View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

Physics Letters B 665 (2008) 344–348

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Analytical estimate of the effect of spherical inhomogeneities on luminosity distance and redshift

N. Brouzakis, N. Tetradis *

Department of Physics, University of Athens, University Campus, Zographou 157 84, Greece

ARTICLE INFO

Article history:
Received 11 March 2008
Received in revised form 11 June 2008
Accepted 13 June 2008
Available online 19 June 2008
Editor: G.F. Giudice

ABSTRACT

We provide an analytical estimate of the effect of a spherical inhomogeneity on light beams that travel through it. We model the interior of the inhomogeneity through the Lemaitre–Tolman–Bondi metric. We assume that the beam source is located outside the inhomogeneity. We study the relative deviations of travelling time, redshift, beam area and luminosity distance from their values in a homogeneous cosmology. They depend on the ratio $\bar{H} = Hr_0$ of the radius r_0 of the inhomogeneity to the horizon distance $1/H$. For an observer located at the center, the deviations are of order $\bar{H}^2$. For an observer outside the inhomogeneity, the deviations of crossing time and redshift are of order $\bar{H}^3$. The deviations of beam area and luminosity distance are of order $\bar{H}^2$. However, when averaged over all possible locations of the observer outside the inhomogeneity, they also become of order $\bar{H}^3$. We discuss the implications for the possibility of attributing the observed cosmological acceleration to the emergence of large-scale structure.

© 2008 Elsevier B.V. All rights reserved.

1. Introduction

The cause of the perceived acceleration of the present cosmological expansion has not been identified yet. An interesting possibility, that does not require the introduction of new ingredients to Standard Cosmology, is that the growth of inhomogeneities in the matter distribution affects the astrophysical observations similarly to accelerated expansion in a homogeneous background. In particular, the luminosity distance of faraway sources may be increased because of the propagation of light through inhomogeneous regions before reaching the observer.

An unambiguous way to examine this possibility is through the study of the transmission of light in an exact inhomogeneous background. The analytical modelling of the Universe can only be approximate, and depends on the scale of the assumed inhomogeneities. At length scales above $\mathcal{O}(10)h^{-1}$ Mpc the density contrast is at most of $\mathcal{O}(1)$. A popular choice for the background is based on the Lemaitre–Tolman–Bondi (LTB) metric [1]. The background has spherical symmetry, but can be inhomogeneous along the radial direction. The metric can be matched to the Friedmann–Robertson–Walker (FRW) metric at a certain radius r_0. There are two possible choices for the location of the observer, which are consistent with the isotropy of the Cosmic Microwave Background. (a) He/she could be located in the interior of the inhomogeneity, near its center [2]. (b) He/she could be located in the homogeneous region, with the light travelling across several inhomogeneities during its propagation from source to observer [3–5].

In both cases, the size of the inhomogeneity r_0 determines its effect on quantities such as redshift and source luminosity distance. The relevant quantity is the dimensionless ratio $\bar{H} = r_0 H$ of r_0 to the horizon distance $1/H$. Consistency with observations requires that $\bar{H}$ be of $\mathcal{O}(10^{-2})$, even though values larger by an order of magnitude have also been advocated for the explanation of the supernova data [2]. In the following we use perturbation theory in $\bar{H}$ in order to determine the dependence of the photon redshift and source luminosity distance on $\bar{H}$, for both possible locations of the observer.

2. Gravitational background

The LTB metric can be written in the form

$$ds^2 = -dt^2 + \frac{R'^2(t,r)}{1+f(r)}dr^2 + R^2(t,r)\,d\Omega^2, \tag{1}$$

where $d\Omega^2$ is the metric of a two-sphere, the prime denotes differentiation with respect to r, and $f(r)$ is an arbitrary function. The function $R(t,r)$ describes the location of a shell of matter marked by r at the time t. Through an appropriate rescaling it can be chosen to satisfy $R(0,r) = r$.

The Einstein equations reduce to

$$\dot{R}^2(t,r) = \frac{1}{8\pi M^2}\frac{\mathcal{M}(r)}{R} + f(r), \tag{2}$$

$$\mathcal{M}'(r) = 4\pi R^2 \rho(t,r) R', \tag{3}$$

* Corresponding author.
E-mail address: ntetrad@phys.uoa.gr (N. Tetradis).

0370-2693/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.physletb.2008.06.032

where the dot denotes differentiation with respect to t, and $G = (16\pi M^2)^{-1}$. The generalized mass function $\mathcal{M}(r)$ of the pressureless fluid with energy density $\rho(t,r)$ can be chosen arbitrarily.

We parametrize the energy density at some arbitrary initial time $t_i = 0$ as $\rho_i(r) = \rho(0,r) = (1+\epsilon(r))\rho_{0,i}$. The initial energy density of the homogeneous background is $\rho_{0,i}$. If the size of the inhomogeneity is r_0, the matching with the homogeneous metric in the exterior requires $4\pi\int_0^{r_0} r^2\epsilon(r)\,dr = 0$, so that $\mathcal{M}(r_0) = 4\pi r_0^3\rho_{0,i}/3$. As we assume that the homogeneous metric is flat, we also have $f(r_0) = 0$. Discontinuities in $f'(r)$ result in discontinuities in the derivatives of the metric functions.

In our modelling we assume that at the initial time $t_i = 0$ the expansion rate $H_i = \dot{R}/R = \dot{R}'/R'$ is given for all r by the standard expression in homogeneous cosmology: $H_i^2 = \rho_{0,i}/(6M^2)$. Then, Eq. (2) with $R(0,r) = r$ implies that

$$f(r) = \frac{\rho_{0,i}}{6M^2} r^2\left(1 - \frac{3\mathcal{M}(r)}{4\pi r^3\rho_{0,i}}\right). \tag{4}$$

For our choice of $f(r)$, overdense regions have positive spatial curvature and tend to contract, while underdense ones negative curvature and expand faster than the average. This is very similar to the initial condition considered in the model of spherical collapse. Even though we work with the particular choice (4) for $f(r)$, we expect that our conclusions are valid for other variations of the LTB metric as well. These may include an arbitrary function $t_0(r)$ resulting from the integration of Eq. (2). As this function appears in the combination $t - t_0(r)$, it becomes irrelevant for large times. Also, the radial coordinate r is often redefined so that ρ_i is constant. As this is only a gauge choice, we do not expect it to affect the physical behaviour. The eventual collapse or fast expansion of a certain region would be determined by its spatial curvature, as in our model.

3. Optical equations

The optical equations [6] can be written as [4]

$$\frac{1}{\sqrt{A}}\frac{d^2\sqrt{A}}{d\lambda^2} = -\frac{1}{4M^2}\rho\left(k^0\right)^2 - \sigma^2, \tag{5}$$

$$\frac{d\sigma}{d\lambda} + \frac{2}{\sqrt{A}}\frac{d\sqrt{A}}{d\lambda}\sigma = \frac{(k^3)^2R^2}{4M^2}\left(\rho - \frac{3\mathcal{M}(r)}{4\pi R^3}\right), \tag{6}$$

where A is the cross section of a light beam, λ an affine parameter along the null trajectory and $k^i = dx^i/d\lambda$. The shear σ is important when the beam passes near regions in which the density exceeds the average one by several orders of magnitude. Within our modelling of large-scale structure, applicable for scales above $\mathcal{O}(10)h^{-1}$ Mpc, the average density contrast is not sufficiently large for the shear to become important [4].

We assume that, even for general backgrounds, the light emission near the source is not affected by the large-scale geometry. By choosing an affine parameter that is locally $\lambda = t$ in the vicinity of the source, we can set $d\sqrt{A}/d\lambda|_{\lambda=0} = \sqrt{\Omega_s}$. The constant Ω_s can be identified with the solid angle spanned by a certain beam when the light is emitted by a point-like isotropic source. This expression, along with $\sqrt{A}|_{\lambda=0} = 0$, provide the initial conditions for the solution of Eq. (5).

In order to define the luminosity distance, we consider photons emitted within a solid angle Ω_s by an isotropic source with luminosity L. These photons are detected by an observer for whom the light beam has a cross-section A_o. The redshift factor is $1+z = \omega_s/\omega_o = k_s^0/k_o^0$, because the frequencies measured at the source and at the observation point are proportional to the values of k^0 at these points. The luminosity distance is $D_L = (1+z)\sqrt{A_o/\Omega_s}$, with A_o the beam area measured by the observer for a beam emitted by the source within a solid angle Ω_s. The beam area can be calculated by solving Eq. (5).

It is convenient to switch to dimensionless variables. We define $\bar{t} = tH_i$, $\bar{r} = r/r_0$, $\bar{R} = R/r_0$, where $H_i^2 = \rho_{0,i}/(6M^2)$ is the initial homogeneous expansion rate and r_0 gives the size of the inhomogeneity in comoving coordinates. The evolution equation becomes

$$\frac{\dot{\bar{R}}^2}{\bar{R}^2} = \frac{3\bar{\mathcal{M}}(\bar{r})}{4\pi\bar{R}^3} + \frac{\bar{f}(\bar{r})}{\bar{R}^2}, \tag{7}$$

with $\bar{\mathcal{M}} = \mathcal{M}/(\rho_{0,i}r_0^3)$ and $\bar{f} = 6M^2 f/(\rho_{0,i}r_0^2) = f/\bar{H}_i^2$, $\bar{H}_i = H_i r_0$. The dot now denotes a derivative with respect to $\bar{t}$. We take the affine parameter λ to have the dimension of time and we define the dimensionless variables $\bar{\lambda} = H_i\lambda$, $\bar{k}^0 = k^0$, $\bar{k}^1 = k^1/\bar{H}_i$, $\bar{k}^3 = r_0 k^3$. The geodesic equations maintain their form, with the various quantities replaced by barred ones, and the combination $1+f$ replaced by $\bar{H}_i^{-2} + \bar{f}$. The optical equations take the form

$$\frac{1}{\sqrt{\bar{A}}}\frac{d^2\sqrt{\bar{A}}}{d\bar{\lambda}^2} = -\frac{3}{2}\bar{\rho}\left(\bar{k}^0\right)^2 - \bar{\sigma}^2, \tag{8}$$

$$\frac{d\bar{\sigma}}{d\bar{\lambda}} + \frac{2}{\sqrt{\bar{A}}}\frac{d\sqrt{\bar{A}}}{d\bar{\lambda}}\bar{\sigma} = \frac{3}{2}\left(\bar{k}^3\right)^2\bar{R}^2\left(\bar{\rho} - \frac{3\bar{\mathcal{M}}}{4\pi\bar{R}^3}\right), \tag{9}$$

with $\bar{\rho} = \rho/\rho_{0,i}$ and $\bar{\sigma} = \sigma/H_i$. The initial conditions become $d\sqrt{\bar{A}}/d\bar{\lambda}|_{\bar{\lambda}=0} = \sqrt{\bar{\Omega}_s}/\bar{H}_i = \sqrt{\Omega_s}$ and $\sqrt{\bar{A}}|_{\lambda=0} = 0$, with $\bar{A} = H_i^2 A$ and $\bar{\Omega} = \bar{H}_i^2\Omega$.

The effect of the inhomogeneity on the characteristics of the light beam can be calculated analytically for perturbations with size much smaller than the distance to the horizon. These have $\bar{H}_i \ll 1$. In the following we use $\bar{H}_i$ as a small parameter in a perturbative calculation of the luminosity distance and redshift. For small inhomogeneities, the variation of the Hubble parameter during the crossing by the light beam is very small. As a result $\bar{H}_i$ is almost identical with the value $\bar{H}$ at the time of detection of the beam. We consider beams with $k^3 = 0$ that pass through the center of the spherical inhomogeneity. Beams with $k^3 \neq 0$ can also be considered along the same lines, even though the calculation is much more involved.

4. Travelling time and redshift

The travelling time for a beam that propagates across the inhomogeneity has been calculated in Ref. [4] up to $\mathcal{O}(\bar{H}_i^2)$. We denote by $\bar{r}_s$ the location of the source and by $\bar{t}_s$ the emission time of the beam. The travelling time is

$$\begin{aligned}\bar{t} - \bar{t}_s = {} & \pm\bar{H}_i\left(\bar{R}(\bar{t}_s,\bar{r}_s) - \bar{R}(\bar{t}_s,\bar{r})\right) + \bar{H}_i^2\int_{\bar{r}}^{\bar{r}_s}\bar{R}'(\bar{t}_s,\bar{r})\dot{\bar{R}}(\bar{t}_s,\bar{r})\,d\bar{r} \\ & - \bar{H}_i^2\left(\bar{R}(\bar{t}_s,\bar{r}_s) - \bar{R}(\bar{t}_s,\bar{r})\right)\dot{\bar{R}}(\bar{t}_s,\bar{r}) + \mathcal{O}\left(\bar{H}_i^3\right)\end{aligned} \tag{10}$$

for incoming and outgoing beams, respectively. The leading term in the above expression, of $\mathcal{O}(\bar{H}_i)$, is the standard Doppler shift. It is non-zero whenever the observer has a peculiar velocity relative to a source in the homogeneous region.

We can make a comparison with the propagation of light in a FRW background. In this case we have $\bar{R}(\bar{t},\bar{r}) = a(\bar{t})\bar{r} = \bar{R}(\bar{t},1)\bar{r}$. Let us consider light signals emitted at $\bar{r}_s = 1$ and observed at the center ($\bar{r}_o = 0$) of the inhomogeneity. The peculiar velocity of such an observer is zero and the term of $\mathcal{O}(\bar{H}_i)$ vanishes. The difference in propagation time within the LTB and FRW backgrounds is

$$\begin{aligned}\bar{t}_o - (\bar{t}_o)_{\mathrm{FRW}} = {} & \bar{H}_i^2\int_0^1\bar{R}'(\bar{t}_s,\bar{r})\dot{\bar{R}}(\bar{t}_s,\bar{r})\,d\bar{r} \\ & - \frac{\bar{H}_i^2}{2}\bar{R}(\bar{t}_s,1)\dot{\bar{R}}(\bar{t}_s,1) + \mathcal{O}\left(\bar{H}_i^3\right).\end{aligned} \tag{11}$$

For signals originating at $\bar{r}_s = 0$ and detected at $\bar{r}_o = 1$ the time difference has the opposite sign. As a result, the time difference for signals that cross the inhomogeneity is of $\mathcal{O}(\bar{H}_i^3)$.

A similar expression can be derived for the redshift of a light beam that passes through the center of the inhomogeneity. One finds [5]

$$\ln(1+z) = \pm\bar{H}_i\big(\dot{\bar{R}}(\bar{t}_s,\bar{r}_s) - \dot{\bar{R}}(\bar{t}_s,\bar{r})\big) + \bar{H}_i^2 \int_{\bar{r}}^{\bar{r}_s} \ddot{\bar{R}}'(\bar{t}_s,\bar{r})\big(\bar{R}(\bar{t}_s,\bar{r}_s) - \bar{R}(\bar{t}_s,\bar{r})\big)\,d\bar{r} + \mathcal{O}\big(\bar{H}_i^3\big) \tag{12}$$

for incoming and outgoing beams, respectively.

For signals originating at $\bar{r}_s = 1$ and detected at $\bar{r}_o = 0$ the redshifts obey

$$\ln\left(\frac{1+z}{1+z_{\mathrm{FRW}}}\right) = \bar{H}_i^2 \int_0^1 \ddot{\bar{R}}'(\bar{t}_s,\bar{r})\big(\bar{R}(\bar{t}_s,1) - \bar{R}(\bar{t}_s,\bar{r})\big)\,d\bar{r} - \frac{\bar{H}_i^2}{2}\ddot{\bar{R}}'(\bar{t}_s,1)\bar{R}(\bar{t}_s,1) + \mathcal{O}\big(\bar{H}_i^3\big). \tag{13}$$

For signals originating at $\bar{r}_s = 0$ and detected at $\bar{r}_o = 1$ the r.h.s. of the above equation has the opposite sign. As a result, the redshift difference for signals that cross the inhomogeneity is of $\mathcal{O}(\bar{H}_i^3)$.

5. Beam area

The beam area obeys the second-order differential equation (8), whose solution depends crucially on the initial conditions. In certain situations, the symmetry of the problem permits an exact solution. For example, for signals emitted from some point $\bar{r}_s$ at a time $\bar{t} = \bar{t}_s$ and observed at $\bar{r}_o = 0$ we have [7]

$$\sqrt{\bar{A}} = (1+z)\bar{R}(\bar{t}_s,\bar{r}_s)\sqrt{\bar{\Omega}}. \tag{14}$$

Similarly, for signals emitted from the center $\bar{r}_s = 0$ and observed at $\bar{r}_o$ at a time $\bar{t}_o$ we have

$$\sqrt{\bar{A}} = \bar{R}(\bar{t}_o,\bar{r}_o)\sqrt{\bar{\Omega}}. \tag{15}$$

However, for a signal that crosses the inhomogeneity we need to integrate Eq. (8) from $\bar{r} = 0$ to $\bar{r}_o$ with initial conditions determined by the propagation from $\bar{r}_s$ to $\bar{r} = 0$. These include not only $\sqrt{\bar{A}}$, but $d\sqrt{\bar{A}}/d\bar{r}$ as well. An exact analytical solution is not possible in this case, and we have to resort to perturbation theory in $\bar{H}_i$. We have checked that the expressions (14) and (15) are reproduced correctly by our results, up to second order in $\bar{H}_i$.

The optical equations (8), (9) can be written in the form

$$\frac{d^2\sqrt{\bar{A}}}{d\bar{r}^2} + \frac{1}{(\bar{k}^1)^2}\frac{d\bar{k}^1}{d\bar{\lambda}}\frac{d\sqrt{\bar{A}}}{d\bar{r}} = -\frac{3}{2}\bar{\rho}\left(\frac{\bar{k}^0}{\bar{k}^1}\right)^2\sqrt{\bar{A}} - \left(\frac{\bar{\sigma}}{\bar{k}^1}\right)^2\sqrt{\bar{A}}, \tag{16}$$

$$\frac{d}{d\bar{r}}\left(\frac{\bar{\sigma}}{\bar{k}^1}\right) + \frac{1}{(\bar{k}^1)^2}\frac{d\bar{k}^1}{d\bar{\lambda}}\frac{\bar{\sigma}}{\bar{k}^1} + \frac{2}{\sqrt{\bar{A}}}\frac{d\sqrt{\bar{A}}}{d\bar{r}}\frac{\bar{\sigma}}{\bar{k}^1} = \frac{3}{2}\left(\frac{\bar{R}\bar{k}^3}{\bar{k}^1}\right)^2\left(\bar{\rho} - \frac{3\bar{\mathcal{M}}}{4\pi\bar{R}^3}\right). \tag{17}$$

The first term in the r.h.s. of Eq. (16) is of $\mathcal{O}(\bar{H}_i^2)$ because $\bar{\rho} = \mathcal{O}(1)$ and $\bar{k}^0/\bar{k}^1 = d\bar{t}/d\bar{r} = \bar{H}_i dt/dr = \mathcal{O}(\bar{H}_i)$. The term in the r.h.s. of Eq. (17) is also of $\mathcal{O}(\bar{H}_i^2)$ because $\bar{R}\bar{k}^3/\bar{k}^1 = \bar{H}_i R d\phi/dr = \mathcal{O}(\bar{H}_i)$. As a result, the second term in the r.h.s. of Eq. (16) is $\mathcal{O}(\bar{H}_i^4)$ and, therefore, negligible. The shear plays no role, except for cases in which the light passes very close to an extremely dense concentration of mass. At the length scales that we are considering the energy density is smoothly distributed, and the shear can be neglected. As the first term in the r.h.s. of Eq. (16) generates the deviations of the luminosity distance from its value in a homogeneous background, we expect the overall effect to be of $\mathcal{O}(\bar{H}_i^2)$. In the following we confirm this expectation through an explicit calculation, assuming a simplified form of the energy density.

We consider beam trajectories that start at the boundary of the inhomogeneity, pass through its center and exit from the other side. These have $\bar{k}^3 = 0$. We express $d\bar{k}_1/d\bar{\lambda}$ in Eq. (16) using the geodesic equation [4], and omit the shear. As the FRW metric is special case of the LTB one, no change of coordinates is necessary. In this way we obtain

$$\frac{d^2\sqrt{\bar{A}}}{d\bar{r}^2} + \left(\pm\frac{2\bar{H}_i\dot{\bar{R}}'}{\sqrt{1+\bar{H}_i^2\bar{f}}} - \frac{\bar{R}''}{\bar{R}'} + \frac{\bar{H}_i^2\bar{f}'}{2(1+\bar{H}_i^2\bar{f})}\right)\frac{d\sqrt{\bar{A}}}{d\bar{r}} = -\frac{3}{2}\bar{\rho}\frac{R'^2}{1+\bar{H}_i^2\bar{f}}\sqrt{\bar{A}}, \tag{18}$$

where the positive sign in the second term corresponds to ingoing and the negative sign to outgoing geodesics.

We use the expansion

$$\sqrt{\bar{A}} = \sqrt{\bar{A}^{(0)}} + \bar{H}_i\sqrt{\bar{A}^{(1)}} + \bar{H}_i^2\sqrt{\bar{A}^{(2)}} + \mathcal{O}\big(\bar{H}_i^3\big), \tag{19}$$

and calculate $\sqrt{\bar{A}^{(i)}}$ in each order of perturbation theory. The travelling time is given by Eq. (10). We can set $t_s = 0$ so the geodesic inside the inhomogeneity is $\bar{t} = -\bar{H}_i(\bar{r}-1)$ for ingoing, and $\bar{t} = \bar{H}_i(\bar{r}+1)$ for outgoing geodesics. We treat $\bar{t}$ as an $\mathcal{O}(\bar{H}_i)$ quantity.

6. Central underdensity

We identify the initial time in the background evolution with the time of light emission: $\bar{t}_i = \bar{t}_s = 0$. This implies that $R'(0,r) = \bar{R}'(0,\bar{r}) = 1$. Also $\dot{\bar{R}}'(0,\bar{r}) = 1$. The initial configuration that we consider has $\bar{\rho}_i(0,\bar{r}) = 0$ for $\bar{r} < \bar{r}_1$ and $\bar{\rho}_i(0,\bar{r}) = 1/(1-\bar{r}_1^3)$ for $\bar{r} > \bar{r}_1$. From (7) we can calculate various derivatives of $\bar{R}$ at $\bar{t} = 0$:

$$\dot{\bar{R}}'(\bar{t},\bar{r}) = \dot{\bar{R}}'(0,\bar{r}) + \bar{t}\ddot{\bar{R}}'(0,\bar{r}) + \mathcal{O}\big(\bar{H}_i^2\big) = 1 + \bar{t}\ddot{\bar{R}}'(0,\bar{r}) + \mathcal{O}\big(\bar{H}_i^2\big), \tag{20}$$

$$\frac{\bar{R}''}{\bar{R}'}(\bar{t},\bar{r}) = \frac{\bar{t}^2}{2}\ddot{\bar{R}}''(0,\bar{r}) + \mathcal{O}\big(\bar{H}_i^3\big). \tag{21}$$

For $\bar{r} > \bar{r}_1$ we have

$$\ddot{\bar{R}}'(0,\bar{r}) = \frac{r^3 + 2\bar{r}_1^3}{2r^3(\bar{r}_1^3 - 1)}, \qquad \ddot{\bar{R}}''(0,\bar{r}) = -\frac{3\bar{r}_1^3}{r^4(\bar{r}_1^3 - 1)}. \tag{22}$$

For $\bar{r} < \bar{r}_1$ both $\ddot{\bar{R}}'(0,\bar{r})$ and $\ddot{\bar{R}}''(0,\bar{r})$ are zero. For the initial configuration that we assume, $\ddot{\bar{R}}$ is a continuous function of $\bar{r}$. However, $\ddot{\bar{R}}'$ is discontinuous at $\bar{r} = \bar{r}_1$ and $\bar{r} = 1$, while $\ddot{\bar{R}}''$ has δ-function singularities at the same points.

The initial conditions for the solution of Eq. (18) for an ingoing beam can be taken $\sqrt{\bar{A}}(1) = 0$, $d\sqrt{\bar{A}}(1)/d\bar{r} = -1$, without loss of generality. To zeroth order in $\bar{H}_i$, Eq. (18) becomes $d^2\sqrt{\bar{A}^{(0)}}/d\bar{r}^2 = 0$, with solution $\sqrt{\bar{A}^{(0)}}(\bar{r}) = -(r-1)$ for ingoing and $\sqrt{\bar{A}^{(0)}}(\bar{r}) = r+1$ for outgoing beams. To first order in $\bar{H}_i$, Eq. (18) gives $d\sqrt{\bar{A}^{(1)}}/d\bar{r} = -2$, with solution $\sqrt{\bar{A}^{(1)}}(\bar{r}) = r^2 - 2r + 1$ for ingoing and $\sqrt{\bar{A}^{(1)}}(\bar{r}) = r^2 + 2r + 1$ for outgoing beams. These results are the same as for the case of a homogeneous background.

The effect of the inhomogeneity appears in second order in $\bar{H}_i$. We obtain

$$\frac{d^2\sqrt{\bar{A}^{(2)}}}{d\bar{r}^2} + \left(\pm 2\bar{t}(\bar{r})\ddot{\bar{R}}'(0,\bar{r}) - \frac{\bar{t}^2}{2}\ddot{\bar{R}}''(0,\bar{r}) + \frac{\bar{f}'(\bar{r})}{2}\right)\frac{d\sqrt{\bar{A}^{(0)}}}{d\bar{r}} \pm 2\frac{d\sqrt{\bar{A}^{(1)}}}{d\bar{r}} = -\frac{3}{2}\bar{\rho}(0,\bar{r})\sqrt{\bar{A}^{(0)}}, \tag{23}$$

with the upper sign corresponding to ingoing and the lower one to outgoing geodesics. As we have already mentioned, for $\bar{r} < \bar{r}_1$ we have $\bar{\rho}_i(0,\bar{r}) = \ddot{\bar{R}}'(0,\bar{r}) = \ddot{\bar{R}}''(0,\bar{r}) = 0$.

The above equation can be solved analytically through simple integration, with the values at the end of each interval determining the initial conditions for the next one. The only non-trivial point is that the δ-function singularities of $\ddot{\bar{R}}''$ at $\bar{r} = \bar{r}_1$ and $\bar{r} = 1$ induce discontinuities in the values of $d\sqrt{\bar{A}^{(2)}}/d\bar{r}$ at these points. These must be taken into account in a consistent calculation. The discontinuities can be easily determined through the integration of Eq. (23) in an infinitesimal interval around each of these points. The remaining calculation is straightforward. It must be emphasized that the discontinuous density profiles that we are considering can be viewed as limiting cases of continuous ones, when the transition regions become infinitesimally thin. The integration of Eq. (23) around the corresponding values of r picks up the leading contributions arising from the transition regions. Including these contributions is necessary in order to reproduce correctly the exact expressions (14), (15).

For a photon beam that starts from the boundary at $\bar{r} = 1$, travels through the center of an underdensity at $\bar{r} = 0$, and exits at the diametrical point with $\bar{r} = 1$, we find

$$\sqrt{\bar{A}^{(2)}}(\bar{r} = 0) = 1 - \frac{3}{4}\frac{\bar{r}_1 + 1}{\bar{r}_1^2 + \bar{r}_1 + 1} \tag{24}$$

and

$$\sqrt{\bar{A}^{(2)}}(\bar{r} = 1) = 5 - \frac{3}{\bar{r}_1^2 + \bar{r}_1 + 1}. \tag{25}$$

Putting everything together, we find that, when the photon exits the inhomogeneity at $\bar{r} = 1$,

$$\sqrt{\bar{A}}(\bar{r} = 1) = 2 + 4\bar{H}_i + \left(5 - \frac{3}{\bar{r}_1^2 + \bar{r}_1 + 1}\right)\bar{H}_i^2 + \mathcal{O}(\bar{H}_i^3). \tag{26}$$

The expressions for a homogeneous universe are obtained by setting $\bar{r}_1 = 0$. The beam area and the luminosity distance are increased by the presence of the inhomogeneity ($\bar{r}_1 \neq 0$).

We also mention that, if the beam is emitted at $\bar{r} = 0$, it exits the inhomogeneity with $\sqrt{\bar{A}^{(2)}}(\bar{r} = 1) = 1/4$ and $\sqrt{\bar{A}^{(2)}}'(\bar{r} = 1) = 3/4$, in agreement with Eq. (15).

7. Central overdensity

The initial configuration that we consider has $\bar{\rho}_i(0,\bar{r}) = 1/\bar{r}_1^3$ for $\bar{r} < \bar{r}_1$ and $\bar{\rho}_i(0,\bar{r}) = 0$ for $\bar{r} > \bar{r}_1$.

For $\bar{r} > \bar{r}_1$ we have

$$\ddot{\bar{R}}'(0,\bar{r}) = \frac{1}{\bar{r}^3}, \qquad \ddot{\bar{R}}''(0,\bar{r}) = -\frac{3}{\bar{r}^4}, \tag{27}$$

while for $\bar{r} < \bar{r}_1$ we have

$$\ddot{\bar{R}}'(0,\bar{r}) = -\frac{1}{2\bar{r}_1^3}, \qquad \ddot{\bar{R}}''(0,\bar{r}) = 0. \tag{28}$$

The expressions for $\sqrt{\bar{A}^{(0)}}$ and $\sqrt{\bar{A}^{(1)}}$ are the same as in the case of a central underdensity, as they are not affected by the inhomogeneity. For $\sqrt{\bar{A}^{(2)}}$ we find

$$\sqrt{\bar{A}^{(2)}}(\bar{r} = 0) = 1 - \frac{3}{4}\frac{1}{\bar{r}_1} \tag{29}$$

and

$$\sqrt{\bar{A}^{(2)}}(\bar{r} = 1) = 5 - \frac{3}{\bar{r}_1^2}. \tag{30}$$

Putting everything together, we find that, when the photon exits the inhomogeneity at $\bar{r} = 1$,

$$\sqrt{\bar{A}}(\bar{r} = 1) = 2 + 4\bar{H}_i + \left(5 - \frac{3}{\bar{r}_1^2}\right)\bar{H}_i^2 + \mathcal{O}(\bar{H}_i^3). \tag{31}$$

The expressions for a homogeneous universe are obtained by setting $\bar{r}_1 = 1$. In this case, the beam area and the luminosity distance are reduced by the presence of the inhomogeneity ($\bar{r}_1 \neq 1$). The singularity for $\bar{r}_1 \to 0$ is an artifact of the perturbative expansion. Clearly, the expansion in $\bar{H}_i$ breaks down when the coefficient of $\bar{H}_i^2$ diverges.

The increase of the beam area by a central underdensity with a certain $\bar{r}_1$ can always be compensated by the decrease because of an overdensity with a different value $\bar{r}_1'$. If one requires that $\bar{r}_1$ and $\bar{r}_1'$ be equal, the solution is $\bar{r}_1 = \bar{r}_1' = 2^{-1/3}$. In this case, the central underdensity and its surrounding overdense shell, as well as the compensating central overdensity and its surrounding underdense shell, all have equal volumes.

If the beam is emitted at $\bar{r} = 0$, it exits the inhomogeneity with $\sqrt{\bar{A}^{(2)}}(\bar{r} = 1) = 1/4$ and $\sqrt{\bar{A}^{(2)}}'(\bar{r} = 1) = 3/4$, exactly as in the case of a central underdensity.

8. Flux conservation

We have seen that, when a light beam crosses a certain inhomogeneity, the deviations of the travelling time $\bar{t}_o$ and redshift z from their values in a homogeneous background are of $\mathcal{O}(\bar{H}_i^3)$, while the deviation of $\bar{A}$ is of $\mathcal{O}(\bar{H}_i^2)$. As a result, the effect on the luminosity distance is of $\mathcal{O}(\bar{H}_i^2)$. This conclusion holds for any beam going through the inhomogeneity, even if the crossing is not central. The analytical estimate has been verified through the numerical solution of the optical equations [4,5]. In particular, a central crossing of a void-like inhomogeneity (with a central underdensity) results in the increase of the luminosity distance by an amount of $\mathcal{O}(\bar{H}_i^2)$ [4,5]. This result is in agreement with the analysis of Ref. [8], in which a sequence of central crossings is assumed during the propagation of light from source to observer. On the other hand, if the inhomogeneity is crossed through the overdense region near its surface a decrease of the luminosity distance by an amount of $\mathcal{O}(\bar{H}_i^2)$ takes place [4,5].

The conclusion that the redshift is affected by an amount of $\mathcal{O}(\bar{H}_i^3)$ has a very important implication. If the redshift is not altered significantly by the propagation in the inhomogeneous background, the conservation of the total flux requires that the average luminosity distance be the same as in the homogeneous case. The energy flux may be redistributed in various directions but the total flux must be the same as in the homogeneous case [9,10]. The maximal deviation from exact flux conservation is determined by the effect of the inhomogeneity on the redshift, which is of $\mathcal{O}(\bar{H}_i^3)$. As a result, even though the effect on the luminosity distance for a single crossing is of $\mathcal{O}(\bar{H}_i^2)$, the *maximal average* effect for beams originating in the same source and crossing the inhomogeneity at various angles is of $\mathcal{O}(\bar{H}_i^3)$. In the case of an underdensity, the increase of the luminosity distance for central beam crossings is compensated by a reduction for beams that travel mainly through the peripheral overdense shell. The opposite happens in the case of a central overdensity.

The above conclusion has been verified numerically in Ref. [5], both for central underdensities and overdensities. An equivalent conclusion is that the maximal statistical effect for light signals received from randomly distributed sources in the sky should be of $\mathcal{O}(\bar{H}_i^3)$, similarly to the effect on the redshift. The statistical analysis of Ref. [5] confirms this expectation.

9. Conclusions

The effect of spherical inhomogeneities on light emitted by a distance source depends on $\bar{H} = r_0 H$. For an observer located at the center of a spherical inhomogeneity, the deviations of travelling time, redshift, beam area and luminosity distance from their values in a homogeneous background are of $\mathcal{O}(\bar{H}^2)$. The luminosity distance is increased by the presence of a central underdensity, while it is reduced by a central overdensity. The increase in the luminosity distance if the observer is located near the center of a large void can by employed for the explanation of the supernova data [2]. An increase of $\mathcal{O}(10\%)$, as required by the data, would imply the existence of a void with size of $\mathcal{O}(10^3)h^{-1}$ Mpc. Numerical factors can reduce the required size, depending on the details of the particular cosmological model employed [2]. However, a typical void with size of $\mathcal{O}(10)h^{-1}$ Mpc leads to a negligible increase of the luminosity distance.

If the observer is located at a random position within the homogeneous region, the beam can cross several inhomogeneities before its detection. Each crossing produces an effect of $\mathcal{O}(\bar{H}^3)$ for the travelling time and the redshift. For the beam area and the luminosity distance the effect is of $\mathcal{O}(\bar{H}^2)$. However, flux conservation implies that positive and negative contributions to the beam area cancel during multiple crossings. The size of the *maximal average* effect of each crossing on the beam area and luminosity distance is set by the effect on the redshift, which is of $\mathcal{O}(\bar{H}^3)$ [5,9,10]. Photons with redshift ~ 1 pass through $\sim (1/H)/r_0 = \bar{H}^{-1}$ inhomogeneities before arrival, assuming that these are tightly packed. As a result, the expectation is that the maximal final effect for a random position of the observer is of $\mathcal{O}(\bar{H}^2)$ for all quantities. This conclusion is supported by the numerical analysis [4,5].

We mention at this point that, even for a random position of the observer, there is a bias in the residual effect on the luminosity distance for a limited sample of sources. The bias is towards increased values if the Universe is dominated by void-like configurations. We did not discuss this point in this Letter, as we assumed that the data sample is large. A detailed study can be found in Ref. [5], to which we refer the reader for the details.

We conclude that the presence of spherical inhomogeneities does not influence sufficiently the propagation of light in order to provide an explanation for the supernova data, unless their size becomes comparable to the horizon distance. It is possible, however, that relaxing the assumption of spherical symmetry for the inhomogeneities may increase the influence of the local geometry on the beam characteristics and provide an effect at a lower order in $\bar{H}$. The crucial question is whether the influence of the inhomogeneities on the redshift can become larger than the effect of $\mathcal{O}(\bar{H}^3)$ predicted by our model and the Rees–Sciama estimate [11]. The modelling of the Universe as an ensemble of inhomogeneities, glued together by a homogeneous region (the "Swiss-cheese" model), may be too constraining. Photons that cross an inhomogeneity enter an evolving Newtonian potential from a homogeneous region, to which they subsequently return. Within this modelling, the residual effect cannot be much larger than of $\mathcal{O}(\bar{H}^3)$. The elimination of the intermediate homogeneous region may be necessary in order to produce a larger effect. This possibility poses formidable technical difficulties, but merits further investigation.

Acknowledgement

This work was supported by the research programs "Kapodistrias" of the University of Athens and "Pythagoras II" (grant 70-03-7992) of the Greek Ministry of National Education, partially funded by the European Union.

References

[1] G. Lemaitre, Gen. Relativ. Gravit 29 (1997) 641;
R.C. Tolman, Proc. Natl. Acad. Sci. 20 (1934) 169;
H. Bondi, Mon. Not. R. Astron. Soc. 107 (1947) 410.
[2] N. Mustapha, C. Hellaby, G.F.R. Ellis, Mon. Not. R. Astron. Soc. 292 (1997) 817, gr-qc/9808079;
M.N. Celerier, Astron. Astrophys. 353 (2000) 63, astro-ph/9907206;
H. Iguchi, T. Nakamura, K.i. Nakao, Prog. Theor. Phys. 108 (2002) 809, astro-ph/0112419;
J.W. Moffat, JCAP 0510 (2005) 012, astro-ph/0502110;
J.W. Moffat, astro-ph/0505326;
H. Alnes, M. Amarzguioui, O. Gron, JCAP 0701 (2007) 007, astro-ph/0506449;
H. Alnes, M. Amarzguioui, O. Gron, Phys. Rev. D 73 (2006) 083519, astro-ph/0512006;
K. Bolejko, astro-ph/0512103;
R. Mansouri, astro-ph/0512605;
C.H. Chuang, J.A. Gu, W.Y. Hwang, astro-ph/0512651;
R.A. Vanderveld, E.E. Flanagan, I. Wasserman, Phys. Rev. D 74 (2006) 023506, astro-ph/0602476;
P.S. Apostolopoulos, N. Brouzakis, N. Tetradis, E. Tzavara, JCAP 0606 (2006) 009, astro-ph/0603234;
D. Garfinkle, Class. Quantum Grav. 23 (2006) 4811, gr-qc/0605088;
T. Biswas, R. Mansouri, A. Notari, astro-ph/0606703;
D.J.H. Chung, A.E. Romano, Phys. Rev. D 74 (2006) 103507, astro-ph/0608403;
K. Enqvist, T. Mattsson, JCAP 0702 (2007) 019, astro-ph/0609120;
H. Alnes, M. Amarzguioui, Phys. Rev. D 75 (2007) 023506, astro-ph/0610331.
[3] T. Biswas, A. Notari, astro-ph/0702555.
[4] N. Brouzakis, N. Tetradis, E. Tzavara, JCAP 0702 (2007) 013, astro-ph/0612179.
[5] N. Brouzakis, N. Tetradis, E. Tzavara, JCAP 0804 (2008) 008, astro-ph/0703586.
[6] R.K. Sachs, Proc. R. Soc. London A 264 (1961) 309.
[7] M.H. Partovi, B. Mashhoon, Astrophys. J. 276 (1984) 4;
N.P. Humphreys, R. Maartens, D.R. Matravers, Astrophys. J. 477 (1997) 47, astro-ph/9602033.
[8] V. Marra, E.W. Kolb, S. Matarrese, Phys. Rev. D 77 (2008) 023003, arXiv: 0710.5505 [astro-ph].
[9] S. Weinberg, Astrophys. J. 208 (1976) L1.
[10] H.G. Rose, Astrophys. J. 560 (2001) L15, astro-ph/0106489.
[11] M.J. Rees, D.W. Sciama, Nature 217 (1968) 511.