \dim_k \operatorname{Ext}_A(V_\alpha, V_i).$$

For $\alpha, \beta, \lambda \in \mathscr{P}$, let $g_{\alpha\beta}^{\lambda}$ be the number of submodules B of V_λ such that $B \cong V_\beta$ and $V_\lambda / B \cong V_\alpha$. More generally, given $\alpha_1, \dots, \alpha_t, \lambda \in \mathscr{P}$, let $g_{\alpha_1, \dots, \alpha_t}^{\lambda}$ be the number of filtrations

$$V_\lambda = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_t = 0$$

such that M_{i-1}/M_i is isomorphic to V_{α_i} for all $1 \le i \le t$. We use a_α to denote the order of the automorphism group of V_α for $\alpha \in \mathscr{P}$.

2.8.3. LEMMA. *Let $i \in I$ be a sink of Ω and V_α and V_β be in rep-$\mathscr{S}\langle i \rangle$. Then we have*

$$g_{\beta i}^{\alpha} = \frac{a_\alpha}{a_\beta} g_{i \sigma_i^+ \alpha}^{\sigma_i^+ \beta}.$$

Moreover, $a_\beta g_{\beta t i}^{\alpha} = a_\alpha g_{t i \sigma_i^+ \alpha}^{\sigma_i^+ \beta}$.

Proof. See [R5, 5.2]. ∎

2.9. Let $q = |k|$, $v = \sqrt{q}$ (hence $q = v^2$), and $\mathbb{Q}(v)$ be the rational function field of v. The Hall algebra $\mathfrak{h}(A)$ is by definition the free $\mathbb{Q}(v)$-module with the basis $\{u_\alpha \mid \alpha \in \mathscr{P}\}$ and the multiplication given by

$$u_\alpha u_\beta = v^{\langle \alpha, \beta \rangle} \sum_{\lambda \in \mathscr{P}} g_{\alpha\beta}^{\lambda} u_\lambda$$

for all $\alpha, \beta \in \mathscr{P}$.

Let A be the tensor algebra of a k-species $\mathscr{S}$. We can identify mod-A = rep-$\mathscr{S}$; therefore, $\mathfrak{h}(A)$ can be viewed as being defined for rep-$\mathscr{S}$. Also, we denote by $\sigma_i A$ the tensor algebra of $\sigma_i \mathscr{S}$. We define $\mathfrak{h}(A)\langle i \rangle$ to be the $\mathbb{Q}(v)$-subspace of $\mathfrak{h}(A)$ generated by u_α with $V_\alpha \in$ rep-$\mathscr{S}\langle i \rangle$. If i is a sink or source, since rep-$\mathscr{S}\langle i \rangle$ is closed under extensions, $\mathfrak{h}(A)\langle i \rangle$ is a subalgebra of $\mathfrak{h}(A)$. Because σ_i^+ : rep-$\mathscr{S}\langle i \rangle \to$ rep-$\sigma_i\mathscr{S}\langle i \rangle$ is an exact equivalent and induces isomorphisms on both Hom and Ext, it is not difficult to see the following result of Ringel [R3, Theorem 5].

PROPOSITION. *Let i be a sink. The functor σ_i^+ yields an $\mathbb{Q}(v)$-algebra isomorphism $\sigma_i : \mathfrak{h}(A)\langle i \rangle \to \mathfrak{h}(\sigma_i A)\langle i \rangle$ with $\sigma_i(u_\alpha) = u_{\sigma_i^+ \alpha}$ for any $V_\alpha \in$ rep-$\mathscr{S}\langle i \rangle$.*

Of course, we have a dual statement for i being a source.

2.10. In the quantum group and the Hall algebra, the following notations and relations are often used:

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}} = v^{n-1} + v^{n-3} + \cdots + v^{-n+1},$$

$$[n]! = \prod_{r=1}^{n} [r], \qquad \begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[r]![n-r]!},$$

and also

$$|n] = \frac{q^n - 1}{q - 1}(q^{n-1} + \cdots + q + 1) = v^{n-1}[n],$$

$$|n]! = \prod_{t=1}^{n} |t] = v^{\binom{n}{2}}[n]!,$$

$$\left|\begin{matrix} n \\ t \end{matrix}\right] = \frac{|n]!}{|t]!|n-t]!} = v^{t(n-t)}\begin{bmatrix} n \\ t \end{bmatrix}.$$

The following equations are basic ones.

LEMMA. *For $n > 0$, we have*

$$\sum_{t=0}^{n} (-1)^t v^{t(t-1)} \left|\begin{matrix} n \\ t \end{matrix}\right] = 0 \qquad \textit{and} \qquad \sum_{t=0}^{n} (-1)^t v^{t(n-1)} \begin{bmatrix} n \\ t \end{bmatrix} = 0.$$

If $f(v)$ is a rational function of v, then $f(v)_\alpha$ means $f(v^{\varepsilon(\alpha)})$.

2.11. We denote by $\mathfrak{c}(A)$ the $\mathbb{Q}(v)$-subalgebra of $\mathfrak{h}(A)$ which is generated by u_i, $i \in I$, where $\{V_i \mid i \in I\}$ is a complete set of paradise non-isomorphic simple A-modules, and $\mathfrak{c}(A)$ is called the composition algebra. The following well-known result of Green and Ringel (see [G, R1]) laid down a base for our investigation.

THEOREM. *There exists an isomorphism $\eta : U^+ \to \mathfrak{c}(A)$ of $\mathbb{Q}(v)$-algebras such that $\eta(E_i) = u_i$ for $i \in I$, where U^+ is the positive part of the quantum group $U_q(\mathfrak{g})$.*

3. DOUBLE RINGEL–HALL ALGEBRAS AND SOME DERIVATIONS

3.1. For the basic concepts of Hopf algebras, the readers can be referred to [A]. Given $\mathbb{Q}(v)$-Hopf algebras $\mathscr{H}^+$ and $\mathscr{H}^-$, a skew-Hopf pairing of $\mathscr{H}^+$ and $\mathscr{H}^-$ is a $\mathbb{Q}(v)$-bilinear function $\varphi : \mathscr{H}^+ \times \mathscr{H}^- \to \mathbb{Q}(v)$ satisfying

(i) $\varphi(1, b) = \varepsilon(b)$, $\varphi(a, 1) = \varepsilon(a)$,

(ii) $\varphi(a, bb') = \varphi(\Delta(a), b \otimes b')$,

(iii) $\varphi(aa', b) = \varphi(a \otimes a', \Delta^{opp}(b))$,

(iv) $\varphi(S(a), b) = \varphi(a, S^{-1}(b))$,

where Δ, ε, and S are the comultiplication, counit, and antipode, respectively. The $\mathscr{H}^+ \otimes \mathscr{H}^-$ has the induced Hopf algebra structure in the following sense, which is the so-called Drinfeld double of $(\mathscr{H}^+, \mathscr{H}^-, \varphi)$, denoted by $\mathscr{D}(\mathscr{H}^+, \mathscr{H}^-)$.

(1) Multiplications,

(A) $(a \otimes 1)(a' \otimes 1) = aa' \otimes 1$,

(B) $(1 \otimes b)(1 \otimes b') = 1 \otimes bb'$

(C) $(a \otimes 1)(1 \otimes b) = a \otimes b$

(D) $(1 \otimes b)(a \otimes 1) = \sum_{(a),(b)} \varphi(a_1, S(b_1))a_2 \otimes b_2 \varphi(a_3, b_3)$

for all $a, a' \in \mathscr{H}^+, b, b' \in \mathscr{H}^-$, where

$$\Delta^2(a) = \sum_{(a)} a_1 \otimes a_2 \otimes a_3 \qquad \text{and} \qquad \Delta^2(b) = \sum_{(b)} b_1 \otimes b_2 \otimes b_3.$$

The unit is $1 \otimes 1$.

(2) Comultiplications,

$$\Delta_{\mathscr{H}^+ \otimes \mathscr{H}^-}(a \otimes b) = \sum_{(a),(b)} (a_1 \otimes b_1) \otimes (a_2 \otimes b_2),$$

and the co-unit is $\varepsilon_{\mathscr{H}^+} \otimes \varepsilon_{\mathscr{H}^-}$.

(3) Antipode,

$$S_{\mathscr{H}^+ \otimes \mathscr{H}^-}(a \otimes b) = (1 \otimes S(b))(S(a) \otimes 1).$$

For a proof of the above, see [Jo], for example.

By a routine technique of Hopf algebras, the hypothesis (D) of (1) can be replaced by

$$(D') \qquad \sum_{(a),(b)} b_2 \otimes a_2 \varphi(a_1, b_1) = \sum_{(a),(b)} a_1 \otimes b_1 \varphi(a_2, b_2),$$

for all $a \in \mathscr{H}^+$ and $b \in \mathscr{H}^-$, where $\Delta(a) = \sum_{(a)} a_1 \otimes a_2$ and $\Delta(b) = \sum_{(b)} b_1 \otimes b_2$.

3.2. Let A be the tensor algebra of a k-species $\mathscr{S}$, $\mathscr{P}_1 = \mathscr{P} - \{0\}$. In the Ringel–Hall algebra $\mathfrak{h}(A)$, we write $\langle u_\alpha \rangle = v^{-\dim V_\alpha + \varepsilon(\alpha)} u_\alpha$ for each $\alpha \in \mathscr{P}$ (noting that $\langle u_i \rangle = u_i$ for all $i \in I$). Then it is easy to see the multiplication of $\mathfrak{h}(A)$ can be replaced by

$$\langle u_\alpha \rangle \langle u_\beta \rangle = v^{-\langle \beta, \alpha \rangle} \sum_{\lambda \in \mathscr{P}} g_{\alpha\beta}^{\lambda} \langle u_\lambda \rangle \qquad \text{for all } \alpha, \beta \in \mathscr{P}.$$

Furthermore, we can introduce the extended Ringel–Hall algebra $\mathscr{H}(A)$. Let $\mathscr{H}(A)$ be the free $\mathbb{Q}(v)$-module with the basis

$$\{K_\alpha \langle u_\lambda \rangle \mid \alpha \in \mathbb{Z}[I], \lambda \in \mathscr{P}\}.$$

Then we can rewrite Theorem 4.5 of [X] as follows

THEOREM. *The Hopf algebra structure of $\mathscr{H}(A)$ is given by the following operations.*

(a) (Ringel) *Multiplication,*

$$\langle u_\alpha\rangle\langle u_\beta\rangle = v^{-\langle \beta, \alpha\rangle} \sum_{\lambda\in\mathscr{P}} g^\lambda_{\alpha\beta}\langle u_\lambda\rangle \qquad \textit{for all } \alpha, \beta \in \mathscr{P}$$

$$K_\alpha\langle u_\beta\rangle = v^{(\alpha,\beta)}\langle u_\beta\rangle K_\alpha \qquad \textit{for all } \alpha \in \mathbb{Z}[I], \beta\in\mathscr{P},$$

$$K_\alpha K_\beta = K_{\alpha+\beta} \qquad \textit{for all } \alpha, \beta \in \mathbb{Z}[I],$$

with unit $1 = u_0 = K_0$.

(b) (Green) *Comultiplication,*

$$\Delta(\langle u_\lambda\rangle) = \sum_{\alpha,\beta\in\mathscr{P}} v^{\langle\alpha,\beta\rangle} g^\lambda_{\alpha\beta}\frac{a_\alpha a_\beta}{a_\lambda} K_\beta\langle u_\alpha\rangle \otimes \langle u_\beta\rangle \qquad \textit{for all } \lambda\in\mathscr{P},$$

$$\Delta(K_\alpha) = K_\alpha\otimes K_\alpha \qquad \textit{for all } \alpha\in\mathbb{Z}[I]$$

with co-unit $\varepsilon(\langle u_\lambda\rangle) = 0$ *for* $\lambda \neq 0$ *in* $\mathscr{P}$ *and* $\varepsilon(K_\alpha) = 1$ *for all* $\alpha \in \mathbb{Z}[I]$.

(c) *Antipode,*

$$S(\langle u_\lambda\rangle) = \delta_{\lambda 0} + \sum_{m\geq 1}(-1)^m \sum_{\lambda_1,\dots,\lambda_m\in\mathscr{P}_1} v^{\sum_{i<j}\langle\lambda_i,\lambda_j\rangle} g^\lambda_{\lambda_1\lambda_2\cdots\lambda_m}\frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_\lambda}$$

$$\times\left(K_{-\lambda_1}\langle u_{\lambda_1}\rangle\right)\cdots\left(K_{-\lambda_m}\langle u_{\lambda_m}\rangle\right), \qquad \textit{for all } \lambda\in\mathscr{P}.$$

$$S(K_\alpha) = K_{-\alpha} \qquad \textit{for all } \alpha\in\mathbb{Z}[I],$$

where $\delta_{\lambda 0}$ *is the Kronecker sign.*

3.3. For any $\mu \in \mathbb{N}[I]$, let $\mathscr{H}_\mu$ be the $\mathbb{Q}(v)$-submodule of $\mathscr{H}(A)$ with the basis $\{K_\alpha\langle u_\mu\rangle \mid \alpha\in\mathbb{Z}[I], \mathbf{dim}\, V_\mu = \mu\}$. So $\mathscr{H}(A) = \bigoplus_{\mu\in\mathbb{N}[I]}\mathscr{H}_\mu$ is an $\mathbb{N}[I]$-graded algebra and Theorem 3.2 implies that for any $\mu\in\mathbb{N}[I]$

$$\Delta(\mathscr{H}_\mu) \subseteq \bigoplus_{0\leq\eta\leq\mu}\mathscr{H}_{\mu-\eta}\otimes\mathscr{H}_\mu.$$

Accordingly, we can define for any $\alpha\in\mathscr{P}$, the following operations on $\mathfrak{h}(A)$

$$r_\alpha(\langle u_\lambda\rangle) = \sum_{\beta\in\mathscr{P}} v^{\langle\beta,\alpha\rangle+(\alpha,\beta)} g^\lambda_{\beta\alpha}\frac{a_\beta a_\alpha}{a_\lambda}\langle u_\beta\rangle$$

$$r'_\alpha(\langle u_\lambda\rangle) = \sum_{\beta\in\mathscr{P}} v^{\langle\alpha,\beta\rangle+(\alpha,\beta)} g^\lambda_{\alpha\beta}\frac{a_\alpha a_\beta}{a_\lambda}\langle u_\beta\rangle$$

for all $\lambda \in \mathscr{P}$. In particular, $r_i(1) = r_i'(1) = 0$ and $r_i(\langle u_j \rangle) = \delta_{ij} = r_i'(\langle u_j \rangle)$ for all $i, j \in I$.

PROPOSITION. *For any $i \in I$ and $\lambda_1, \lambda_2 \in \mathscr{P}$, we have*

(1) $r_i(\langle u_{\lambda_1} \rangle \langle u_{\lambda_2} \rangle) = \langle u_{\lambda_1} \rangle r_i(\langle u_{\lambda_2} \rangle) + v^{(i,\lambda_2)} r_i(\langle u_{\lambda_1} \rangle) \langle u_{\lambda_2} \rangle$

(2) $r_i'(\langle u_{\lambda_1} \rangle \langle u_{\lambda_2} \rangle) = v^{(i,\lambda_1)} \langle u_{\lambda_1} \rangle r_i'(\langle u_{\lambda_2} \rangle) + r_i'(\langle u_{\lambda_1} \rangle) \langle u_{\lambda_2} \rangle$.

Proof. It is more or less the same as [CX, Proposition 3.2]. ∎

For this reason, the operations r_i and r_i' are called the right and left derivations on $\mathfrak{h}(A)$, respectively, for any $i \in I$ (see [L1, 1.2.13]).

3.4. Let $\mathscr{H}^+(A)$ just be the Hopf algebra $\mathscr{H}(A)$ but we write $\langle u_\lambda^+ \rangle$ for $\langle u_\lambda \rangle$ for all $\lambda \in \mathscr{P}$. Therefore the Hopf algebra structure of $\mathscr{H}^+(A)$ is given as in Theorem 3.2 and the operations r_α and r_α' for $\alpha \in \mathscr{P}$ are defined as in 3.3.

Dually, let $\mathscr{H}^-(A)$ be the free $\mathbb{Q}(v)$-module with the basis $\{K_\alpha \langle u_\lambda^- \rangle \mid \alpha \in \mathbb{Z}[I], \lambda \in \mathscr{P}\}$. The Hopf algebra of $\mathscr{H}^-(A)$ can be given as follows:

(a) Multiplication,

$$\langle u_\alpha^- \rangle \langle u_\beta^- \rangle = v^{-\langle \beta, \alpha \rangle} \sum_{\lambda \in \mathscr{P}} g_{\alpha\beta}^\lambda \langle u_\lambda^- \rangle \qquad \text{for all } \alpha, \beta \in \mathscr{P}$$

$$K_\alpha \langle u_\beta^- \rangle = v^{-(\alpha,\beta)} \langle u_\beta^- \rangle K_\alpha \qquad \text{for all } \alpha \in \mathbb{Z}[I], \beta \in \mathscr{P},$$

$$K_\alpha K_\beta = K_{\alpha+\beta} \qquad \text{for all } \alpha, \beta \in \mathbb{Z}[I],$$

with unit $1 = u_0 = K_0$.

(b) Comultiplication,

$$\Delta(\langle u_\lambda^- \rangle) = \sum_{\alpha, \beta \in \mathscr{P}} v^{\langle \alpha, \beta \rangle} g_{\alpha\beta}^\lambda \frac{a_\alpha a_\beta}{a_\lambda} \langle u_\beta^- \rangle \otimes K_{-\beta} \langle u_\alpha^- \rangle \qquad \text{for all } \lambda \in \mathscr{P},$$

$$\Delta(K_\alpha) = K_\alpha \otimes K_\alpha \qquad \text{for all } \alpha \in \mathbb{Z}[I]$$

with co-unit $\varepsilon(\langle u_\lambda^- \rangle) = 0$ for $\lambda \in \mathscr{P}_1$ and $\varepsilon(K_\alpha) = 1$ for all $\alpha \in \mathbb{Z}[I]$.

(c) Antipode,

$$S(\langle u_\lambda^- \rangle) = \delta_{\lambda 0} + \sum_{m \geq 1} (-1)^m \sum_{\lambda_1, \dots, \lambda_m \in \mathscr{P}_1} v^{\sum_{i<j} \langle \lambda_i, \lambda_j \rangle} g_{\lambda_1 \lambda_2 \cdots \lambda_m}^\lambda \frac{a_{\lambda_1} \cdots a_{\lambda_m}}{a_\lambda}$$

$$\times \langle u_{\lambda_m}^- \rangle \cdots \langle u_{\lambda_1}^- \rangle K_\lambda, \qquad \text{for all } \lambda \in \mathscr{P}$$

$$S(K_\alpha) = K_{-\alpha} \qquad \text{for all } \alpha \in \mathbb{Z}[I],$$

3.5. Of course, we also have the following operations for all $\alpha \in \mathscr{P}$,

$$r_\alpha(\langle u_\lambda^- \rangle) = \sum_{\beta \in \mathscr{P}} v^{\langle \alpha, \beta \rangle + (\alpha, \beta)} g_{\alpha\beta}^\lambda \frac{a_\alpha a_\beta}{a_\lambda} \langle u_\beta^- \rangle$$

$$r'_\alpha(\langle u_\lambda^- \rangle) = \sum_{\beta \in \mathscr{P}} v^{\langle \beta, \alpha \rangle + (\alpha, \beta)} g_{\beta\alpha}^\lambda \frac{a_\beta a_\alpha}{a_\lambda} \langle u_\beta^- \rangle,$$

for any $\lambda \in \mathscr{P}$. In particular

$$r_i(1) = r'_i(1) = 0 \qquad \text{and} \qquad r_i(\langle u_j^- \rangle) = r'_i(\langle u_j^- \rangle) = \delta_{ij}$$

for all $j \in I$. Similarly, we have the following

PROPOSITION. *For $i \in I$ and $\lambda_1, \lambda_2 \in \mathscr{P}$,*

(1) $r_i(\langle u_{\lambda_1}^- \rangle \langle u_{\lambda_2}^- \rangle) = v^{(i, \lambda_1)} \langle u_{\lambda_1}^- \rangle r_i(\langle u_{\lambda_2}^- \rangle) + r_i(\langle u_{\lambda_1}^- \rangle) \langle u_{\lambda_2}^- \rangle$.

(2) $r'_i(\langle u_{\lambda_1}^- \rangle \langle u_{\lambda_2}^- \rangle) = \langle u_{\lambda_1}^- \rangle r'_i(\langle u_{\lambda_2}^- \rangle) + v^{(i, \lambda_2)} r'_i(\langle u_{\lambda_1}^- \rangle) \langle u_{\lambda_2}^- \rangle$.

3.6. In view of Proposition 5.3 in [X], the bilinear form $\varphi : \mathscr{H}^+(A) \times \mathscr{H}^-(A) \to \mathbb{Q}(v)$, defined by

$$\varphi(K_\alpha \langle u_\beta^+ \rangle, K_{\alpha'} \langle u_{\beta'}^- \rangle) = v^{-(\alpha, \alpha') - (\beta, \alpha') + (\alpha, \beta') + (\beta, \beta)} a_\beta^{-1} \delta_{\beta\beta'}$$

for all $\alpha, \alpha' \in \mathbb{Z}[I]$, $\beta, \beta' \in \mathscr{P}$, is a skew Hopf pairing. Therefore, we have the Drinfeld double $D(\mathscr{H}^+(A), \mathscr{H}^-(A))$ of φ, which is a Hopf algebra structure of $\mathscr{H}^+(A) \otimes \mathscr{H}^-(A)$ (see 3.1). It is clear that the ideal of $D(\mathscr{H}^+(A), \mathscr{H}^-(A))$ generated by $K_\alpha \otimes K_{-\alpha} - 1$, or equivalently, by $K_\alpha \otimes 1 - 1 \otimes K_\alpha$ for all $\alpha \in \mathbb{Z}[I]$, is a Hopf ideal. The corresponding quotient inherits a Hopf algebra structure, which is called the reduced Drinfeld double of A and denoted by $\mathscr{D}(A)$.

As an associative algebra, $\mathscr{D}(A)$ is given by the following defining relations:

(1) $K_0 = u_0 = 1$, $K_\gamma K_\eta = K_{\gamma + \eta}$

(2) $\langle u_\alpha^+ \rangle \langle u_\beta^+ \rangle = v^{-\langle \beta, \alpha \rangle} \sum_{\lambda \in \mathscr{P}} g_{\alpha\beta}^\lambda \langle u_\lambda^+ \rangle$

(3) $\langle u_\alpha^- \rangle \langle u_\beta^- \rangle = v^{-\langle \beta, \alpha \rangle} \sum_{\lambda \in \mathscr{P}} g_{\alpha\beta}^\lambda \langle u_\lambda^- \rangle$

(4) $K_\gamma \langle u_\beta^+ \rangle = v^{(\gamma, \beta)} \langle u_\beta^+ \rangle K_\gamma$

(5) $K_\gamma \langle u_\beta^- \rangle = v^{-(\gamma, \beta)} \langle u_\beta^- \rangle K_\gamma$

(6) $\sum_{\alpha', \alpha} v^{\langle \alpha', \alpha \rangle + (\alpha, \alpha)} (a_{\alpha'}/a_{\lambda'}) g_{\alpha'\alpha}^{\lambda'} K_{-\alpha} \langle u_{\alpha'}^- \rangle r'_\alpha(\langle u_\lambda^+ \rangle) = \sum_{\alpha, \beta} v^{\langle \alpha, \beta \rangle + (\beta, \beta)} (a_\alpha/a_\lambda) g_{\alpha\beta}^\lambda K_\beta \langle u_\alpha^+ \rangle r_\beta(\langle u_{\lambda'}^- \rangle)$

for all $\lambda, \lambda', \alpha, \beta \in \mathscr{P}$, and $\gamma, \eta \in \mathbb{Z}[I]$. The relations (1)–(5) are obtained from the defining relations of $\mathscr{H}^+(A)$ and $\mathscr{H}^-(A)$, and the relation (6) follows from the relation (D') in 3.1 which is equivalent to the relation (D) in 3.1.

It is immediate to see the following

3.7. LEMMA. *For any* $\lambda_1, \lambda_2 \in \mathscr{P}$ *and* $i \in I$, *we have*

(1) $\varphi(\langle u_{\lambda_1}^+\rangle, \langle u_i^-\rangle\langle u_{\lambda_2}^-\rangle) = \varphi(\langle u_i^+\rangle, \langle u_i^-\rangle)\varphi(r_i'(\langle u_{\lambda_1}^+\rangle), \langle u_{\lambda_2}^-\rangle)$ *and* $\varphi(\langle u_{\lambda_1}^+\rangle, \langle u_{\lambda_2}^-\rangle\langle u_i^-\rangle) = \varphi(\langle u_i^+\rangle, \langle u_i^-\rangle)\varphi(r_i(\langle u_{\lambda_1}^+\rangle), \langle u_{\lambda_2}^-\rangle)$.

(2) $\varphi(\langle u_i^+\rangle\langle u_{\lambda_1}^+\rangle, \langle u_{\lambda_2}^-\rangle) = \varphi(\langle u_i^+\rangle, \langle u_i^-\rangle)\varphi(\langle u_{\lambda_1}^+\rangle, r_i(\langle u_{\lambda_2}^-\rangle))$ *and* $\varphi(\langle u_{\lambda_1}^+\rangle\langle u_i^+\rangle, \langle u_{\lambda_2}^-\rangle) = \varphi(\langle u_i^+\rangle, \langle u_i^-\rangle)\varphi(\langle u_{\lambda_1}^+\rangle, r_i'(\langle u_{\lambda_2}^-\rangle))$.

The following formulae for the commutators seem well known

3.8. PROPOSITION. *For any* $\lambda \in \mathscr{P}$ *and* $i \in I$, *we have the formulae in* $\mathscr{D}(A)$

(1) $\langle u_i^-\rangle\langle u_\lambda^+\rangle - \langle u_\lambda^+\rangle\langle u_i^-\rangle = \varphi(\langle u_i^+\rangle, \langle u_i^-\rangle)(r_i(\langle u_\lambda^+\rangle)K_i - K_{-i}r_i'(\langle u_\lambda^+\rangle))$

(2) $\langle u_\lambda^-\rangle\langle u_i^+\rangle - \langle u_i^+\rangle\langle u_\lambda^-\rangle = \varphi(\langle u_i^+\rangle, \langle u_i^-\rangle)(K_i r_i(\langle u_\lambda^-\rangle) - r_i'(\langle u_\lambda^-\rangle)K_{-i})$.

Proof. It is straightforward or see Corollary 5.5.2 of [X]. ∎

3.9. We consider the composition algebra $\mathscr{C}^+(A)$, which is the subalgebra of $\mathscr{H}^+(A)$ generated by the elements $\langle u_i^+\rangle$, $i \in I$, and K_α, $\alpha \in \mathbb{Z}[I]$. Dually, the composition algebra $\mathscr{C}^-(A)$ is the subalgebra of $\mathscr{H}^-(A)$ generated by the elements $\langle u_i^-\rangle$, $i \in I$, and K_α, $\alpha \in \mathbb{Z}[I]$. Obviously, they are Hopf subalgebras of $\mathscr{H}^+(A)$ and $\mathscr{H}^-(A)$, respectively. We can restrict the pairing

$$\varphi : \mathscr{H}^+(A) \times \mathscr{H}^-(A) \to \mathbb{Q}(v)$$

to their composition algebras; this restriction

$$\varphi : \mathscr{C}^+(A) \times \mathscr{C}^-(A) \to \mathbb{Q}(v)$$

is easily seen to be a skew-Hopf pairing belonging to the Cartan datum $\Delta = (I, (\ ,\ ))$ (see Section 2 of [X]). Therefore we have the reduced Drinfeld double of the skew-Hopf pairing $(\mathscr{C}^+(A), \mathscr{C}^-(A), \varphi)$, which we denote by $\mathscr{D}_c(A)$. Obviously $\mathscr{D}_c(A)$ is a Hopf subalgebra of $\mathscr{D}(A)$ generated by $u_i^\pm$, $i \in I$, and K_α, $\alpha \in \mathbb{Z}[I]$. By construction one has the triangular decomposition

$$\mathscr{D}_c(A) = \mathfrak{c}^-(A) \otimes T \otimes \mathfrak{c}^+(A),$$

where $\mathfrak{c}^-(A)$ is the subalgebra generated by u_i^-, $i \in I$, $\mathfrak{c}^+(A)$ the subalgebra generated by u_i^+, $i \in I$, and T the torus algebra.

3.10. Let $\Delta = (I, (\,,\,))$ be a Cartan datum, A a finite dimensional hereditary k-algebra corresponding to Δ, and $U_q(\mathfrak{g})$ the quantum group corresponding to Δ. The Green–Ringel Theorem in 2.11 can be generalized to the Drinfeld double (see [X, Theorem 5.8]).

THEOREM. *The map $\theta : \mathscr{D}_c(A) \to U_q(\mathfrak{g})$ by sending*

$$\langle u_i^+ \rangle \to E_i, \langle u_i^- \rangle \to -v_i F_i, K_i \to \tilde{K}_i$$

for all $i \in I$ induces an isomorphism as Hopf $\mathbb{Q}(v)$-algebras, where $\tilde{K}_i = K_i^{\varepsilon_i}$ and the notations for elements of $U_q(\mathfrak{g})$ are as in [L1, Chap. 3].

4. BGP-REFLECTION OPERATORS FOR DOUBLE RINGEL–HALL ALGEBRAS

4.1. All notations are conserved as noted above. Throughout this section except in 4.6, we always assume that i is a sink for Ω, and σ_i^+ the Bernstein–Gelfand–Ponomarev reflection functor as defined as in 2.6. Then $\sigma_i^+ : \text{rep-}\mathscr{S}\langle i\rangle \to \text{rep-}\sigma_i\mathscr{S}\langle i\rangle$ is an equivalence. Therefore, by Proposition 2.9 (of Ringel), the morphism $T_i : \mathfrak{h}\langle i\rangle \to \mathfrak{h}(\sigma_i A)\langle i\rangle$ by taking $T_i(\langle u_\lambda\rangle) = \langle u_{\sigma_i^+\lambda}\rangle$ for $\lambda \in \mathscr{P}$ is a $\mathbb{Q}(v)$-algebra isomorphism.

The aim of this section is to extend the map T_i to the whole reduced Drinfeld double $\mathscr{D}(A)$.

4.2. Let $\overline{K}_i = v^{-\varepsilon(i)}K_i, \langle u_\alpha\rangle^{(t)} = \langle u_\alpha\rangle^t/([t]!)_\alpha$ for $\alpha \in \mathscr{P}$ and $t \in \mathbb{N}$. If $V_\lambda = V_\alpha \oplus V_\beta$ in rep-$\mathscr{S}$ and $\text{Hom}(V_\beta, V_\alpha) = 0 = \text{Ext}(V_\alpha, V_\beta)$, then it is easy to see that $\langle u_\lambda\rangle = v^{\langle \beta, \alpha\rangle}\langle u_\alpha\rangle\langle u_\beta\rangle$ in $\mathfrak{h}(A)$, and if $\text{Ext}(V_\alpha, V_\alpha) = 0$, then $\langle u_{t\alpha}\rangle = \langle u_\alpha\rangle^{(t)}$, where $u_{t\alpha}$ is the vector of the t copies of V_α in $\mathfrak{h}(A)$ (see [R3]).

For $\lambda \in \mathscr{P}$, assume that $V_\lambda = V_{\lambda_0} \oplus tV_i$ and V_{λ_0} contains no direct summand isomorphic to V_i. Then $\text{Hom}(V_{\lambda_0}, V_i) = 0$ and $\text{Ext}(V_i, V_{\lambda_0}) = 0$ since i is a sink of $\mathscr{S}$. In this case,

$$\langle u_\lambda^+\rangle = v^{\langle \lambda_0, ti\rangle}\langle u_i^+\rangle^{(t)}\langle u_{\lambda_0}^+\rangle$$

in $\mathfrak{h}^+(A)$. We define a morphism $T_i : \mathfrak{h}^+(A) \to \mathscr{D}(\sigma_i A)$ given by

$$T_i(\langle u_\lambda^+\rangle) = \frac{v^{\langle \lambda_0, ti\rangle}}{[t]!_i}\big(\langle u_i^-\rangle \overline{K}_i\big)^t\langle u_{\sigma_i^+\lambda_0}^+\rangle$$

$$= v^{\langle \lambda, ti\rangle}K_{ti}\langle u_i^-\rangle^{(t)}\langle u_{\sigma_i^+\lambda_0}^+\rangle$$

for all $\lambda \in \mathscr{P}$, where $V_\lambda = V_{\lambda_0} \oplus tV_i$ and V_{λ_0} contains no direct summand isomorphic to V_i. For convenience, we write σ_i for σ_i^+ below. By definition we have

$$T_i\big(\langle u_i^+\rangle^{(t)}\langle u_{\lambda_0}^+\rangle\big) = T_i\big(\langle u_i^+\rangle^{(t)}\big)T_i\big(\langle u_{\lambda_0}^+\rangle\big).$$

In particular

$$T_i(\langle u_i^+\rangle) = \langle u_i^-\rangle \bar{K}_i.$$

In fact, we have the following

4.2.1. LEMMA. *For any $\lambda \in \mathscr{P}$ and $m \in \mathbb{N}$,*

$$T_i\big(\langle u_i^+\rangle^{(m)}\langle u_\lambda^+\rangle\big) = T_i\big(\langle u_i^+\rangle^{(m)}\big)T_i\big(\langle u_\lambda^+\rangle\big).$$

Proof. We write $V_\lambda = V_{\lambda_0} \oplus tV_i$ as above. Then

$$\begin{aligned}
T_i\big(\langle u_i^+\rangle^{(m)}\langle u_\lambda^+\rangle\big) &= v^{\langle \lambda_0, ti\rangle} T_i\Big(\langle u_i^+\rangle^{(m)}\langle u_i^+\rangle^{(t)}\langle u_{\lambda_0}^+\rangle\Big) \\
&= v^{\langle \lambda_0, ti\rangle}\begin{bmatrix} s+t \\ m \end{bmatrix}_i T_i\Big(\langle u_i^+\rangle^{(m+t)}\langle u_{\lambda_0}^+\rangle\Big) \\
&= \frac{v^{\langle \lambda_0, ti\rangle}}{[m]!_i[t]!_i}\big(\langle u_i^-\rangle\bar{K}_i\big)^{m+t}\langle u_{\sigma_i^+\lambda_0}^+\rangle\big) \\
&= \frac{1}{[m]!_i}\big(\langle u_i^-\rangle\bar{K}_i\big)^m\left(\frac{v^{\langle \lambda_0, ti\rangle}}{[t]!_i}\big(\langle u_i^-\rangle\bar{K}_i\big)^t\langle u_{\sigma_i^+\lambda_0}^+\rangle\right) \\
&= T_i\big(\langle u_i^+\rangle^{(m)}\big)T_i\big(\langle u_\lambda^+\rangle\big).
\end{aligned}$$

∎

4.2.2. LEMMA. *For any $\beta \in \mathscr{P}$ and $m \in \mathbb{N}$,*

$$T_i\big(\langle u_\beta^+\rangle\langle u_i^+\rangle^{(m)}\big) = T_i\big(\langle u_\beta^+\rangle\big)T_i\big(\langle u_i^+\rangle^{(m)}\big).$$

Proof. By Lemma 4.2.1, it suffices to prove the lemma for the case where V_β does not contain V_i as a direct summand. So we assume that V_i is not a direct summand of V_β. It is easy to see that (see [R3, Theorem 1])

$$\langle u_\beta^+\rangle\langle u_i^+\rangle = v^{(i,\beta)}\langle u_i^+\rangle\langle u_\beta^+\rangle + v^{-\langle i,\beta\rangle}\sum_{\alpha \neq \beta \oplus i} g_{\beta i}^{\alpha}\langle u_\alpha^+\rangle.$$

Therefore,

$$T_i\big(\langle u_\beta^+\rangle\langle u_i^+\rangle\big) = v^{(i,\beta)}T_i\big(\langle u_i^+\rangle\big)T_i\big(\langle u_\beta^+\rangle\big) + v^{-\langle i,\beta\rangle}\sum_{\alpha \neq \beta \oplus i} g_{\beta i}^{\alpha}T_i\big(\langle u_\alpha^+\rangle\big).$$

On the other hand,

$$T_i\big(\langle u_\beta^+\rangle\big)T_i\big(\langle u_i^+\rangle\big) = \langle u_{\sigma_i\beta}^+\rangle\langle u_i^-\rangle\bar{K}_i$$

and

$$T_i(\langle u_i^+\rangle)T_i(\langle u_\beta^+\rangle) = (\langle u_i^-\rangle\overline{K}_i)\langle u_{\sigma_i\beta}^+\rangle.$$

Thus, to prove $T_i(\langle u_\beta^+\rangle\langle u_i^+\rangle) = T_i(\langle u_\beta^+\rangle)T_i(\langle u_i^+\rangle)$, it suffices to show that

$$\langle u_{\sigma_i\beta}^+\rangle\langle u_i^-\rangle\overline{K}_i - \langle u_i^-\rangle\langle u_{\sigma_i\beta}^+\rangle\overline{K}_i = v^{-\langle i,\beta\rangle}\sum_{\alpha\neq\beta\oplus i} g_{\beta i}^{\alpha}\langle u_{\sigma_i\alpha}^+\rangle,$$

where we use the fact that

$$\overline{K}_i\langle u_{\sigma_i\beta}^+\rangle = v^{(i,s_i\beta)}\langle u_{\sigma_i\beta}^+\rangle\overline{K}_i = v^{-(i,\beta)}\langle u_{\sigma_i\beta}^+\rangle\overline{K}_i \qquad \text{(by Lemma 2.8.1)}$$

and if $g_{\beta i}^{\alpha}\neq 0$ and $V_\alpha\neq V_\beta\oplus V_i$, then V_α contains no direct summand isomorphic to V_i. Hence we have to show that

$$\langle u_{\sigma_i\beta}^+\rangle\langle u_i^-\rangle - \langle u_i^-\rangle\langle u_{\sigma_i\beta}^+\rangle = v^{-\langle i,\beta\rangle}\sum_{\alpha\in\text{rep-}\mathscr{S}\langle i\rangle} g_{\beta i}^{\alpha}\langle u_{\sigma_i\alpha}^+\rangle\overline{K}_{-i},$$

where $\overline{K}_{-i} = v^{\varepsilon(i)}K_{-i}$. In rep-$\sigma_i\mathscr{S}$, V_i is a simple injective and $V_{\sigma_i\beta}\in$ rep-$\sigma_i\mathscr{S}\langle i\rangle$, so $g_{\gamma i}^{\sigma_i\beta}=0$ for all $V_\gamma\in$ rep-$\sigma_i\mathscr{S}$. By Proposition 3.8 we have

$$\begin{aligned}
&\langle u_{\sigma_i\beta}^+\rangle\langle u_i^-\rangle - \langle u_i^-\rangle\langle u_{\sigma_i\beta}^+\rangle\\
&\quad = \varphi(\langle u_i^+\rangle,\langle u_i^-\rangle)K_{-i}r_i'(\langle u_{\sigma_i\beta}^+\rangle)\\
&\quad = \frac{v^{2\varepsilon(i)}}{a_i}K_{-i}\sum_{\alpha\in\text{rep-}\mathscr{S}\langle i\rangle}\frac{a_i a_{\sigma_i\alpha}}{a_{\sigma_i\beta}}v^{\langle i,s_i\alpha\rangle+(i,s_i\alpha)}g_{i\sigma_i\alpha}^{\sigma_i\beta}\langle u_{\sigma_i\alpha}^+\rangle\\
&\quad = v^{2\varepsilon(i)}\sum_{\alpha\in\text{rep-}\mathscr{S}\langle i\rangle} g_{\beta i}^{\alpha}v^{-\langle i,\alpha\rangle}\langle u_{\sigma_i\alpha}^+\rangle K_{-i} \qquad \text{(by Lemma 2.8.3)}\\
&\quad = v^{-\langle i,\beta\rangle}\sum_{\alpha\in\text{rep-}\mathscr{S}\langle i\rangle} g_{\beta i}^{\alpha}\langle u_{\sigma_i\alpha}^+\rangle\overline{K}_{-i}.
\end{aligned}$$

This shows that $T_i(\langle u_\beta^+\rangle\langle u_i^+\rangle) = T_i(\langle u_\beta^+\rangle)T_i(\langle u_i^+\rangle)$. By the induction, it is easy to see that

$$T_i(\langle u_\beta^+\rangle\langle u_i^+\rangle^{(m)}) = T_i(\langle u_\beta^+\rangle)T_i(\langle u_i^+\rangle^{(m)}).$$

∎

Combining Proposition 2.9 and Lemmas 4.2.1 and 4.2.2, the following is a consequence.

PROPOSITION. *For any $\alpha,\beta\in\mathscr{P}$, we have*

$$T_i(\langle u_\alpha^+\rangle\langle u_\beta^+\rangle) = T_i(\langle u_\alpha^+\rangle)T_i(\langle u_\beta^+\rangle).$$

4.3. Symmetrically we define a morphism $T_i : \mathfrak{h}^-(A) \to \mathscr{D}(\sigma_i A)$ given by

$$T_i(\langle u_\lambda^- \rangle) = \frac{v^{\langle \lambda_0, ti \rangle}}{[t]!_i} \left(\overline{K}_{-i} \langle u_i^+ \rangle \right)^t \langle u_{\sigma_i \lambda_0}^- \rangle$$
$$= v^{\langle \lambda, ti \rangle} K_{-ti} \langle u_i^+ \rangle^{(t)} \langle u_{\sigma_i \lambda_0}^- \rangle$$

for all $\lambda \in \mathscr{P}$, where we write $V_\lambda = V_{\lambda_0} \oplus tV_i$ and V_{λ_0} contains no direct summand isomorphic to V_i. By definition we have

$$T_i\left(\langle u_i^- \rangle^{(t)} \langle u_\lambda^- \rangle \right) = T_i\left(\langle u_i^- \rangle^{(t)} \right) T_i(\langle u_\lambda^- \rangle)$$

for any $\lambda \in \mathscr{P}$. In particular,

$$T_i(\langle u_i^- \rangle) = v_i K_{-i} \langle u_i^+ \rangle.$$

Similarly, we have

PROPOSITION. *For any $\alpha, \beta \in \mathscr{P}$, then*

$$T_i\left(\langle u_\alpha^- \rangle \langle u_\beta^- \rangle \right) = T_i(\langle u_\alpha^- \rangle) T_i\left(\langle u_\beta^- \rangle \right).$$

4.4. Of course, we can extend the action of T_i to the torus algebra, by setting

$$T_i(K_\alpha) = K_{s_i \alpha}$$

for $\alpha \in \mathbb{Z}[I]$. It is obvious that

$$T_i\left(K_\alpha \langle u_\lambda^\pm \rangle \right) = T_i(K_\alpha) T_i\left(\langle u_\lambda^\pm \rangle \right)$$

for all $\alpha \in \mathbb{Z}[I]$ and $\lambda \in \mathscr{P}$. We also have the following relations in the Double Ringel–Hall algebras.

PROPOSITION. *For all $\lambda \in \mathscr{P}$, we have*

(1) $T_i(\langle u_i^- \rangle \langle u_\lambda^+ \rangle - \langle u_\lambda^+ \rangle \langle u_i^- \rangle) = T_i(\langle u_i^- \rangle) T_i(\langle u_\lambda^+ \rangle) - T_i(\langle u_\lambda^+ \rangle) \times T_i(\langle u_i^- \rangle)$

(2) $T_i(\langle u_\lambda^- \rangle \langle u_i^+ \rangle - \langle u_i^+ \rangle \langle u_\lambda^- \rangle) = T_i(\langle u_\lambda^- \rangle) T_i(\langle u_i^+ \rangle) - T_i(\langle u_i^+ \rangle) \times T_i(\langle u_\lambda^- \rangle)$.

Proof. We only prove (1). First, if $\lambda = i$, it is easily checked since

$$\langle u_i^- \rangle \langle u_i^+ \rangle - \langle u_i^+ \rangle \langle u_i^- \rangle = \frac{v^{2\varepsilon(i)}}{a_i} (K_i - K_{-i}).$$

Second, if V_λ has no direct summand isomorphic to V_i, then $g_{i\beta}^{\lambda} = 0$, hence $r_i'(\langle u_\lambda^+ \rangle) = 0$ since i is a sink of $\mathscr{S}$. Therefore, by Proposition 3.8 and the definition of r_i in 3.3,

$$\langle u_i^- \rangle \langle u_\lambda^+ \rangle - \langle u_\lambda^+ \rangle \langle u_i^- \rangle = v^{2\varepsilon(i)} \sum_{\beta \in \mathscr{P}} v^{\langle \beta, i \rangle + (\beta, i)} \frac{a_\beta}{a_\lambda} g_{\beta i}^{\lambda} \langle u_\beta^+ \rangle K_i,$$

one sees that $V_\beta \in$ rep-$\mathscr{S}\langle i \rangle$ automatically. It follows that

$$\begin{aligned} &T_i\big(\langle u_i^- \rangle \langle u_\lambda^+ \rangle - \langle u_\lambda^+ \rangle \langle u_i^- \rangle\big) \\ &\quad = v^{2\varepsilon(i)} \sum_{\beta \in \mathscr{P}} v^{\langle \beta, i \rangle + (\beta, i)} \frac{a_\beta}{a_\lambda} g_{\beta i}^{\lambda} T_i\big(\langle u_\beta^+ \rangle\big) K_{-i} \\ &\quad = v^{2\varepsilon(i)} \sum_{\beta \in \mathscr{P}} v^{-\langle s_i\beta, i \rangle - (s_i\beta, i)} g_{i\sigma_i\lambda}^{\sigma_i\beta} \langle u_{\sigma_i\beta}^+ \rangle K_{-i} \\ &\qquad \text{(by Lemmas 2.8.1 and 2.8.3)} \\ &\quad = v^{-\varepsilon(i)} \sum_{\beta \in \mathscr{P}} v^{-\langle s_i\lambda, i \rangle - (s_i\lambda, i)} g_{i\sigma_i\lambda}^{\sigma_i\beta} \langle u_{\sigma_i\beta}^+ \rangle K_{-i}. \end{aligned}$$

On the other hand, since i is a source of $\sigma_i\mathscr{S}$ and V_i is a injective module in $\sigma_i\mathscr{S}$, if $V_\lambda \in$ rep-$\mathscr{S}\langle i \rangle$, then $\mathrm{Hom}(V_\lambda, V_i) = \mathrm{Ext}(V_i, V_\lambda) = 0$ in rep-$\mathscr{S}$; correspondingly, $\mathrm{Hom}(V_i, V_{\sigma_i\lambda}) = \mathrm{Ext}(V_{\sigma_i\lambda}, V_i) = 0$ in rep-$\sigma_i\mathscr{S}$. Thus,

$$\langle u_i^+ \rangle \langle u_{\sigma_i\lambda}^+ \rangle = v^{(i, s_i\lambda)} \langle u_{\sigma_i\lambda}^+ \rangle \langle u_i^+ \rangle + \sum_{\beta \in \mathscr{P}} v^{-\langle s_i\lambda, i \rangle} g_{i\sigma_i\lambda}^{\sigma_i\beta} \langle u_{\sigma_i\beta}^+ \rangle.$$

It follows that

$$\begin{aligned} &T_i\big(\langle u_i^- \rangle\big) T_i\big(\langle u_\lambda^+ \rangle\big) - T_i\big(\langle u_\lambda^+ \rangle\big) T_i\big(\langle u_i^- \rangle\big) \\ &\quad = \overline{K}_{-i} \langle u_i^+ \rangle \langle u_{\sigma_i\lambda}^+ \rangle - \big(\langle u_{\sigma_i\lambda}^+ \rangle\big) \overline{K}_{-i} \langle u_i^+ \rangle \\ &\quad = v^{-\varepsilon(i) - (s_i\lambda, i)} \langle u_i^+ \rangle \langle u_{\sigma_i\lambda}^+ \rangle K_{-i} - v^{-\varepsilon(i)} \langle u_{\sigma_i\lambda}^+ \rangle \langle u_i^+ \rangle K_{-i} \\ &\quad = v^{-\varepsilon(i)} \sum_{\beta \in \mathscr{P}} v^{-(s_i\lambda, i) - \langle s_i\lambda, i \rangle} g_{i\sigma_i\lambda}^{\sigma_i\beta} \langle u_{\sigma_i\beta}^+ \rangle K_{-i}. \end{aligned}$$

Therefore,

$$T_i\big(\langle u_i^- \rangle \langle u_\lambda^+ \rangle - \langle u_\lambda^+ \rangle \langle u_i^- \rangle\big) = T_i\big(\langle u_i^- \rangle T_i\big(\langle u_\lambda^+ \rangle\big) - T_i\big(\langle u_\lambda^+ \rangle\big) T_i\big(\langle u_i^- \rangle\big).$$

By Propositions 3.8 and 4.2, for all $\lambda \in \mathscr{P}$ we easily see that

$$\begin{aligned} &T_i\big(\langle u_i^+ \rangle \big(\langle u_i^- \rangle \langle u_\lambda^+ \rangle - \langle u_\lambda^+ \rangle \langle u_i^- \rangle\big)\big) \\ &\quad = T_i\big(\langle u_i^+ \rangle\big) T_i\big(\langle u_i^- \rangle \langle u_\lambda^+ \rangle - \langle u_\lambda^+ \rangle \langle u_i^- \rangle\big). \end{aligned}$$

Then, by the induction the proof is finished. ∎

4.5. Now the main result of the section can be stated as follows.

THEOREM. *Let i be a sink. For all $\lambda \in \mathscr{P}$ and $\alpha \in \mathbb{Z}[I]$, we write $V_\lambda = V_{\lambda_0} \oplus tV_i$ where V_{λ_0} has no direct summand isomorphic to V_i. Then operator T_i defined as*

$$T_i(\langle u_\lambda^+ \rangle) = v^{\langle \lambda, ti \rangle} K_{ti} \langle u_i^- \rangle^{(t)} \langle u_{\sigma_i \lambda_0}^+ \rangle,$$

$$T_i(\langle u_\lambda^- \rangle) = v^{\langle \lambda, ti \rangle} K_{-ti} \langle u_i^+ \rangle^{(t)} \langle u_{\sigma_i \lambda_0}^- \rangle,$$

$$T_i(K_\alpha) = K_{s_i(\alpha)},$$

induces a $\mathbb{Q}(v)$-algebra isomorphism: $\mathscr{D}_c(A) \to \mathscr{D}_c(\sigma_i A)$.

Proof. One sees that $T_i(T) \subseteq \mathscr{D}_c(\sigma_i A)$. For any $j \in I$, if $j = i$, of course,

$$T_i(\langle u_i^+ \rangle) = v_i^{-1} K_i \langle u_i^- \rangle \in \mathscr{D}_c(\sigma_i A);$$

if $j \neq i$, then $T_i(\langle u_j^+ \rangle) = \langle u_{\sigma_i(j)}^+ \rangle$. Note that $V_{\sigma_i(j)}$ is an exceptional object in rep-$\sigma_i \mathscr{S}$, so $\langle u_{\sigma_i(j)}^+ \rangle \in \mathscr{C}^+(\sigma_i A)$ by a result of Ringel (for example, see [Z; CX, 5.2] or by Theorem 6.3). Therefore, $T_i(\mathscr{C}^+(A)) \subseteq \mathscr{D}_c(\sigma_i A)$; similarly, $T_i(\mathscr{C}^-(A)) \subseteq \mathscr{D}_c(\sigma_i A)$. In view of Propositions 4.2 and 4.3, to prove T_i is a homomorphism, it suffices to verify that T_i preserves the relations

$$\langle u_j^- \rangle \langle u_k^+ \rangle - \langle u_k^+ \rangle \langle u_j^- \rangle = \delta_{kj} \frac{v^{2\varepsilon(j)}}{a_j} (K_j - K_{-j})$$

for all $k, j \in I$.

If $j = i$ or $k = i$, we have shown that, by Proposition 4.4, this relation is preserved by the operator T_i. If none of j and k is i, according to the formulae in Proposition 5.5 and Theorem 5.10 in [X] (noting that i is a source of $\sigma_i \mathscr{S}$) we have the relation

$$u_{\sigma_i(j)}^- u_{\sigma_i(k)}^+ - u_{\sigma_i(k)}^+ u_{\sigma_i(j)}^- = \delta_{kj} \frac{|V_{\sigma_i(j)}|}{a_{\sigma_i(j)}} (K_{s_i(j)} - K_{-s_i(j)}).$$

Since $|V_{\sigma_i(j)}| = v^{2 \dim_k V_{\sigma_i(j)}}$ and $a_{\sigma_i(j)} = a_j$, the above relation is

$$\langle u_{\sigma_i(j)}^- \rangle \langle u_{\sigma_i(k)}^+ \rangle - \langle u_{\sigma_i(k)}^+ \rangle \langle u_{\sigma_i(j)}^- \rangle = \delta_{kj} \frac{v^{2\varepsilon(j)}}{a_j} (K_{s_i(j)} - K_{-s_i(j)}).$$

This exactly means that

$$T_i(\langle u_j^- \rangle) T_i(\langle u_k^+ \rangle) - T_i(\langle u_k^+ \rangle) T_i(\langle u_j^- \rangle) = T_i(\langle u_j^- \rangle \langle u_k^+ \rangle - \langle u_k^+ \rangle \langle u_j^- \rangle).$$

We have shown that T_i is a $\mathbb{Q}(v)$-algebra homomorphism from $\mathscr{D}_c(A)$ to $\mathscr{D}_c(\sigma_i A)$.

4.6. Let i be a source of $\mathscr{S}$. For all $\lambda \in \mathscr{P}$ and $\alpha \in \mathbb{Z}[I]$, we write $V_\lambda = V_{\lambda_0} \oplus tV_i$ where V_{λ_0} has no direct summand isomorphic to V_i. We can define the operator T_i' as

$$T_i'(\langle u_\lambda^+ \rangle) = \frac{v^{\langle ti, \lambda_0 \rangle}}{[t]!_i} \langle u_{\sigma_i \lambda_0}^+ \rangle \big(v^{-\varepsilon(i)} K_{-i} \langle u_i^- \rangle \big)^t$$

$$= v^{\langle ti, \lambda \rangle} \langle u_{\sigma_i \lambda_0}^+ \rangle \langle u_i^- \rangle^{(t)} K_{-ti},$$

$$T_i'(\langle u_\lambda^- \rangle) = \frac{v^{\langle ti, \lambda_0 \rangle}}{[t]!_i} \langle u_{\sigma_i \lambda_0}^- \rangle \big(v^{\varepsilon(i)} \langle u_i^+ \rangle K_i \big)^t$$

$$= v^{\langle ti, \lambda \rangle} \langle u_{\sigma_i \lambda_0}^- \rangle \langle u_i^+ \rangle^{(t)} K_{ti},$$

$$T_i'(K_\alpha) = K_{s_i(\alpha)}.$$

By a similar way, we can prove that T_i' induces a $\mathbb{Q}(v)$-algebra homomorphism from $\mathscr{D}_c(A)$ to $\mathscr{D}_c(\sigma_i A)$.

4.7. Now, we come back to the situation where i is a sink of $\mathscr{S}$. Then i is a source of $\sigma_i \mathscr{S}$. Therefore we have the induced $\mathbb{Q}(v)$-algebra homomorphism $T_i' : \mathscr{D}_c(\sigma_i A) \to \mathscr{D}_c(A)$. It is easily seen that $T_i T_i' = 1$ and $T_i' T_i = 1$. So we have shown that $T_i : \mathscr{D}_c(A) \to \mathscr{D}_c(\sigma_i A)$ is a $\mathbb{Q}(v)$-algebra isomorphism, whose inverse is $T_i' : \mathscr{D}_c(\sigma_i A) \to \mathscr{D}_c(A)$. The proof of Theorem 4.5 is finished. ∎

4.8. By a similar method as in [SV], it can be verified that the operator T_i preserves the relation (6) in 3.6. So we have the following result due to Sevenhant and Van den Bergh.

THEOREM. *Let i be a sink. Then the operator T_i gives a $\mathbb{Q}(v)$-algebra isomorphism $\mathscr{D}(A) \to \mathscr{D}(\sigma_i A)$.*

Proof. It remains to show that T_i preserves the relation

$$\text{(4.8.1)} \qquad \sum_{\alpha', \alpha} v^{\langle \alpha', \alpha \rangle + (\alpha, \alpha)} \frac{a_{\alpha'}}{a_{\lambda'}} g_{\alpha' \alpha}^{\lambda'} K_{-\alpha} \langle u_{\alpha'}^- \rangle r_\alpha'(\langle u_\lambda^+ \rangle)$$

$$= \sum_{\alpha, \beta} v^{\langle \alpha, \beta \rangle + (\beta, \beta)} \frac{a_\alpha}{a_\lambda} g_{\alpha\beta}^{\lambda} K_\beta \langle u_\alpha^+ \rangle r_\beta(\langle u_{\lambda'}^- \rangle),$$

where $V_\lambda, V_{\lambda'} \in$ rep-$\mathscr{S}\langle i \rangle$, respectively. Indeed, the left hand side of (4.8.1) may be rewritten as

$$\text{(4.8.2)} \quad l = \sum_{\alpha', \alpha, \beta} v^{\langle \lambda', \alpha \rangle + \langle \alpha, \lambda \rangle + (\alpha, \beta)} \frac{a_{\alpha'} a_\alpha a_\beta}{a_{\lambda'} a_\lambda} g_{\alpha' \alpha}^{\lambda'} g_{\alpha\beta}^{\lambda} K_{-\alpha} \langle u_{\alpha'}^- \rangle \langle u_\beta^+ \rangle,$$

where we see that $\alpha, \alpha' \in \text{rep-}\mathscr{S}\langle i\rangle$ automatically; however, this is not the case for the V_β. We need the following facts:

(4.8.3)

$$\text{If } i \text{ is a sink, } V_\alpha, V_\beta \in \text{rep-}\mathscr{S}\langle i\rangle, \text{ then } g^{\lambda}_{\alpha,\,\beta\oplus ti} = \sum_{\gamma} g^{\gamma}_{\alpha,\,ti} g^{\lambda}_{\gamma\beta}.$$

(4.8.4) If i is a source and $V_\alpha, V_\beta \in \text{rep-}\mathscr{S}\langle i\rangle$, then $g^{\lambda}_{\alpha\oplus ti,\,\beta} = \sum_{\gamma} g^{\gamma}_{ti,\,\beta} g^{\lambda}_{\alpha\gamma}$.

Now, assume $V_\beta = V_{\beta'} \oplus tV_i$, where $V_{\beta'} \in \text{rep-}\mathscr{S}\langle i\rangle$. Then $\langle u^+_\beta\rangle = v^{\langle \beta', ti\rangle}\langle u^+_i\rangle^{(t)}\langle u^+_{\beta'}\rangle$. Applying T_i to (4.8.2) yields

$$(4.8.5)\quad \sum_{\alpha',\alpha,\beta',t} v^{\langle\lambda',\alpha\rangle+\langle\alpha,\lambda\rangle+(a,\beta)+t^2\langle i,i\rangle-t(i,\alpha')+\langle\beta',ti\rangle}\frac{a_{\alpha'}a_\alpha a_\beta}{a_{\lambda'}a_\lambda}$$
$$\times g^{\lambda'}_{\alpha'\alpha} g^{\lambda}_{\alpha,\beta} K_{ti-s_i\alpha}\langle u^-_{\sigma_i\alpha'}\rangle\langle u^-_i\rangle^{(t)}\langle u^+_{\sigma_i\beta'}\rangle.$$

Noting the fact $a_\beta = a_{\beta'\oplus ti} = a_{\beta'}a_{ti}|\text{Hom}(tV_i, V_{\beta'})| = v^{2\langle ti,\beta'\rangle}a_{\beta'}a_{ti}$, we write (4.8.5) as

$$\sum_{\gamma,\alpha',\alpha,\beta',t} v^{\langle\lambda',\alpha\rangle+\langle\alpha,\lambda\rangle+(\alpha,\beta)+t^2\langle i,i\rangle-(ti,\alpha')+2\langle ti,\beta'\rangle+\langle\beta',ti\rangle} g^{\lambda'}_{\alpha'\alpha} g^{\gamma}_{\alpha ti} g^{\lambda}_{\gamma\beta'}$$
$$\times\frac{a_{\alpha'}a_\alpha a_{\beta'}a_{ti}}{a_{\lambda'}a_\lambda}K_{ti-s_i\alpha}\langle u^-_{\sigma_i\alpha'}\rangle\langle u^-_i\rangle^{(t)}\langle u^+_{\sigma_i\beta'}\rangle.$$

The terms in the last sum can be non-zero only if $\gamma \in \text{rep-}\mathscr{S}\langle i\rangle$. Recall that

$$a_\beta g^{\alpha}_{\beta,ti} = a_\alpha g^{\sigma_i\beta}_{ti,\,\sigma_i\alpha},$$

$$a_{\sigma_i\alpha'\oplus ti} = a_{\sigma_i\alpha'}a_{ti}\left|\text{Hom}\left(V_{\sigma_i\alpha'}, tV_i\right)\right| = v^{-2\langle\alpha',ti\rangle}a_{\beta'}a_{ti}$$

and

$$\langle u^-_{\sigma_i\alpha'\oplus ti}\rangle = v^{\langle ti, s_i\alpha'\rangle}\langle u^-_{\sigma_i\alpha'}\rangle\langle u^-_i\rangle^{(t)} = v^{-\langle ti,\alpha'\rangle}\langle u^-_{\sigma_i\alpha'}\rangle\langle u^-_i\rangle^{(t)}$$

since i is a source for $\sigma_i\Omega$. Therefore, we may rewrite the terms where $\gamma, \beta' \in \mathscr{S}\langle i\rangle$ as

$$\sum_{\gamma,\alpha',\alpha,\beta',t} v^{\langle\lambda',\alpha\rangle+\langle\alpha,\lambda\rangle+(\alpha,\beta)+t^2\langle i,i\rangle-(ti,\alpha')+2\langle ti,\beta'\rangle+\langle\beta',ti\rangle+\langle ti,\alpha'\rangle}$$
$$\times g^{\sigma_i\lambda'}_{\sigma_i\alpha'\sigma_i\alpha} g^{\sigma_i\lambda}_{\sigma_i\gamma\sigma_i\beta'} g^{\sigma_i\alpha}_{ti\,\sigma_i\gamma}\frac{a_{\sigma_i\alpha'}a_{\sigma_i\alpha}a_{\sigma_i\beta'}a_{ti}}{a_{\sigma_i\lambda'}a_{\sigma_i\lambda}}K_{ti-s_i\alpha}\langle u^-_{\sigma_i\alpha'\oplus ti}\rangle\langle u^+_{\sigma_i\beta'}\rangle$$
$$= \sum_{\gamma,\alpha',\alpha,\beta',t} v^{\langle\lambda',\alpha\rangle+\langle\alpha,\lambda\rangle+(\alpha,\beta)+t^2\langle i,i\rangle-(ti,\alpha')+2\langle ti,\beta'\rangle+\langle\beta',ti\rangle+\langle ti,\alpha'\rangle+2\langle\alpha',ti\rangle}$$
$$\times g^{\sigma_i\lambda'}_{\sigma_i\alpha'\oplus ti,\,\sigma_i\gamma} g^{\sigma_i\lambda}_{\sigma_i\gamma\sigma_i\beta'}\frac{a_{\sigma_i\alpha'\oplus ti}a_{\sigma_i\gamma}a_{\sigma_i\beta'}}{a_{\sigma_i\lambda'}a_{\sigma_i\lambda}}K_{ti-s_i\alpha}\langle u^-_{\sigma_i\alpha'\oplus ti}\rangle\langle u^+_{\sigma_i\beta'}\rangle.$$

Noting that

$$s_i\gamma + ti = s_i\alpha, \qquad s_i\beta = s_i\beta' + ti, \qquad \lambda = \alpha + \beta, \qquad \lambda' = \alpha' + \alpha,$$

in $\mathbb{Z}[I]$, we see that

$$v^{\langle \lambda', \alpha\rangle + \langle \alpha, \lambda\rangle + (\alpha, \beta) + t^2\langle i, i\rangle - (ti, \alpha') + 2\langle ti, \beta'\rangle + \langle \beta', ti\rangle + \langle ti, \alpha'\rangle + 2\langle \alpha', ti\rangle}$$
$$= v^{\langle s_i\lambda', s_i\gamma\rangle + \langle s_i\gamma, s_i\lambda\rangle + (s_i\gamma, s_i\beta')}.$$

Hence we obtain

$$T_i(l) = \sum_{\mu, \gamma, \alpha} v^{\langle s_i\lambda', \mu\rangle + \langle \mu, s_i\lambda\rangle + (\mu, \gamma)} \frac{a_\mu a_\gamma a_\alpha}{a_{\sigma_i\lambda'} a_{\sigma_i\lambda}} g_{\alpha\mu}^{\sigma_i\lambda'} g_{\mu,\gamma}^{\sigma_i\lambda} K_{-\mu}\langle u_\alpha^-\rangle\langle u_\gamma^+\rangle.$$

Applying T_i to the right side of (4.8.1), we get the entirely similar form

$$\sum_{\alpha, \mu, \alpha'} v^{\langle s_i\lambda, \gamma\rangle + \langle \gamma, s_i\lambda'\rangle + (\gamma, \alpha')} \frac{a_\alpha a_\gamma a_{\alpha'}}{a_{\sigma_i\lambda'} a_{\sigma_i\lambda}} g_{\alpha\gamma}^{\sigma_i\lambda} g_{\gamma,\alpha'}^{\sigma_i\lambda'} K_\gamma\langle u_\alpha^+\rangle\langle u_{\alpha'}^-\rangle.$$

Thus T_i preserves the relations (6) in 3.6 for $V_\lambda, V_\lambda' \in$ rep-$\mathscr{S}\langle i\rangle$. The proof is completed. ∎

5. SOME PROPERTIES OF BGP-REFLECTION OPERATORS

5.1. Again we assume that i is a sink for $\mathscr{S}$. Let

$$\mathfrak{h}^+(A)\langle i\rangle = \mathfrak{h}^+(A) \mid \text{rep-}\mathscr{S}\langle i\rangle,$$

i.e., the $\mathbb{Q}(v)$-subspace of $\mathfrak{h}^+(A)$ generated by $\langle u_\alpha^+\rangle$ with $V_\alpha \in$ rep-$\mathscr{S}\langle i\rangle$. It is easy to see that $\mathfrak{h}^+(A)\langle i\rangle$ is a $\mathbb{Q}(v)$-subalgebra of $\mathfrak{h}^+(A)$, hence of $\mathscr{H}^+(A)$. Similarly, let

$$\mathfrak{h}^+(\sigma_i A)\langle i\rangle = \mathfrak{h}^+(\sigma_i A) \mid \text{rep-}\sigma_i\mathscr{S}\langle i\rangle$$

the $\mathbb{Q}(v)$-subalgebra of $\mathfrak{h}^+(\sigma_i A)$ generated by $\langle u_\alpha^+\rangle$ with $V_\alpha \in$ rep-$\sigma_i\mathscr{S}\langle i\rangle$. Obviously, we have

$$\text{(5.1.1)} \qquad \mathfrak{h}^+(A)\langle i\rangle = \{x \in \mathfrak{h}^+(A) \mid T_i(x) \in \mathscr{H}^+(\sigma_i A)\}$$
$$\mathfrak{h}^+(\sigma_i A)\langle i\rangle = \{x \in \mathfrak{h}^+(\sigma_i A) \mid T_i'(x) \in \mathscr{H}^+(A)\}.$$

Let T be the torus algebra of $\mathbb{Z}[I]$, $\mathscr{H}^+(A)\langle i\rangle = T\mathfrak{h}^+(A)\langle i\rangle$, and $\mathscr{H}^+(\sigma_i A)\langle i\rangle = T\mathfrak{h}^+(\sigma_i A)\langle i\rangle$. We have

$$\mathfrak{h}^+(A) = \sum_{t\geq 0}\langle u_i^+\rangle^{(t)}\mathfrak{h}^+(A)\langle i\rangle, \qquad \text{and}$$

$$\mathscr{H}^+(A) = \sum_{t\geq 0}\langle u_i^+\rangle^{(t)}\mathscr{H}^+(A)\langle i\rangle$$

$$\mathfrak{h}^+(\sigma_i A) = \sum_{t\geq 0}\mathfrak{h}^+(\sigma_i A)\langle i\rangle\langle u_i^+\rangle^{(t)}, \qquad \text{and}$$

$$\mathscr{H}^+(\sigma_i A) = \sum_{t\geq 0}\mathscr{H}^+(\sigma_i A)\langle i\rangle\langle u_i^+\rangle^{(t)}.$$

Dually, the subalgebras $\mathfrak{h}^-(A)\langle i\rangle$, $\mathfrak{h}^-(\sigma_i A)\langle i\rangle$, $\mathscr{H}^-(A)\langle i\rangle$, and $\mathscr{H}^-(\sigma_i A)\langle i\rangle$ can be defined and the same relations as above can be obtained.

By the definition of the derivations, it is easy to check

(5.1.2)
$$\mathfrak{h}^+(A)\langle i\rangle = \{x\in\mathfrak{h}^+(A) \mid r_i'(x) = 0\},$$
$$\mathfrak{h}^+(\sigma_i A)\langle i\rangle = \{x\in\mathfrak{h}^+(\sigma_i A) \mid r_i(x) = 0\}.$$

Indeed, if $r_i'(\langle u_\lambda^+\rangle) = 0$, then $g_{i\beta}^{\lambda} = 0$ for all $\beta\in\mathscr{P}$. There is no extension of the form

$$0\to V_\beta\to V_\lambda\to V_i\to 0.$$

This implies that V_i is not a direct summand of V_λ. It follows that $\langle u_\lambda^+\rangle\in\mathfrak{h}^+(A)\langle i\rangle$. Conversely, if $V_\lambda\in\text{rep-}\mathscr{S}\langle i\rangle$ then $g_{i\beta}^{\lambda} = 0$ for all $\beta\in\mathscr{P}$ since i is a sink of $\mathscr{S}$. Therefore, $r_i'(\langle u_\lambda^+\rangle) = 0$. The first relation in (5.1.2) is verified. It is similar for the second. ∎

Since $T_i : \mathfrak{h}^{\pm}(A)\langle i\rangle\to\mathfrak{h}^{\pm}(\sigma_i A)\langle i\rangle$ are isomorphisms, therefore

(5.1.3)
$$T_i(\mathfrak{h}^+(A)\langle i\rangle) = \mathfrak{h}^+(\sigma_i A)\langle i\rangle$$
$$T_i(\mathfrak{h}^-(A)\langle i\rangle) = \mathfrak{h}^-(\sigma_i A)\langle i\rangle.$$

5.2. The following property is our main concern in this section.

PROPOSITION. *Let i be a sink and $\varphi : \mathscr{H}^+(A)\times\mathscr{H}^-(A)\to\mathbb{Q}(v)$ the skew Hopf pairing defined as in* 3.6. *Then*

$$\varphi(T_i(x), T_i(y)) = \varphi(x, y)$$

for all $x\in\mathscr{H}^+(A)\langle i\rangle$ and $y\in\mathscr{H}^-(A)\langle i\rangle$.

Proof. Let V_β and $V_{\beta'}$ belong to rep-$\mathscr{S}\langle i\rangle$, $\alpha, \alpha' \in \mathbb{Z}[I]$. Then

$$\begin{aligned}
&\varphi\big(T_i\big(K_\alpha\langle u_\beta^+\rangle\big), T_i\big(K_{\alpha'}\langle u_{\beta'}^+\rangle\big)\big)\\
&\quad= \varphi\big(K_{s_i(\alpha)}\langle u_{\sigma_i\beta}^+\rangle, K_{s_i(\alpha')}\langle u_{\sigma_i\beta'}^-\rangle\big)\\
&\quad= v^{-(s_i(\alpha), s_i(\alpha'))-(s_i(\beta), s_i(\alpha'))+(s_i(\alpha), s_i(\beta'))+(s_i(\beta), s_i(\beta'))} a_{\sigma_i(\beta)}^{-1}\delta_{\sigma_i(\beta)\sigma_i(\beta')}\\
&\quad= v^{-(\alpha,\alpha')-(\beta,\alpha')+(\alpha,\beta')+(\beta,\beta)} a_\beta^{-1}\delta_{\beta\beta'}\\
&\quad= \varphi\big(K_\alpha\langle u_\beta^+\rangle, K_{\alpha'}\langle u_{\beta'}^+\rangle\big).
\end{aligned}$$

The result follows. ∎

5.3. It is easy to see that

$$\varphi\big(\langle u_i^+\rangle^t, \langle u_i^-\rangle^t\big) = v_i^{t(t+1)/2}[t]!_i\big(v_i - v_i^{-1}\big)^{-t} \tag{5.3.1}$$

for all $i \in I$ and $t \in \mathbb{N}$, where $v_i = v^{\varepsilon(i)}$, $[t]!_i = [t]!_{\varepsilon(i)}$. Therefore we have the following

PROPOSITION. *Let $x \in \mathfrak{h}^+(A)\langle i\rangle$, $y \in \mathfrak{h}^-(A)\langle i\rangle$. Then*

(1) $\varphi(T_i(x)\langle u_i^+\rangle^t, T_i(y)\langle u_i^-\rangle^t) = \varphi(x, y)v_i^{t(t+1)/2}[t]!_i(v_i - v_i^{-1})^{-t}$.

(2) $\varphi(\langle u_i^+\rangle^t T_i(x), \langle u_i^-\rangle^t T_i(y)) = \varphi(x, y)v_i^{t(t+1)/2}[t]!_i(v_i - v_i^{-1})^{-t}$.

Proof. It suffices to show the equations for $\langle u_\alpha^+\rangle$ and $\langle u_\beta^-\rangle$ with V_α, $V_\beta \in$ rep-$\mathscr{S}\langle i\rangle$. To prove Eq. (1), we only need to show

$$\varphi\Big(\langle u_{\sigma_i\alpha}^+\rangle\langle u_i^+\rangle^t, \langle u_{\sigma_i\beta}^-\rangle\langle u_i^-\rangle^t\Big) = \varphi\big(\langle u_\alpha^+\rangle, \langle u_\beta^-\rangle\big)\varphi\big(\langle u_i^+\rangle^t, \langle u_i^-\rangle^t\big) \tag{5.3.2}$$

since $\varphi(T_i(\langle u_\alpha^+\rangle), T_i(\langle u_\beta^-\rangle)) = \varphi(\langle u_{\sigma_i\alpha}^+\rangle, \langle u_{\sigma_i\beta}^-\rangle) = \varphi(\langle u_\alpha^+\rangle, \langle u_\beta^-\rangle)$. It is not difficult to obtain that

$$r_i\big(\langle u_i^+\rangle^t\big) = v_i^{t-1}[t]_i\langle u_i^+\rangle^{t-1}. \tag{5.3.3}$$

We prove Eq. (5.3.2) by using induction. For $t = 1$, note that $r_i(\langle u_{\sigma_i\alpha}^+\rangle) = 0$ since $V_{\sigma_i\alpha} \in$ rep-$\sigma_i\mathscr{S}\langle i\rangle$ and i is a source of $\sigma_i\mathscr{S}$. According to Lemma 3.7,

$$\begin{aligned}
&\varphi\big(\langle u_{\sigma_i\alpha}^+\rangle\langle u_i^+\rangle, \langle u_{\sigma_i\beta}^-\rangle\langle u_i^-\rangle\big)\\
&\quad= \varphi\big(\langle u_i^+\rangle, \langle u_i^-\rangle\big)\varphi\big(r_i\big(\langle u_{\sigma_i\alpha}^+\rangle\langle u_i^+\rangle\big), \langle u_{\sigma_i\beta}^-\rangle\big)\\
&\quad= \varphi\big(\langle u_i^+\rangle, \langle u_i^-\rangle\big)\varphi\big(\langle u_{\sigma_i\alpha}^+\rangle r_i\big(\langle u_i^+\rangle\big), \langle u_{\sigma_i\beta}^-\rangle\big)\\
&\quad= \varphi\big(\langle u_\alpha^+\rangle, \langle u_\beta^-\rangle\big)\varphi\big(\langle u_i^+\rangle, \langle u_i^-\rangle\big).
\end{aligned}$$

Now, we assume that Eq. (5.3.2) is true for t. Because

$$\varphi\left(\langle u_i^+\rangle^{t+1}, \langle u_i\rangle^{t+1}\right) = \varphi\left(\langle u_i^+\rangle^t, \langle u_i^-\rangle^t\right) v_i^t [t+1]_i \varphi\left(\langle u_i^+\rangle, \langle u_i^-\rangle\right),$$

we have that

$$\begin{aligned}
&\varphi\left(\langle u_{\sigma_i\alpha}^+\rangle\langle u_i^+\rangle^{t+1}, \langle u_{\sigma_i\beta}^-\rangle\langle u_i^-\rangle^{t+1}\right)\\
&\quad = \varphi\left(\langle u_i^+\rangle, \langle u_i^-\rangle\right)\varphi\left(r_i\left(\langle u_{\sigma_i\alpha}^+\rangle\langle u_i^+\rangle^{t+1}\right), \langle u_{\sigma_i\beta}^-\rangle\langle u_i^-\rangle^t\right)\\
&\quad = \varphi\left(\langle u_i^+\rangle, \langle u_i^-\rangle\right)\varphi\left(\langle u_{\sigma_i\alpha}^+\rangle r_i\left(\langle u_i^+\rangle^{t+1}\right), \langle u_{\sigma_i\beta}^-\rangle\langle u_i^-\rangle^t\right)\\
&\quad = \varphi\left(\langle u_i^+\rangle, \langle u_i^-\rangle\right) v_i^t [t+1]_i \varphi\left(\langle u_{\sigma_i\alpha}^+\rangle\langle u_i^+\rangle^t, \langle u_{\sigma_i\beta}^-\rangle\langle u_i^-\rangle\right)\\
&\quad = \varphi\left(\langle u_i^+\rangle, \langle u_i^-\rangle\right) v_i^t [t+1]_i \varphi\left(\langle u_{\sigma_i\alpha}^+\rangle, \langle u_{\sigma_i\beta}^-\rangle\right)\varphi\left(\langle u_i^+\rangle^t, \langle u_i^-\rangle^t\right)\\
&\quad = \varphi\left(\langle u_{\sigma_i\alpha}^+\rangle, \langle u_{\sigma_i\beta}^-\rangle\right)\varphi\left(\langle u_i^+\rangle^{t+1}, \langle u_i^-\rangle^{t+1}\right).
\end{aligned}$$

So Eq. (1) is verified. The proof of (2) is similar. ∎

6. COMPARING BGP-REFLECTION OPERATORS WITH LUSZTIG'S SYMMETRIES

6.1. We have obtained in Theorem 4.5 that if i is a sink (source) of Ω, then T_i (T_i') is a $\mathbb{Q}(v)$-algebra isomorphism: $\mathscr{D}_c(A) \to \mathscr{D}_c(\sigma_i A)$. In this section, we focus on comparing T_i with $T''_{i,1}$. First, we state one observation as follows (also see [R3]).

PROPOSITION 6.1. *Let $i \neq j \in I$, and $n = -\frac{2(i,j)}{(i,i)}$.*

(1) *If i is a sink, then in $\mathscr{H}(A)$ we have*

$$\langle u_\lambda\rangle = \sum_{t=0}^{n} (-1)^t v_i^{-t} \langle u_i\rangle^{(t)}\langle u_j\rangle\langle u_i\rangle^{(n-t)},$$

where $\lambda \in \mathscr{P}$ is the unique class of indecomposable modules with the dimension vector $e_j + ne_i$.

(2) *If i is a source, then in $\mathscr{H}(A)$ we have*

$$\langle u_\lambda\rangle = \sum_{t=0}^{n} (-1)^t v_i^{-t} \langle u_i\rangle^{(n-t)}\langle u_j\rangle\langle u_i\rangle^{(t)},$$

where $\lambda \in \mathscr{P}$ is the unique class of indecomposable modules with the dimension vector $e_j + ne_i$.

Proof. We only prove the case (1); the case (2) is its dual. Assume i is a sink; then $\langle e_i, e_j\rangle = 0$ and $\operatorname{Ext}_A(V_i, V_j) = \operatorname{Ext}_A(V_i, V_i) = 0$. First, it is easy to see that

$$(*)\qquad \langle u_i\rangle^t\langle u_j\rangle\langle u_i\rangle^{n-t} = v_i^{-\binom{n}{2}+tn}\sum_{\lambda\in\mathscr{P}} c_\lambda(t)\langle u_\lambda\rangle,$$

where

$$c_\lambda(t) = g^{\lambda}_{\underbrace{ii\cdots i}_{t}j\underbrace{ii\cdots i}_{n-t}}.$$

Now, fix $\lambda\in\mathscr{P}$. In order that $c_\lambda(t)\neq 0$, V_λ must have a composition series with n factors of the form V_i and one factor isomorphic to V_j. Let N be a direct summand of V_λ of minimal length which has the composition factor V_j. Then $V_\lambda = N\oplus sV_i$ for some $s\geq 0$. By the proof of [R2, Part III. 2, Proposition], we see that

$$(6.1.1)\qquad c_\lambda(t) = \left(\left|\begin{matrix} s\\ s-t\end{matrix}\right] |\,t]!\,|\,n-t]!\right)_i, \text{ if } t\leq s, \qquad \text{and}$$

$$c_\lambda(t) = 0, \text{ if } t > s.$$

If $s > 0$ in (6.1.1), then

$$(6.1.2)\quad \sum_{t=0}^{n}(-1)^t\frac{v_i^{t(t-1)}}{(|\,n-t]!\,|\,t]!)_i}c_\lambda(t) = \sum_{t=0}^{s}(-1)^t v_i^{t(t-1)}\left|\begin{matrix} s\\ s-t\end{matrix}\right]_i = 0.$$

If $s = 0$ in (6.1.1), then V_λ is uniquely determined by V_i and V_j; in fact, V_λ is indecomposable projective in the category of modules which have composition factors isomorphic to V_i and V_j; in this case

$$c_\lambda(t) = 0, \text{ if } t\geq 1, \qquad \text{and} \qquad c_\lambda(0) = |\,n]!_i, \text{ if } t = 0.$$

Thus,

$$\sum_{t=0}^{n}(-1)^t v_i^{-t}\langle u_i\rangle^{(t)}\langle u_j\rangle\langle u_i\rangle^{(n-t)}$$

$$= \sum_{t=0}^{n}(-1)^t\frac{v_i^{-t}}{([n-t]![t]!)_i}\langle u_i\rangle^t\langle u_j\rangle\langle u_i\rangle^{n-t}$$

$$= \sum_{t=0}^{n}(-1)^t\frac{v_i^{t(t-1)+\binom{n}{2}-tn}}{|\,n-t]!_i\,|\,t]!_i}$$

$$\times\left(v_i^{-\binom{n}{2}+tn}\sum_{\lambda\in\mathscr{P}} c_\lambda(t)\langle u_\lambda\rangle\right)\qquad \text{(by Lemma 2.10 and } (*)\text{)}$$

$$= \sum_{\lambda \in \mathscr{P}} \sum_{t=0}^{n} (-1)^t \frac{v_i^{t(t-1)}}{|n-t]!_i |t]!_i} c_\lambda(t) \langle u_\lambda \rangle$$

$$= \sum_{s \neq 0} \left(\sum_{t=0}^{n} (-1)^t \frac{v_i^{t(t-1)}}{|n-t]!_i |t]!_i} c_\lambda(t) \right) \langle u_\lambda \rangle$$

$$+ \frac{1}{|n]!_i} c_\lambda(0) \langle u_\lambda \rangle \qquad (\text{the case } s = 0)$$

$$= \langle u_\lambda \rangle \qquad (\text{by } (6.1.2)).$$

∎

6.2. Recall that we have the Green–Ringel isomorphism $\mathscr{D}_c(A) \to U_q(\mathfrak{g})$ in (3.10). So we have the canonical isomorphism $\mathscr{D}_c(A) \to \mathscr{D}_c(\sigma_i A)$ by mapping $\langle u_i^{\pm} \rangle \to \langle u_i^{\pm} \rangle$ and $K_i \to K_i$ for a sink $i \in I$. Therefore we can identify $\mathscr{D}_c(\sigma_i A)$ with $\mathscr{D}_c(A)$ under this canonical isomorphism. Then T_i is an automorphism $\mathscr{D}_c(A) \to \mathscr{D}_c(A)$.

6.3. THEOREM. (a) *Let i be a sink. Then the isomorphism $T_i : \mathscr{D}_c(A) \to \mathscr{D}_c(A)$ coincides with $T''_{i,1}$. Namely, $T_i = T''_{i,1}$ if we identify $\langle u_j^+ \rangle$ with E_j, $\langle u_j^- \rangle$ with $-v_j F_j$, and K_j with $\tilde{K}_j$ for $j \in I$.*

(b) *Let i be a source. Then the isomorphism $T_i : \mathscr{D}_c(A) \to \mathscr{D}_c(A)$ coincides with $T'_{i,-1}$ (see the definition of $T''_{i,1}$ and $T'_{i,-1}$ in* [L1, Chap. 37]).

Proof. (a) Assume i is a sink in Ω. Then it is a source in $\sigma_i \Omega$, and $V_{\sigma_i(j)}$ is a unique indecomposable module in rep-$\sigma_i \mathscr{S}\langle i \rangle$ with dimension vector

$$\mathbf{dim}\, V_{\sigma_i(j)} = e_j + n e_i,$$

where $n = -\frac{2(i,j)}{(i,i)}$. Thus, we have by Proposition 6.1,

$$\langle u_{\sigma_i(j)} \rangle = \sum_{r+s=n} (-1)^r v_i^{-r} \langle u_i \rangle^{(s)} \langle u_j \rangle \langle u_i \rangle^{(r)}.$$

Hence

$$T_i\big(\langle u_j^{\pm} \rangle\big) = \sum_{r+s=n} (-1)^r v_i^{-r} \langle u_i^{\pm} \rangle^{(s)} \langle u_j^{\pm} \rangle \langle u_i^{\pm} \rangle^{(r)}.$$

To prove T_i coincides with $T''_{i,1}$, it suffices to show that $T''_{i,1}$ and T_i have the same effect on generating sets. It can be easily seen by identification of

$$\big\{\langle u_j^+ \rangle\big\}_{j \in I} \text{ with } \{E_j\}_{j \in I}, \qquad \big\{\langle u_j^- \rangle\big\}_{j \in I} \text{ with } \{-v_j F_j\}_{j \in I}, \qquad \text{and}$$

$$\{K_j\}_{j \in I} \text{ with } \big\{\tilde{K}_j\big\}_{j \in I}.$$

The proof of (b) is similar. ∎

7. ACTIONS ON INTEGRABLE MODULES

7.1. Let (Γ, d) be a valued graph, Ω an orientation of it. Let $\mathscr{S}$ be a reduced k-species of (Γ, d, Ω), and A the tensor algebra of $\mathscr{S}$. An orientation Ω of (Γ, d) is said to be admissible if there is an ordering $k_1, k_2, \dots, k_n$ of Γ such that each vertex k_t is a sink with respect to the orientation $s_{k_{t-1}} \cdots s_{k_2} s_{k_1}\Omega$ for all $1 \le t \le n$. Such an ordering is called an admissible ordering for Ω. Now, let $k_1, k_2, \dots, k_n$ be an admissible ordering of Γ with respect to Ω. Then, according to Section 4, T_{k_1} is defined on $\mathscr{D}_c(A)$, T_{k_2} is defined on $\mathscr{D}(\sigma_{k_1} A)$, and, in general, T_{k_t} is defined on $\mathscr{D}(\sigma_{k_{t-1}} \cdots \sigma_{k_2}\sigma_{k_1} A)$ for $1 \le t \le n$.

As a $\mathbb{Q}(v)$-algebra, $\mathscr{D}_c(A)$ is the subalgebra of $\mathscr{D}(A)$ generated by $\{\langle u_i^+\rangle\}_{i=1}^n$, $\{\langle u_i^-\rangle\}_{i=1}^n$, and $\{K_{\pm i}\}_{i=1}^n$. It follows that from Green–Ringel Theorem in 3.10 that we have the canonical $\mathbb{Q}(v)$-algebra isomorphism

$$\mathscr{D}_c(A) \to \mathscr{D}_c\big(\sigma_{k_t} \cdots \sigma_{k_1} A\big) \tag{7.1.1}$$

by setting $u_i^+ \to u_i^+$, $u_i^- \to u_i^-$ and $K_i \to K_i$ for all $i \in I$. Therefore, we identify $\mathscr{D}_c(A)$ with $\mathscr{D}_c(\sigma_{k_t} \cdots \sigma_{k_1} A)$ for $1 \le t \le n$ along these canonical isomorphisms.

According to Theorem 4.5, we have the isomorphisms $T_i : \mathscr{D}_c(A) \to \mathscr{D}_c(\sigma_i A)$ and $T_i' : \mathscr{D}_c(\sigma_i A) \to \mathscr{D}_c(A)$ if i is a sink. So by the canonical isomorphism (7.1.1), we can view T_i and T_i' as automorphisms of $\mathscr{D}_c(A)$.

7.2. Lusztig first defined operators $T''_{i,e}$ on any integrable $U_q(\mathfrak{g})$-module V ($e = \pm 1$); then, he deduced the corresponding automorphisms $T''_{i,e}$ of $U_q(\mathfrak{g})$ and showed that the $T''_{i,e}$ satisfy braid group relations. In this section, we first define an action T_i on all integrable simple modules $L(\lambda)$ and ${}^{\omega}L(\lambda)$ in a global way. Then we give the formula $T_i(\eta)$ for $\eta \in L(\lambda)$, which is the same as the one defined by Lusztig.

7.3. A weight $\lambda \in \Lambda$ is said to be dominant if $(\lambda, i) \ge 0$ for all $i \in I$. Given a dominant weight λ, let

$$\begin{aligned} J_\lambda = {} & \sum_{i\in I} \mathscr{D}_c(A)\langle u_i^+\rangle + \sum_{i\in I} \mathscr{D}_c(A)\langle u_i^-\rangle^{n_i+1} \\ & + \sum_{i\in I} \mathscr{D}_c(A)\big(K_i - v^{(\lambda, i)}\big), \end{aligned} \tag{7.3.1}$$

where $n_i = \frac{2(\lambda, i)}{(i,i)}$ and the quotient module

$$L(\lambda) = \frac{\mathscr{D}_c(A)}{J_\lambda}. \tag{7.3.2}$$

According to the theory of quantum groups, $L(\lambda)$ is a simple integrable module of $\mathscr{D}_c(A)$ and is uniquely determined by λ. We denote by η_λ the coset of 1, which is a highest vector of weight λ.

We now define an action T_i and T_i', $i \in I$, on integrable simple modules $L(\lambda)$ of $\mathscr{D}_c(A)$ with λ dominant. Fix any $i \in I$; we define a map $T_i : L(\lambda) \to L(\lambda)$ by

$$T_i(x \cdot \eta_\lambda) = T_i(x) \cdot \langle u_i^- \rangle^{(n_i)} \eta_\lambda \tag{7.3.3}$$

for any $x \in \mathscr{D}_c(A)$. This is well defined. Indeed, $L(\lambda)$ can be made into a new $\mathscr{D}_c(A)$-module defined as

$$x * \eta = T_i(x) \cdot \eta \tag{7.3.4}$$

for all $x \in \mathscr{D}_c(A)$ and $\eta \in L(\lambda)$. We denote this module by $L(\lambda)^*$; note that as a space $L(\lambda)^* = L(\lambda)$. On the other hand, let $\eta_\lambda^* = \langle u_i^- \rangle^{(n_i)} \eta_\lambda$. Then clearly $\eta_\lambda^* \neq 0$ in $L(\lambda)^*$. If $j \neq i$, by Theorem 6.3 we have

$$T_i\big(\langle u_j^+ \rangle\big) = \sum_{r+s=-a_{ij}} (-1)^r v_i^{-r} \langle u_i^+ \rangle^{(s)} \langle u_j^+ \rangle \langle u_i^+ \rangle^{(r)}.$$

According to the formulae

$$\langle u_i^+ \rangle^{(r)} \langle u_j^- \rangle^{(s)} = \langle u_j^- \rangle^{(s)} \langle u_i^+ \rangle^{(r)}$$

and

$$\langle u_i^+ \rangle^{(r)} \langle u_i^- \rangle^{(s)} \eta_\lambda = \sum_{t \geq 0} (-1)^t v_i^t \begin{bmatrix} r - s + 2\dfrac{(\lambda, i)}{(i,i)} \\ t \end{bmatrix}_i \times \langle u_i^- \rangle^{(s-t)} \langle u_i^+ \rangle^{(r-t)} \eta_\lambda \tag{7.3.5}$$

(cf. [L1, 3.4.2]), we see that

$$\begin{aligned} \langle u_j^+ \rangle * \eta_\lambda^* &= T_i\big(\langle u_j^+ \rangle\big) \eta_\lambda^* \\ &= \sum_{r+s=-a_{ij}} (-1)^r v_i^{-r} \langle u_i^+ \rangle^{(s)} \langle u_j^+ \rangle \langle u_i^+ \rangle^{(r)} \langle u_i^- \rangle^{(n_i)} \eta_\lambda \\ &= 0. \end{aligned}$$

If $j = i$, then

$$\langle u_i^+ \rangle * \eta_\lambda^* = v^{-\varepsilon(i)} \langle u_i^- \rangle K_i \langle u_i^- \rangle^{(n_i)} \eta_\lambda = 0,$$

since $\langle u_i^- \rangle^{(n_i+1)} \eta_\lambda = 0$. Also

$$K_\mu * \eta_\lambda^* = T_i\big(K_\mu\big) \eta_\lambda^* = v^{(s_i \mu, s_i \lambda)} \eta_\lambda^* = v^{(\lambda, \mu)} \eta_\lambda^*$$

for any $\mu \in \mathbb{Z}[I]$ since $\eta_\lambda^* \in L(\lambda)_{s_i(\lambda)}$. This means that $\eta_\lambda^* \in L(\lambda)_\lambda^*$ is a highest vector of weight λ. Therefore, there is a unique homomorphism $T_i : L(\lambda) \to L(\lambda)^*$ as $\mathscr{D}_c(A)$-modules such that

$$T_i(\eta_\lambda) = \eta_\lambda^*.$$

Because both $L(\lambda)$ and $L(\lambda)^*$ are simple,

$$T_i : L(\lambda) \to L(\lambda)^*$$

is an isomorphism as desired. One sees that $T_i(x\eta) = T_i(x)T_i(\eta)$ for all $x \in \mathscr{D}_c(A)$ and $\eta \in L(\lambda)$.

In a similar way, we may define another action T_i' on $L(\lambda)$. Explicitly,

$$T_i'(x \cdot \eta_\lambda) = (-1)^{n_i} v_i^{-n_i} T_i'(x) \langle u_i^- \rangle^{(n_i)} \eta_\lambda.$$

One can prove that T_i' is just the inverse of T_i. Indeed,

$$\begin{aligned} T_i' T_i(\eta_\lambda) &= T_i'\big(\langle u_i^- \rangle^{(n_i)} \eta_\lambda\big) \\ &= T_i'\big(\langle u_i^- \rangle^{(n_i)}\big) T_i'(\eta_\lambda) \\ &= v_i^{-n_i^2} K_{n_i i} \langle u_i^+ \rangle^{(n_i)} (-1)^{n_i} v_i^{-n_i} \langle u_i^- \rangle^{(n_i)} \eta_\lambda \\ &= (-1)^{n_i} v_i^{-n_i^2 - n_i} (-1)^{n_i} v_i^{n_i} K_{n_i i} \eta_\lambda \ (\text{by } (7.3.5)) \\ &= \eta_\lambda. \end{aligned}$$

Similarly, $T_i T_i'(\eta_\lambda) = \eta_\lambda$.

7.4. *Remark.* It is easy to see that

$$\omega : \langle u_i^+ \rangle \to \langle u_i^- \rangle, \qquad \langle u_i^- \rangle \to \langle u_i^+ \rangle, \qquad \text{and} \qquad K_i \to K_{-i}$$

induce a unique automorphism of $\mathscr{D}_c(A)$. Let $L(\lambda)$ be an integrable simple $\mathscr{D}_c(A)$-module with λ a dominant weight; we define a new module ${}^{\omega}L(\lambda)$ as follows: the ${}^{\omega}L(\lambda)$ have the same underlying $\mathbb{Q}(v)$-space as $L(\lambda)$. By definition $({}^{\omega}L(\lambda))_\mu = L(\lambda)_{-\mu}$ for any weight. For any $u \in \mathscr{D}_c(A)$, the operator u on ${}^{\omega}L(\lambda)$ coincides with the operator $\omega(u)$ on $L(\lambda)$. It is easy to see that

$${}^{\omega}L(\lambda) \cong \frac{\mathscr{D}_c(A)}{\sum_i \mathscr{D}_c(A)\langle u_i^- \rangle + \sum_i \mathscr{D}_c(A)\langle u_i^+ \rangle^{n_i+1} + \sum_i \mathscr{D}_c(A)(K_i - v_i^{-n_i})},$$

as $\mathscr{D}_c(A)$-modules. One sees that when η_λ is considered as an element of ${}^{\omega}L(\lambda)$, then $\eta_\lambda \in {}^{\omega}L(\lambda)_{-\lambda}$. Since T_i is well defined on $L(\lambda)$, T_i can be defined on ${}^{\omega}L(\lambda)$ in the natural way. Of course, $T_i(x\eta) = T_i(x)T_i(\eta)$ for all $\eta \in {}^{\omega}L(\lambda)$ and $x \in \mathscr{D}_c(A)$.

7.5. We have constructed the $\mathbb{Q}(v)$-linear map $T_i : L(\lambda) \to L(\lambda)$ for any dominant weight λ such that

$$T_i(\eta_\lambda) = \langle u_i^- \rangle^{(n_i)} \eta_\lambda \qquad \text{and} \qquad T_i(x\eta) = T_i(x) \cdot T_i(\eta),$$

for all $x \in \mathscr{D}_c(A)$ and $\eta \in L(\lambda)$, where λ is a dominant weight and η_λ is a highest weight vector, and $n_i = \frac{2(i,\lambda)}{(i,i)}$. Now, we have the following theorem.

THEOREM. *For any integrable simple $\mathscr{D}_c(A)$-module $L(\lambda)$ with λ dominant we have*

$$(*) \qquad T_i(\eta) = \sum_{a,b,c \geq 0,\, a-b+c=m} v_i^{-ac} \langle u_i^- \rangle^{(a)} \langle u_i^+ \rangle^{(b)} \langle u_i^- \rangle^{(c)} \eta,$$

for all $\eta \in L(\lambda)_\mu$ and $m = \frac{2(i,\mu)}{(i,i)}$.

7.5.1. *Remark.* By a similar discussion, one sees that

$$T_i(\eta) = \sum_{a,b,c \geq 0;\, -a+b-c=m} v_i^{-ac} \langle u_i^+ \rangle^{(a)} \langle u_i^- \rangle^{(b)} \langle u_i^+ \rangle^{(c)} \eta$$

for all $\eta \in ({}^{\omega}L(\lambda))_{-\mu}$ (note that $\eta \in L(\lambda)_\mu$) with $m = -\frac{2(i,\mu)}{(i,i)}$. In fact, this is just the another form of $(*)$ in $L(\lambda)$.

7.5.2. *Remark.* Given an integrable $\mathscr{D}_c(A)$-module V, it is also an integrable $U_q(\mathfrak{g})$-module defined by $x\eta = \theta^{-1}(x)\eta$ for $x \in U_q(\mathfrak{g})$ and $\eta \in V$, where θ is the isomorphism in (3.10). Now, let $\eta \in L(\lambda)$. Then we have

$$\begin{aligned} T_i(\eta) &= \sum_{a,b,c \geq 0;\, a-b+c=m} v_i^{-ac} \langle u_i^- \rangle^{(a)} \langle u_i^+ \rangle^{(b)} \langle u_i^- \rangle^{(c)} \eta \\ &= (-1)^m v_i^m \sum_{a,b,c \geq 0;\, a-b+c=m} \left((-1)^b v_i^{b-ac} F_i^{(a)} E_i^{(b)} F_i^{(c)} \right) \eta \\ &= T''_{i,1}(\eta). \end{aligned}$$

This means that the operator T_i on $L(\lambda)$ coincides with the operator $T''_{i,1}$ on $L(\lambda)$.

To prove Theorem 7.5, we first introduce some basic lemmas.

7.6. Let

$$b(m) = \sum_{k=0}^{m} (-1)^k v_i^{k(m-1-r)} \langle u_i^- \rangle^{(m-k)} \langle u_j^- \rangle \langle u_i^- \rangle^{(k)}$$

for all $m \geq 0$. One sees that for all $m > r$, $b(m) = 0$ by the quantum Serre's relation, where $r = -\frac{2(i,j)}{(i,i)}$. By Theorem 6.3, $b(r) = T_i(\langle u_j^- \rangle)$; it is easy to see that

$$\langle u_i \rangle \langle u_{\sigma_i(j)} \rangle = v_i^{-r} \langle u_{\sigma_i(j)} \rangle \langle u_i \rangle$$

since there is no non-trivial extension of V_i by $V_{\sigma_i(j)}$. It follows that

$$b(r) \langle u_i^- \rangle^{(k)} = v_i^{-kr} \langle u_i^- \rangle^{(k)} b(r). \tag{7.6.1}$$

LEMMA. *We have for all integers m, $k \geq 0$,*

$$b(m) \langle u_i^- \rangle^{(k)} = \sum_{t=0}^{k} (-1)^t \begin{bmatrix} m+t \\ t \end{bmatrix}_i \times v_i^{k(r-2m)-t(k-1)} \langle u_i^- \rangle^{(k-t)} b(m+t) \tag{7.6.2}$$

$$b(m) \langle u_i^+ \rangle^{(k)} = \sum_{t=0}^{k} \begin{bmatrix} r-m+t \\ t \end{bmatrix}_i v_i^{kt} \langle u_i^+ \rangle^{(k-t)} b(m-t) K_i^t. \tag{7.6.3}$$

Proof. Let

$$b'(m) = \sum_{p=0}^{m} (-1)^p v_i^{p(m-1-r)} F_i^{(m-p)} F_j F_i^{(p)}$$

for all $m \geq 0$. By Theorem 3.10, Eqs. (7.6.2) and (7.6.3) are equivalent to the following, which are given in [Jan, 8.9], respectively,

$$b'(m) F_i^{(k)} = \sum_{p=0}^{k} (-1)^p \begin{bmatrix} m+p \\ m \end{bmatrix}_i v_i^{k(r-2m)-p(k-1)} \times F_i^{(k-p)} b'(m+p) \tag{7.6.4}$$

$$b'(m) E_i^{(k)} = \sum_{p=0}^{k} (-1)^p \begin{bmatrix} r-m+p \\ p \end{bmatrix}_i v_i^{p(k-r)} \times E_i^{(k-p)} b'(m-p) K_i^p \tag{7.6.5}$$

and we have the result. ∎

7.7. LEMMA. *If*

$$T_i(\eta) = \sum_{a,b,c\geq 0,\, a-b+c=s} v_i^{-ac}\langle u_i^-\rangle^{(a)}\langle u_i^+\rangle^{(b)}\langle u_i^-\rangle^{(c)}\eta$$

for $\eta \in L(\lambda)_\mu$, *where* $s = \frac{2(i,\mu)}{(i,i)}$, *then*

$$T_i\big(\langle u_j^-\rangle\eta\big) = \sum_{a,b,c\geq 0,\, a-b+c=s'} v_i^{-ac}\langle u_i^-\rangle^{(a)}\langle u_i^+\rangle^{(b)}\langle u_i^-\rangle^{(c)}\big(\langle u_j^-\rangle\eta\big),$$

where $s' = \frac{2(i,\mu-j)}{(i,i)} = s + r.$

Proof. By definition we have $K_i^p\langle u_i^-\rangle^{(c)}\eta = v_i^{p(s-2c)}\langle u_i^-\rangle^{(c)}\eta$ for all $p, c \geq 0$ and

$$T_i\big(\langle u_j^-\rangle\eta\big) = T_i\big(\langle u_j^-\rangle\big)T_i(\eta) = b(r)T_i(\eta).$$

On the other hand, by the formulae (7.6.1), (7.6.2), and (7.6.3), we have

$$\begin{aligned}
&b(r)T_i(\eta)\\
&\quad= \sum_{a,b,c\geq 0,\, a-b+c=s} v_i^{-ac}b(r)\langle u_i^-\rangle^{(a)}\langle u_i^+\rangle^{(b)}\langle u_i^-\rangle^{(c)}\eta\\
&\quad= \sum_{a,b,c\geq 0,\, a-b+c=s} v_i^{-a(c+r)}\langle u_i^-\rangle^{(a)}b(r)\langle u_i^+\rangle^{(b)}\langle u_i^-\rangle^{(c)}\eta\\
&\quad= \sum_{a,b,c\geq 0,\, a-b+c=s}\ \sum_{p=0}^{\min(b,r)} v_i^{-a(c+r)}\langle u_i^-\rangle^{(a)}v_i^{pb}\langle u_i^+\rangle^{(b-p)}\\
&\qquad\times b(r-p)K_i^p\langle u_i^-\rangle^{(c)}\eta\\
&\quad= \sum_{a,b,c\geq 0,\, a-b+c=s}\ \sum_{p=0}^{\min(b,r)}\ \sum_{q=0}^{c} v_i^{-a(c+r)+p(b+s-2c)}\\
&\qquad\times\langle u_i^-\rangle^{(a)}\langle u_i^+\rangle^{(b-p)}(-1)^q\begin{bmatrix} r-p+q\\ q\end{bmatrix}_i v_i^{c(-r+2p)-q(c-1)}\\
&\qquad\times\langle u_i^-\rangle^{(c-q)}b(r-p+q)\eta\\
&\quad= \sum_{a,b,c\geq 0,\, a-b+c=s}\ \sum_p\sum_q(-1)^q v_i^{-a(c+r)+p(b+s)-cr-q(c-1)}\begin{bmatrix} r-p+q\\ q\end{bmatrix}_i\\
&\qquad\times\langle u_i^-\rangle^{(a)}\langle u_i^+\rangle^{(b-p)}\langle u_i^-\rangle^{(c-q)}b(r-p+q)\eta,
\end{aligned}$$

for all summands $a-(b-p)+(c-q) = s+h$, where $h = p-q$. Note that $b(r-p+q) = 0$ for $q > p$. Replace b by $b+p$ and c by $c+q$.

Then $b(r)T_i(\eta)$ is a linear combinations of terms $\langle u_i^-\rangle^{(a)}\langle u_i^+\rangle^{(b)}\langle u_i^-\rangle^{(c)}b(r-h)\eta$ with $a,b,c,h \geq 0$ and $a-b+c=s+h$. The coefficient of $\langle u_i^-\rangle^{(a)}\langle u_i^+\rangle^{(b)}\langle u_i^-\rangle^{(c)}b(r-h)\eta$ is equal to

$$\sum_{q=0}^{r-h}(-1)^q v_i^{-a(c+q+r)+p(b+p+s)-(c+q)r-q(c+q-1)}\begin{bmatrix} r-p+q \\ q \end{bmatrix}_i$$

$$= v_i^{-a(c+r)-cr+h(b+s+h)}\sum_{q=0}^{r-h}(-1)^q v_i^{-q(r-h-1)}\begin{bmatrix} r-h \\ q \end{bmatrix}_i$$

since $p=h+q$ and $a-b+c=s+h$. However,

$$\sum_{q=0}^{r-h}(-1)^q v_i^{-q(r-h-1)}\begin{bmatrix} r-h \\ q \end{bmatrix}_\alpha = \delta_{r-h},0$$

and for $h=r$, $v_i^{-a(c+r)-cr+h(b+s+h)} = v_i^{-ac}$. So we get

$$b(r)T_i(\eta) = \sum_{a,b,c\geq 0;\, a-b+c=s+r} v_i^{-ac}\langle u_i^-\rangle^{(a)}\langle u_i^-\rangle^{(b)}\langle u_i^-\rangle^{(c)}b(0)\eta,$$

where $b(0)=\langle u_j^-\rangle$ and $\langle u_j^-\rangle\eta \in L(\lambda)_{\mu-j}$ with $s' = \frac{2(\mu-j,i)}{(i,i)} = s+r$. Hence

$$T_i\big(\langle u_j^-\rangle\eta\big) = \sum_{a,b,c\geq 0;\, a-b+c=s'} v_i^{-ac}\langle u_i^-\rangle^{(a)}\langle u_i^+\rangle^{(b)}\langle u_i^-\rangle^{(c)}\big(\langle u_j^-\rangle\eta\big).$$

∎

7.8. LEMMA. *Let $\eta \in L(\lambda)_\mu$ with $m = \frac{2(i,\mu)}{(i,i)}$. If $\langle u_i^+\rangle\eta = 0$, then*

$$\sum_{a,b,c\geq 0,\, a-b+c=m'} v_i^{-ac}\langle u_i^-\rangle^{(a)}\langle u_i^+\rangle^{(b)}\langle u_i^-\rangle^{(c)}\big(\langle u_i^-\rangle^{(p)}\eta\big)$$

$$= (-1)^p v_i^{p(h+1)}\langle u_i^-\rangle^{(h)}\eta,$$

where $m' = \frac{(i,\mu-pi)}{(i,i)} = m-2p$ and $h=m-p$ with $p,h\geq 0$.

Proof. Note that $\langle u_i^-\rangle^{(p)}\eta \in L(\lambda)_{\mu-pi}$. Assume that $a-b+c=m-2p$. We have

$$\langle u_i^-\rangle^{(a)}\langle u_i^+\rangle^{(b)}\langle u_i^-\rangle^{(c)}\langle u_i^-\rangle^{(p)}\eta$$

$$= \begin{bmatrix} c+p \\ c \end{bmatrix}_i \langle u_i^-\rangle^{(a)}\langle u_i^+\rangle^{(b)}\langle u_i^-\rangle^{(c+p)}\eta$$

$$= \sum_{t\geq 0}(-1)^t v_i^t \begin{bmatrix} c+p \\ c \end{bmatrix}_i \begin{bmatrix} b-c+h \\ b \end{bmatrix}_i \langle u_i^-\rangle^{(a)}\langle u_i^-\rangle^{(c+p-t)}\langle u_i^+\rangle^{(b-t)}\eta.$$

Now, $\langle u_i^+\rangle^{(b-t)}\eta = 0$ if $b \neq t$, thus,

$$\langle u_i^-\rangle^{(a)}\langle u_i^+\rangle^{(b)}\langle u_i^-\rangle^{(c)}\big(\langle u_i^-\rangle^{(p)}\eta\big) = (-1)^b v_i^b \begin{bmatrix} c+p \\ c \end{bmatrix}_i \begin{bmatrix} a+p \\ b \end{bmatrix}_i \begin{bmatrix} h \\ a \end{bmatrix}_i \langle u_i^-\rangle^{(h)}\eta.$$

However,

$$\sum_{a,b,c\geq 0;\, a-b+c=h-p} (-1)^b v_i^{b-ac} \begin{bmatrix} c+p \\ p \end{bmatrix} \begin{bmatrix} a+p \\ b \end{bmatrix}_i \begin{bmatrix} h \\ a \end{bmatrix}_i = (-1)^p v_i^{(h+1)p}$$

for any $p, h \geq 0$ (see [L1, Proposition 5.2.2]); it follows that

$$\sum_{a,b,c\geq 0;\, a-b+c=m'} v_i^{-ac}\langle u_i^-\rangle^{(a)}\langle u_i^+\rangle^{(b)}\langle u_i^-\rangle^{(c)}\big(\langle u_i^-\rangle^{(p)}\eta\big) = (-1)^p v_i^{p(h+1)}\langle u_i^-\rangle^{(h)}\eta.$$

∎

7.9. Lemma. *Given $\eta \in L(\lambda)_\mu$, if $\langle u_i^+\rangle\eta = 0$, then $T_i(\eta) = \langle u_i^-\rangle^{(n)}\eta$, where $n = \frac{2(i,\mu)}{(i,i)}$.*

Proof. We use the induction. By definition, we have

$$T_i(\eta_\lambda) = \langle u_i^-\rangle^{(n_i)}\eta_\lambda.$$

Assume $T_i(\eta) = \langle u_i^-\rangle^{(m)}\eta$ provided $\langle u_i^+\rangle\eta = 0$ with $\mu \leq \lambda$ and $\eta \in L(\lambda)_\mu$, where $m = \frac{2(i,\mu)}{(i,i)}$. It follows that $\langle u_i^-\rangle^{(m+1)}\eta = 0$ since $T_i(\langle u_i^+\rangle\eta) = b\langle u_i^-\rangle^{(m+1)}\eta$ for some $b \neq 0$.

First, by assumption and Lemma 7.8 we have

$$T_i(\eta) = \sum_{a,b,c\geq 0,\, a-b+c=m} v_i^{-ac}\langle u_i^-\rangle^{(a)}\langle u_i^+\rangle^{(b)}\langle u_i^-\rangle^{(c)}\eta.$$

Now, consider $\eta' = \langle u_j^-\rangle\eta$ with $\langle u_i^+\rangle\eta' = 0$; clearly, $j \neq i$. If $\eta' = 0$, there is nothing to verify. If $\eta' \neq 0$, then we have by Lemma 7.7

$$T_i(\eta') = \sum_{a-b+c=m'} v_i^{-ac}\langle u_i^-\rangle^{(a)}\langle u_i^+\rangle^{(b)}\langle u_i^-\rangle^{(c)}\eta',$$

where $m' = \frac{2(i,-j+\mu)}{(i,i)} = m + r$. On the other hand, again by Lemma 7.8 we have

$$\langle u_i^-\rangle^{(m')}\eta' = \sum_{a-b+c=m'} v_i^{-ac}\langle u_i^-\rangle^{(a)}\langle u_i^+\rangle^{(b)}\langle u_i^-\rangle^{(c)}\eta'.$$

Hence $T_i(\eta') = \langle u_i^-\rangle^{(m')}\eta'$. The proof is completed. ∎

7.10. Now, we turn to prove Theorem 7.5. Let $\mathscr{D}_c(i)$ be the subalgebra of $\mathscr{D}_c(A)$ generated by $\langle u_i^+\rangle$, $\langle u_i^-\rangle$, and $K_i^{\pm}$.

$L(\lambda)$ can be viewed as an integrable $\mathscr{D}_c(i)$-module by the natural way, thus $L(\lambda)$ is a direct sum of some simple $\mathscr{D}_c(i)$-modules $L(m)$ with integers $m \geq 0$. Such direct summands $L(m)$ of $L(\lambda)$ can be identified with

$$\frac{\mathscr{D}_c(i)}{\mathscr{D}_c(i)\langle u_i^+\rangle + \mathscr{D}_c(i)\langle u_i^-\rangle^{m+1} + \mathscr{D}_c(i)(K_i - v_i^m)}$$

as $\mathscr{D}_c(i)$-modules for various integers $m \in \mathbb{N}$.

Let $0 \neq \eta \in L(m)_m$; then $\langle u_i^+\rangle\eta = 0$ and $T_i(x\eta) = T_i(x)\langle u_i^-\rangle^{(m)}\eta$ for all $x \in \mathscr{D}_c(i)$. It is easy to see that $\eta, \langle u_i^-\rangle\eta, \cdots \langle u_i^-\rangle^{(m)}\eta$ form a basis of $L(m)$. On one hand,

$$\begin{aligned} T_i\big(\langle u_i^-\rangle^{(p)}\eta\big) &= T_i\big(\langle u_i^-\rangle^{(p)}\big)T_i(\eta) \qquad \text{(by (7.3.3))} \\ &= v_i^{p^2}K_{-pi}\langle u_i^+\rangle\big)^{(p)}\langle u_i^-\rangle^{(m)}\eta \qquad \text{(by Lemma 7.9)} \\ &= (-1)^p v_i^{p^2+p}K_{-pi}\langle u_i^-\rangle^{(m-p)}\eta \\ &= (-1)^p v_i^{p^2+p+2p(m-p)-pm}\langle u_i^-\rangle^{(m-p)}\eta \\ &= (-1)^p v_i^{mp-p^2+p}\langle u_i^-\rangle^{(m-p)}\eta \\ &= (-1)^j v_i^{p(h+1)}\langle u_i^-\rangle^{(h)}\eta, \end{aligned}$$

where $h = m - p, h, p \geq 0$. On the other hand, combining Lemma 7.8 one sees that

$$T_i\big(\langle u_i^-\rangle^{(p)}\eta\big) = \sum_{a,b,c\geq 0;\, a-b+c=m-2p} v_i^{-ac}\langle u_i^-\rangle^{(a)}\langle u_i^+\rangle^{(b)}\langle u_i^-\rangle^{(c)}\big(\langle u_i^-\rangle^{(p)}\eta\big).$$

This means that for all $\eta' \in L(m)_n$ with $n \leq m$, then

$$T_i(\eta') = \sum_{a,b,c\geq 0;\, a-b+c=n} v_i^{-ac}\langle u_i^-\rangle^{(a)}\langle u_i^+\rangle^{(b)}\langle u_i^-\rangle^{(c)}\eta'.$$

Thus, we have

$$T_i(\eta) = \sum_{a,b,c\geq 0;\, a-b+c=m} v_i^{-ac}\langle u_i^-\rangle^{(a)}\langle u_i^+\rangle^{(b)}\langle u_i^-\rangle^{(c)}\eta,$$

where $\eta \in L(\lambda)_\mu$ and $m = \frac{2(i,\mu)}{(i,i)}$. The result follows. ∎

7.11. As a special case, if A is finite-type, then $\mathscr{D}_c(A) = \mathscr{D}(A)$ and any integrable $\mathscr{D}(A)$-module is a direct sum of some integrable simple modules $L(\lambda)$ with dominant weights λ. Thus, the definition of T_i in 7.3 can be naturally defined on V. In general cases, the formula in 7.5 provides us with a definition of the symmetries T_i, $i \in I$, in a local way (due to Lusztig). However, it has an advantage over the global definition of T_i in 7.3. Namely it can be used to define T_i acting on every integrable $\mathscr{D}_c(A)$-module.

COROLLARY. *Let V be any integrable $\mathscr{D}_c(A)$-module. For any $u \in \mathscr{D}_c(A)$ and $\eta \in V$, we have $T_i(u\eta) = T_i(u)T_i(\eta)$ for $i \in I$.*

Proof. Apply the slight modification of Lemma 7.7 and the action of ω (cf. 7.4 and 7.5.1). ∎

8. BRAID GROUP RELATIONS

8.1. Set $a_{ij} = \frac{2(i,j)}{(i,i)}$ for $i, j \in I$, where

$$(i,j) = (\mathbf{dim}\, V_i, \mathbf{dim}\, V_j).$$

Then $C = (a_{ij})_{i,j \in I}$ is a symmetrizable generalized Cartan matrix. Therefore, a classical result in the Kac–Moody algebra is that (see [K]), if $d(i,j) = a_{ij}a_{ji} \leq 3$, then the order $m(i,j)$ of $s_i s_j$ is finite. Namely

$$\text{if } d(i,j) = 0, \text{ then } m(i,j) = 2$$

$$\text{if } d(i,j) = 1, \text{ then } m(i,j) = 3$$

$$\text{if } d(i,j) = 2, \text{ then } m(i,j) = 4$$

$$\text{if } d(i,j) = 3, \text{ then } m(i,j) = 6$$

$$\text{if } d(i,j) \geq 4, \text{ then } m(i,j) = \infty.$$

8.2. Let Δ be the Cartan datum corresponding to A. Then the braid group of type Δ is defined by the generators $\{\sigma_i\}_{i \in I}$ and relations

(8.2.1) $$\sigma_i \sigma_j \cdots = \sigma_j \sigma_i \cdots$$

for $i \neq j \in I$ with $m(i,j) < +\infty$ factors on both sides, where $m(i,j)$ is the order of $s_i s_j$ in W. Namely

(8.2.2) $$\text{if } m(i,j) = 2, \text{ then } \sigma_i\sigma_j = \sigma_j\sigma_i$$

$$\text{if } m(i,j) = 3, \text{ then } \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$$

$$\text{if } m(i,j) = 4, \text{ then } \sigma_i\sigma_j\sigma_i\sigma_j = \sigma_j\sigma_i\sigma_j\sigma_i$$

$$\text{if } m(i,j) = 6, \text{ then } \sigma_i\sigma_j\sigma_i\sigma_j\sigma_i\sigma_j = \sigma_j\sigma_i\sigma_j\sigma_i\sigma_j\sigma_i.$$

Note that the second relation yields the classic braid group $B(n)$ for Δ of A_n-type. Our main result in this section is the following.

8.3. THEOREM. *For any $i \neq j$ in I such that $m = m(i, j) < +\infty$, then T_i and T_j satisfy braid group relations* (8.2.2) *as operators acting on* $\mathscr{D}_c(A)$.

Proof. We will show that T_i and T_j satisfy the following braid group relations,

$$T_i T_j \cdots = T_j T_i \cdots, \tag{8.3.1}$$

where we have $m = m(i, j)$ factors on both sides.

It is easy to see that both sides of (8.3.1) coincide on the torus algebra T. Because $(s_i s_j)^{m(i,j)} = 1$ and $s_i^2 = s_j^2 = 1$ in the Weyl group, therefore

$$T_i T_j \cdots (K_\alpha) = K_{s_i s_j \cdots (\alpha)} = K_{s_j s_i \cdots (\alpha)} = T_j T_i \cdots (K_\alpha)$$

for all $\alpha \in \mathbb{Z}[I]$.

Let us first consider the case of rank 2. This means that Eq. (8.3.1) holds on $\langle u_i^\pm \rangle$ and $\langle u_j^\pm \rangle$.

The case $m(i, j) = 2$. Now $d(i, j) = 0$, hence $a_{ij} = a_{ji} = 0$. This means that the vertices i and j are not neighbours, so $s_i(j) = j$ and $s_j(i) = i$. Since $T_i T_j(\langle u_i^+ \rangle) = T_i(\langle u_i^+ \rangle) = \langle u_i^- \rangle \overline{K}_i$. $T_j T_i(\langle u_i^+ \rangle) = T_j(\langle u_i^- \rangle \overline{K}_i) = \langle u_i^- \rangle \overline{K}_i$, we have $T_i T_j(\langle u_i^+ \rangle) = T_j T_i(\langle u_i^+ \rangle)$. In a similar way we get $T_i T_j = T_j T_i$ as acting on $\langle u_i^\pm \rangle$ and $\langle u_j^\pm \rangle$.

The case $m(i, j) = 3$. Now $d(i, j) = 1$, hence $a_{ij} = a_{ji} = -1$. It follows that $s_i s_j(i) = j$ and $s_j s_i(j) = i$ and $\varepsilon(i) = \varepsilon(j)$. We have

$$\langle u_i^+ \rangle \overset{T_j}{\to} \langle u_{\sigma_j(i)}^+ \rangle \overset{T_i}{\to} \langle u_{\sigma_i \sigma_j(i)}^+ \rangle = \langle u_j^+ \rangle.$$

thus, $T_i T_j(\langle u_i^+ \rangle) = \langle u_j^+ \rangle$ and $T_i T_j(\langle u_i^- \rangle) = \langle u_j^- \rangle$. Also $T_j T_i(\langle u_j^+ \rangle) = \langle u_i^+ \rangle$, and $T_j T_i(\langle u_j^- \rangle) = \langle u_i^- \rangle$. Since

$$T_i T_j T_i(\langle u_i^+ \rangle) = T_i T_j\big(\langle u_i^- \rangle \overline{K}_i\big) = \langle u_j^- \rangle v^{-\langle i,i \rangle} K_j = \langle u_j^- \rangle \overline{K}_j,$$

$$T_j T_i T_j(\langle u_i^+ \rangle) = T_j\big(\langle u_j^- \rangle\big) = \langle u_j^- \rangle \overline{K}_j,$$

therefore we have $T_i T_j T_i = T_j T_i T_j$ on $\langle u_i^\pm \rangle$ and $\langle u_j^\pm \rangle$.

The case $m(i, j) = 4$. Now $d(i, j) = 2$, hence $a_{ij} a_{ji} = 2$, and without loss of generality, we assume that $a_{ij} = -2$ and $a_{ji} = -1$. It follows that $s_j s_i s_j(i) = i$ and $s_i s_j s_i(j) = j$. We have

$$\langle u_i^\pm \rangle \overset{T_j}{\to} \langle u_{\sigma_j(i)}^\pm \rangle \overset{T_i}{\to} \langle u_{\sigma_i \sigma_j(i)}^\pm \rangle \overset{T_j}{\to} \langle u_{\sigma_j \sigma_i \sigma_j(i)}^\pm \rangle = \langle u_i^\pm \rangle$$

and

$$\langle u_j^{\pm}\rangle \xrightarrow{T_i} \langle u^{\pm}_{\sigma_i(j)}\rangle \xrightarrow{T_j} \langle u^{\pm}_{\sigma_j\sigma_i(j)}\rangle \xrightarrow{T_i} \langle u^{\pm}_{\sigma_i\sigma_j\sigma_i(j)}\rangle = \langle u_j^{\pm}\rangle.$$

Thus,

$$T_iT_jT_iT_j(\langle u_i^+\rangle) = T_i\big(\langle u_i^-\rangle \overline{K}_i\big) = \langle u_i^-\rangle \overline{K}_i$$

$$T_jT_iT_jT_i(\langle u_i^+\rangle) = T_jT_iT_j\big(\langle u_i^-\rangle \overline{K}_i\big) = \langle u_i^-\rangle v^{-\varepsilon(i)} K_{s_js_is_j(i)} = \langle u_i^-\rangle \overline{K}_i.$$

Similar calculations show that $T_iT_jT_iT_j = T_jT_iT_jT_i$ on $\langle u_i^{\pm}\rangle$ and $\langle u_j^{\pm}\rangle$.

The case $m(i,j) = 6$. Now $d(i,j) = 3$, hence $a_{ij}a_{ji} = 3$ and we may assume that $a_{ij} = -3$ and $a_{ji} = -1$. It follows that $s_js_is_js_is_j(i) = i$ and $s_isjs_is_js_i(j) = j$. We have

$$\langle u_i^{\pm}\rangle \xrightarrow{T_j} \langle u^{\pm}_{\sigma_j(i)}\rangle \xrightarrow{T_i} \langle u^{\pm}_{\sigma_i\sigma_j(i)}\rangle \xrightarrow{T_j} \langle u^{\pm}_{\sigma_j\sigma_i\sigma_j(i)}\rangle$$

$$\xrightarrow{T_i} \langle u^{\pm}_{\sigma_i\sigma_j\sigma_i\sigma_j(i)}\rangle \xrightarrow{T_j} \langle u^{\pm}_{\sigma_j\sigma_i\sigma_j\sigma_i\sigma_j(i)}\rangle = \langle u_i^{\pm}\rangle$$

and

$$\langle u_j^{\pm}\rangle \xrightarrow{T_i} \langle u^{\pm}_{\sigma_i(j)}\rangle \xrightarrow{T_j} \langle u^{\pm}_{\sigma_j\sigma_i(j)}\rangle \xrightarrow{T_i} \langle u^{\pm}_{\sigma_i\sigma_j\sigma_i(j)}\rangle$$

$$\xrightarrow{T_j} \langle u^{\pm}_{\sigma_j\sigma_i\sigma_j\sigma_i(j)}\rangle \xrightarrow{T_i} \langle u^{\pm}_{\sigma_i\sigma_j\sigma_i\sigma_j\sigma_i(j)}\rangle = \langle u_j^{\pm}\rangle.$$

Thus we have $T_iT_jT_iT_jT_iT_j = T_jT_iT_jT_iT_jT_i$ on $\langle u_i^{\pm}\rangle$ and $\langle u_j^{\pm}\rangle$. So we have shown Theorem 8.3 in the case of rank 2. To prove the theorem in general, we should consider the action of T_i, $i \in I$, on the integrable modules over $\mathscr{D}_c(A)$.

8.4. Keep the notations as in Section 7. Assume $s_{i_1}s_{i_2}\cdots s_{i_N}$ is a reduced expression. For an integrable simple module $L(\lambda)$, by the definition in (7.3.3) and induction, it is easy to see that

$$\begin{aligned} T_{i_1}\cdots T_{s_N}(\eta_\lambda) &= \langle u_{i_1}^-\rangle^{(a_1)}\langle u_{i_2}^-\rangle^{(a_2)}\cdots\langle u_{i_N}^-\rangle^{(a_N)}\eta_\lambda \\ &= \langle u_{i_1}^-\rangle^{(a_1)} T_{i_2}\cdots T_{s_N}(\eta_\lambda), \end{aligned}$$

where $a_1 = 2(s_{i_N}\cdots s_{i_2}(i_1), \lambda)/(i_1, i_1)$, $a_2 = 2(s_{i_N}\cdots s_{i_3}(i_2), \lambda)/(i_2, i_2), \dots, a_N = 2(i_N, \lambda)/(i_N, i_N)$. Note that $a_1, a_2, \dots, a_N \in \mathbb{N}$.

The following lemma is crucial for the proof of Theorem 8.3.

LEMMA. *Let $\eta_\lambda \in L(\lambda)$ be the highest vector with dominant λ. If $m(i,j) < +\infty$, then $T_jT_iT_j\cdots(\eta_\lambda) = T_iT_jT_i\cdots(\eta_\lambda)$ where both sides have $m(i,j)$ factors.*

Proof. We set $a = 2(\cdots s_i s_j(i), \lambda)/(i,i)$. The right hand side has $(m(i,j)-1)$ factors s. Note that $a \in \mathbb{N}$ since $\cdots s_j s_i$ (has $m(i,j)$ factors) is reduced. One sees that

$$T_i T_j \cdots (\eta_\lambda) = \langle u_i^- \rangle^{(a)} T_j \cdots (\eta_\lambda).$$

Recall that $T_i'(\langle u_i^- \rangle^{(a)}) = v_i^{-a^2} K_{ai} \langle u_i^+ \rangle^{(a)}$ for all $i \in I$. We consider the following cases:

The case $m(i,j) = 2$. It is trivial since $\langle u_i^- \rangle \langle u_j^- \rangle = \langle u_j^- \rangle \langle u_i^- \rangle$.

The case $m(i,j) = 3$. Then $(i,i) = (j,j)$. One sees that $a = n_j$ since $a = 2(s_i s_j(i), \lambda)/(i,i) = 2(j,\lambda)/(j,j) = n_j$ (see 8.3). Thus,

$$\begin{aligned}
(T_j' T_i' T_j')(T_i T_j T_i(\eta_\lambda)) &= (T_j' T_i' T_j')\left(\langle u_i^- \rangle^{(a)} T_j T_i(\eta_\lambda)\right) \\
&= T_j' T_i' T_j'\left(\langle u_i^- \rangle^{(a)}\right)(T_j' T_i' T_j')(T_j T_i(\eta_\lambda)) \\
&= \left(T_j'\left(\langle u_j^- \rangle^{(a)}\right)(T_j'(\eta_\lambda))\right) \\
&= v_j^{-a^2} K_{aj} \langle u_j^+ \rangle^{(a)} (-1)^{n_j} v_j^{-n_j} \langle u_j^- \rangle^{(n_j)} \eta_\lambda \\
&= (-1)^{n_j} v_j^{-a^2 - n_j} (-1)^{n_j} v_j^{n_j} K_{aj} \eta_\lambda \text{ by (7.3.5)} \\
&= \eta_\lambda .
\end{aligned}$$

It follows that $T_i T_j T_i(\eta_\lambda) = T_j T_i T_j(\eta_\lambda)$.

The case $m(i,j) = 4$ and $m(i,j) = 6$. Then $a = n_i$. This follows from $a = 2(\cdots s_j s_i s_j(i), \lambda)/(i,i)$ and $\cdots s_j s_i s_j(i) = i$ (see 8.3), where $\cdots s_j s_i s_j(i)$ has $m(i,j) - 1$ factors s.

Thus,

$$\begin{aligned}
&\underbrace{(\cdots T_j' T_i' T_j')}_{m(i,j)} \underbrace{(T_i T_j T_i \cdots}_{m(i,j)} (\eta_\lambda)) \\
&= (\cdots T_j' T_i' T_j')\left(\langle u_i^- \rangle^{(a)} T_j T_i \cdots (\eta_\lambda)\right) \\
&= \cdots T_j' T_i' T_j'\left(\langle u_i^- \rangle^{(a)}\right)(\cdots T_j' T_i' T_j')(T_j T_i \cdots (\eta_\lambda)) \\
&= \left(T_i'\left(\langle u_i^- \rangle^{(a)}\right)(T_i'(\eta_\lambda))\right) \\
&= v_i^{-a^2} K_{ai} \langle u_i^+ \rangle^{(a)} (-1)^{n_i} v_i^{-n_i} \langle u_i^- \rangle^{(n_i)} \eta_\lambda \\
&= (-1)^{n_i} v_i^{-n_i^2 - n_i} (-1)^{n_i} v_i^{n_i} K_{n_i i} \eta_\lambda \\
&= \eta_\lambda .
\end{aligned}$$

Hence $\cdots T_i T_j T_i(\eta_\lambda) = \cdots T_j T_i T_j(\eta_\lambda)$. The lemma is proved. ∎

8.5. PROPOSITION. *Let V be any integrable $\mathscr{D}_c(A)$-module. For any $i \neq j$ in I such that $m(i, j) < +\infty$, the actions of T_i and T_j on V (defined in* 7.11) *satisfy the braid group relations* (8.2.2).

Proof. Let $\mathscr{S}(i, j)$ be the full subspecies of (Γ, d, Ω) generated by the vertices i and j, $A(i, j)$ the tensor algebra of $\mathscr{S}(i, j)$. It is a subalgebra of A. Therefore $\mathscr{D}_c(A(i, j))$ is a subalgebra of $\mathscr{D}_c(A)$. For any integrable $\mathscr{D}_c(A)$-module V, we restrict V to an $\mathscr{D}_c(A(i, j))$-module in an obvious way. Without loss of generality, we assume that $A(i, j)$ is of finite type, that is, $d(i, j) \leq 3$. Then V is a direct sum of integrable hightest simple $\mathscr{D}_c(A(i, j))$-modules. So, according to 7.3.3, we can define the linear operators, denoted by t_i, $i \in I$, on V (as $\mathscr{D}_c(A(i, j))$-modules). We first prove that the linear operators t_i, $i \in I$ satisfy the braid relations (8.2.2). Indeed, V is generated by a family of hightest vectors as $\mathscr{D}_c(A(i, j))$-modules. Take any $u \in \mathscr{D}_c(A(i, j))$ and $\eta_\lambda \in V$ to be any highest vector (over $\mathscr{D}_c(A(i, j))$). We have by Definition 7.3,

$$t_i t_j t_i \cdots (u\eta_\lambda) = t_i t_j t_i \cdots (u) t_i t_j t_i \cdots (\eta_\lambda),$$

$$t_j t_i t_j \cdots (u\eta_\lambda) = t_j t_i t_j \cdots (u) t_j t_i t_j \cdots (\eta_\lambda).$$

But we have proved $t_i t_j t_i \cdots (u) = t_j t_i t_j \cdots (u)$ and $t_i t_j t_i \cdots (\eta_\lambda) = t_j t_i t_j \cdots (\eta_\lambda)$ (both products have $m(i, j)$ factors) in 8.3 and 8.4. Therefore, $t_i t_j t_i \cdots = t_j t_i t_j \cdots$ as linear operators on V. According to Theorem 7.5, $T_i = t_i$ on V for any $i \in I$. So we have

$$T_i T_j T_i \cdots = T_j T_i T_j \cdots$$

on V, where both products have $m(i, j)$ factors. ∎

8.6. Now, we turn to prove Theorem 8.3 in general. Let V be any integrable $\mathscr{D}_c(A)$-module. For any $u \in \mathscr{D}_c(A)$ and $\eta \in V$, by definition and Corollary 7.11,

$$T_i T_j T_i \cdots (u) T_i T_j T_i \cdots (\eta) = T_i T_j T_i \cdots (u\eta)$$

$$T_j T_i T_j \cdots (u) T_j T_i T_j \cdots (\eta) = T_i T_j T_i \cdots (u\eta).$$

Since $T_i T_j T_i \cdots : V \to V$ is an isomorphism, it follows from Proposition 8.5 that $T_i T_j T_i \cdots (u) - T_j T_i T_j \cdots (u)$ acts as zero on V. It is well known that if $a \in \mathscr{D}_c(A)$ annihilates all integrable $\mathscr{D}_c(A)$-modules, then $a = 0$. Therefore, $T_i T_j T_i \cdots (u) = T_j T_i T_j \cdots (u)$ for any $u \in \mathscr{D}_c(A)$, where both products have $m(i, j)$ factors. Theorem 8.3 is proved finally. ∎

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