Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 206 (2006) 455–465
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Coding of the dimension group

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Received 28 April 2004; accepted 11 October 2005
Available online 17 November 2005
Communicated by Alain Connes

Abstract

We study an aspect of dimension group theory, linked to coding. The dimension group that we consider is built on a given square primitive integer matrix M satisfying the conditions that $|\det M| \geqslant 2$ and that the characteristic polynomial of M is irreducible. The coding is based on iteration of what could be seen as a generalization to $\mathbb{Z}^d$ of the Euclidean algorithm induced by the matrix M and in a natural way we define a binary operation of addition in the coding group.

The set B of symbols is a subset of $\mathbb{Z}^d$, and if we denote by ρ the Perron–Frobenius eigenvalue of M and by v a left eigenvector associated to ρ, we define a function $\mathbb{Z}^d \times B^{\mathbb{N}^*} \to \mathbb{R}$ which assigns to the element $(p, b_1, b_2, \ldots)$ the series

$$\langle v, p\rangle + \frac{1}{\rho}\langle v, b_1\rangle + \frac{1}{\rho^2}\langle v, b_2\rangle + \cdots$$

(in case $M = (10)$, this is the decimal expansion) and the restriction of this function to finite codes is the classical embedding of the dimension group into $\mathbb{R}$.

Finally, and under some suitable conditions, we prove that the last function is surjective and this allows the coding of real numbers and consequently the dimension group embedded into $\mathbb{R}$ appears as the set of decimal numbers.

Keywords: Dimension group; Integer matrices; Coding

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0001-8708/$ – see front matter
doi:10.1016/j.aim.2005.10.001

1. Introduction

There exists an increasing literature devoted to the study of dimension groups, which were introduced in [6] for the purpose of classifying certain non-commutative rings. An ordered abelian group is called a dimension group if it can be expressed as the inductive limit of a sequence of finite ordered group direct sums of copies of the integers, with non-decreasing morphisms [2,3,5]. In 1980, Effros et al. [5] gave an axiomatic characterization of dimension groups, namely, as countable ordered abelian groups which are unperforated and have the Riesz interpolation property. Their result made it possible to produce easily explicit examples of dimension groups [1,4] and to make certain inroads into the classification problem. We adopt in this paper a new point of view by studying an aspect of dimension groups linked to coding.

A few words on terminology are in order. An *inductive system* of abelian groups is a sequence $(E_n, f_{n+1})_{n\geqslant 0}$ of abelian groups and morphisms written as

$$E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} E_2 \xrightarrow{f_3} \cdots.$$

An *inductive cone* based on the inductive system $(E_n, f_{n+1})_{n\geqslant 0}$ is a sequence of morphisms $(g_n : E_n \to G)_{n\geqslant 0}$ such that for any integer n, one has $g_{n+1} f_{n+1} = g_n$. An *inductive limit* $(i_n : E_n \to E)_{n\geqslant 0}$ is an initial inductive cone, i.e. it satisfies the following condition:

for each inductive cone $(g_n : E_n \to G)_{n\geqslant 0}$ *there exists a unique morphism* $\Phi : E \to G$ *such that* $\Phi i_n = g_n$.

One has the following characterization:

An inductive cone $(i_n : E_n \to L)_{n\geqslant 0}$ *is an inductive limit of the system* $(E_n, f_{n+1})_{n\geqslant 0}$ *if and only if the two following conditions are satisfied*:

(a) *For each* x *in* L*, there exists a couple* (p, y) *such that* $y \in E_p$ *and* $i_p(y) = x$.
(b) *If* $y \in E_p$ *and* $z \in E_q$ *satisfy* $i_p(y) = i_q(z)$*, then there exists* $l \geqslant p, q$ *such that* $f_{pl}(y) = f_{ql}(z)$*, where* $f_{ij} = f_j f_{j-1} \dots f_{i+1}$ *in the case* $i < j$ *and* $f_{ii} = f_i$.

Let us introduce the order. Consider an inductive system of abelian groups

$$\mathbb{Z}^{d_0} \xrightarrow{M_1} \mathbb{Z}^{d_1} \xrightarrow{M_2} \mathbb{Z}^{d_2} \xrightarrow{M_3} \cdots,$$

each of which is equipped with the order defined by the natural positive cone associated with the canonical basis. The morphisms are non-decreasing exactly when their matrices are non-negative. The positive cone of the inductive limit ordered abelian group is

$$\bigcup_{n\geqslant 0} i_n\left(\mathbb{N}^{d_n}\right).$$

2. Coding

We are now ready to define the matrix with which we will be dealing throughout the remainder of this paper. Let M be a non-negative $d \times d$ integer matrix satisfying $|\det M| \geqslant 2$, and think of it as representing a linear map

$$\mathbb{Z}^d \xrightarrow{M} \mathbb{Z}^d.$$

The aim is to describe the corresponding dimension group G_D, which is a limit of the inductive system of ordered groups

$$\mathbb{Z}^d \xrightarrow{M} \mathbb{Z}^d \xrightarrow{M} \mathbb{Z}^d \xrightarrow{M} \cdots.$$

The matrix M has an inverse in the ring of $d \times d$ rational matrices, and the inductive cone

$$\begin{array}{ccc} \mathbb{Z}^d & \xrightarrow{\quad M \quad} & \mathbb{Z}^d \\ & {\scriptstyle M^{-n}} \searrow \qquad \swarrow {\scriptstyle M^{-n-1}} & \\ & \mathbb{Q}^d & \end{array}$$

has the inductive limit

$$G_D = \bigcup_{n \geqslant 0} M^{-n}\left(\mathbb{Z}^d\right) \subseteq \mathbb{Q}^d$$

with the order defined by the positive cone

$$\bigcup_{n \geqslant 0} M^{-n}\left(\mathbb{N}^d\right).$$

Let us take one particular case. Suppose that the matrix M is just 1×1 and is equal to (10):

$$\mathbb{Z} \xrightarrow{10} \mathbb{Z} \xrightarrow{10} \mathbb{Z} \longrightarrow \cdots.$$

Then it is easily seen that the dimension group obtained by this procedure is the group of decimal numbers coded by $\{0, 1, \ldots, 9\}$. Generalizing this classical code will be one of our aims in this work. The analogue of an element of $\{0, 1, \ldots, 9\}$ is a set $B \subset \mathbb{Z}^d$ containing $|\det M|$ elements and defined as follows:

The matrix M realizes an automorphism of the vector space $\mathbb{R}^d$ and is one-to-one on $\mathbb{Z}^d$. We denote by

$$C = \sum_{i=1}^{d} [0, 1[e_i$$

the hypercube $[0,1[^d$ of $\mathbb{R}^d$. The family of translated hypercubes $(p+C)_{p\in\mathbb{Z}^d}$ defines a partition of $\mathbb{R}^d$. Moreover, if we denote by M_i, $1\leqslant i\leqslant n$, the columns of the matrix M, then the image of C under M is

$$R=\sum_{i=1}^{d}[0,1[M_i.$$

If we set $B=R\cap\mathbb{Z}^d$, then the family $(Mp+B)_{p\in\mathbb{Z}^d}$ is a partition of $\mathbb{Z}^d$.

To bring out the structure of code, one notes that in the inductive system

$$\mathbb{Z}^d \xrightarrow{M} \mathbb{Z}^d \xrightarrow{M} \mathbb{Z}^d \xrightarrow{M} \cdots$$

the one-to-one mapping from each group $\mathbb{Z}^d$ to the next one can be viewed as a magnification: the element p of $\mathbb{Z}^d$ is replaced by the finite set

$$Mp+B. \tag{$*$}$$

Example. Let M be

$$\begin{pmatrix}2 & 2\\ 1 & 4\end{pmatrix}.$$

The set B is $\{(0,0),(1,1),(1,2),(2,2),(2,3),(3,4)\}$, and the vector $\binom{1}{1}$ in the lattice $\mathbb{Z}^2$ at step n is replaced by the set

$$M\begin{pmatrix}1\\1\end{pmatrix}+B=\bigl\{(4,5),(5,6),(5,7),(6,7),(6,8),(7,9)\bigr\}$$

in the lattice $\mathbb{Z}^2$ at step $n+1$ where each one of the six elements appears as part of a magnification of $\binom{1}{1}$.

Since the family $(Mp+B)_{p\in\mathbb{Z}^d}$ is a partition of $\mathbb{Z}^d$, one obtains a generalized Euclidean algorithm:

Proposition 1. *For each q in $\mathbb{Z}^d$, there exists a unique element (p,b) in $\mathbb{Z}^d\times B$ such that $q=Mp+b$.*

In other words, the mapping from $\mathbb{Z}^d\times B$ to $\mathbb{Z}^d$, which assigns $Mp+b$ to the element (p,b), is bijective. The term B which appeared in $(*)$ conveys the extra information brought to the elements of $\mathbb{Z}^d$ by the magnification effect induced by M.

On iterating, one obtains

Proposition 2. *For each $n\geqslant 1$, the mapping*

$$\begin{aligned}\mathbb{Z}^d \times B^n &\longrightarrow \mathbb{Z}^d\\ (p, b_1, \ldots, b_n) &\longmapsto M^n p + M^{n-1} b_1 + \cdots + b_n\end{aligned}$$

is bijective.

If we denote by c_n its inverse (coding at level n),

$$c_n : \mathbb{Z}^d \longrightarrow \mathbb{Z}^d \times B^n,$$

we deduce an isomorphism of inductive systems:

$$\begin{array}{ccccccc}\mathbb{Z}^d & \xrightarrow{M} & \mathbb{Z}^d & \xrightarrow{M} & \mathbb{Z}^d & \longrightarrow & \cdots\\ \wr\big\downarrow c_0 & & \wr\big\downarrow c_1 & & \wr\big\downarrow c_2 & & \\ \mathbb{Z}^d & \xrightarrow{i_0} & \mathbb{Z}^d \times B & \xrightarrow{i_1} & \mathbb{Z}^d \times B^2 & \longrightarrow & \cdots\end{array}$$

where

$$i_n(p, b_1, \ldots, b_n) = (p, b_1, \ldots, b_n, 0).$$

The limit of the system

$$\mathbb{Z}^d \xrightarrow{i_0} \mathbb{Z}^d \times B \xrightarrow{i_1} \mathbb{Z}^d \times B^2 \xrightarrow{i_2} \cdots$$

is $\mathbb{Z}^d \times B^{(\mathbb{N}^*)}$, where $B^{(\mathbb{N}^*)}$ is the set of sequences $(b_n)_{n\geqslant 1}$ of elements of B which eventually vanish. It results that one obtains, in the limit, an isomorphism

$$c_\infty : G_D \longrightarrow \mathbb{Z}^d \times B^{(\mathbb{N}^*)}$$

which makes the following diagram commutative:

$$\begin{array}{ccc}\mathbb{Z}^d & \xrightarrow{M^{-n}} & G_D\\ \wr\big\downarrow c_n & & \wr\big\downarrow c_\infty\\ \mathbb{Z}^d \times B^n & \xrightarrow{i_{n,\infty}} & \mathbb{Z}^d \times B^{(\mathbb{N}^*)}\end{array}$$

where $i_{n,\infty}(p, b_1, \ldots, b_n) = (p, b_1, \ldots, b_n, 0, \ldots)$, so that one associates to the code $(p, b_1, \ldots, b_n, 0, \ldots)$ the element $p + M^{-1}b_1 + \cdots + M^{-n}b_n$ in G_D.

We proceed to define a binary operation of addition in $\mathbb{Z}^d \times B^{(\mathbb{N}^*)}$, compatible with that in G_D. The operation of addition in $\mathbb{Z}^d \times B^{(\mathbb{N}^*)}$ uses carrying. More precisely, to add two elements $(b_1, \ldots, b_n)$ and $(v_1, \ldots, v_n)$ in B^n we take their pre-images in $\mathbb{Z}^d$

$$M^{n-1}b_1 + M^{n-2}b_2 + \cdots + b_n \quad \text{and} \quad M^{n-1}v_1 + M^{n-2}v_2 + \cdots + v_n$$

respectively, we write

$$M^{n-1}(b_1+v_1)+\dots+(b_n+v_n)=M^n p+M^{n-1}w_1+M^{n-2}w_2+\dots+w_n,$$

and we set

$$(b_1,\dots,b_n)\oplus(v_1,\dots,v_n)=(w_1,\dots,w_n).$$

Let us look at the simple case $n=1$. Let (b_1,v_1) be an element of $B\times B$. There exists a unique element of $\mathbb{Z}^d\times B$ written $(f(b_1,v_1),b_1\oplus v_1)$ such that

$$b_1+v_1=M\big(f(b_1,v_1)\big)+b_1\oplus v_1.$$

Thus, we define the operation of addition $\oplus$ on the group B and a function $f:B\times B\to\{0,1\}^d$ named carrying. In the last example, addition on B is given by the table below:

$\oplus$	(0, 0)	(1, 1)	(1, 2)	(2, 2)	(2, 3)	(3, 4)
(0, 0)	(0, 0)	(1, 1)	(1, 2)	(2, 2)	(2, 3)	(3, 4)
(1, 1)		(2, 2)	(2, 3)	(1, 2)	(3, 4)	(0, 0)
(1, 2)			(0, 0)	(3, 4)	(1, 1)	(2, 2)
(2, 2)				(2, 3)	(0, 0)	(1, 1)
(2, 3)					(2, 2)	(1, 2)
(3, 4)						(2, 3).

As for the carrying map $f:B\times B\to\{0,1\}^2$, one has the following table:

f	(0, 0)	(1, 1)	(1, 2)	(2, 2)	(2, 3)	(3, 4)
(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)
(1, 1)		(0, 0)	(0, 0)	(1, 0)	(0, 0)	(1, 1)
(1, 2)			(0, 1)	(0, 0)	(0, 1)	(0, 1)
(2, 2)				(1, 0)	(1, 1)	(1, 1)
(2, 3)					(0, 1)	(1, 1)
(3, 4)						(1, 1).

In general, carrying is performed from right to left and this provides the element p.

3. Embedding the dimension group into $\mathbb{R}$

By following [2,5], we consider M as the matrix of an endomorphism of $\mathbb{R}^d$. If ρ is a non-zero eigenvalue and $v \in (\mathbb{R}^d)^*$ a left eigenvector of M associated to ρ, then the commutative diagrams

$$\begin{array}{ccc} \mathbb{Z}^d & \xrightarrow{\ M\ } & \mathbb{Z}^d \\ & {\scriptstyle \rho^{-n}\langle v,.\rangle} \searrow \quad \swarrow {\scriptstyle \rho^{-n-1}\langle v,.\rangle} & \\ & \mathbb{R} & \end{array}$$

define an inductive cone which induces a mapping $\tau : G_D \to \mathbb{R}$, sending the code $(p, b_1, \ldots, b_n)$ to the element

$$\langle v, p\rangle + \langle v, M^{-1}b_1\rangle + \cdots + \langle v, M^{-n}b_n\rangle = \langle v, p\rangle + \sum_{i=1}^{n} \frac{1}{\rho^i}\langle v, b_i\rangle.$$

In the case that the mapping $\langle v, .\rangle : \mathbb{Z}^d \to \mathbb{R}$ is one-to-one, the real number

$$x = \langle v, p\rangle + \frac{1}{\rho}\langle v, b_1\rangle + \cdots + \frac{1}{\rho^n}\langle v, b_n\rangle$$

has a unique code $(p, b_1, \ldots, b_n, 0, \ldots)$.

From now on, we suppose that the matrix M is primitive [7]. *We denote by ρ the Perron–Frobenius eigenvalue of M and by v a left eigenvector associated to ρ.*

It is well known that the following statements are equivalent:

(i) *The characteristic polynomial of M is irreducible*;
(ii) *The components of v are $\mathbb{Z}$-free*;
(iii) *The mapping $\langle v, .\rangle : \mathbb{Z}^d \to \mathbb{R}$ is one-to-one.*

We denote by (H_0) the following hypothesis: *the matrix M is integer, non-negative, primitive, satisfies $|\det M| \geqslant 2$ and its characteristic polynomial is irreducible.*

Thus, under the hypothesis (H_0), the dimension group embedded into $\mathbb{R}$ is the totally ordered subgroup of $\mathbb{R}$ defined by

$$\bigcup_{n \geqslant 0} \frac{1}{\rho^n}\langle v, \mathbb{Z}^d\rangle.$$

Indeed, the range of the mapping $\rho^{-n}\langle v, .\rangle : \mathbb{Z}^d \to \mathbb{R}$ is the subgroup $\rho^{-n}\langle v, \mathbb{Z}^d\rangle$ of $\mathbb{R}$ which is included in $\rho^{-n-1}\langle v, \mathbb{Z}^d\rangle$; thus we get a non-decreasing sequence of subgroups of $\mathbb{R}$ each of which is dense in $\mathbb{R}$, because at least one of the components of v is irrational.

Moreover, each element of the dimension group embedded into $\mathbb{R}$ can be written in a unique manner as

$$\langle p, v\rangle + \frac{1}{\rho}\langle v, b_1\rangle + \cdots + \frac{1}{\rho^n}\langle v, b_n\rangle.$$

Indeed, let $x \in \bigcup_{n\geqslant 0} \rho^{-n}\langle v, \mathbb{Z}^d\rangle$. For $n \leqslant m$, one takes in $\mathbb{Z}^d$ pre-images $\bar{x}$ and $\bar{y}$ of x at levels n and m respectively:

$$\begin{array}{ccc} \mathbb{Z}^d & \xrightarrow{\;M^{m-n}\;} & \mathbb{Z}^d \\ & {\scriptstyle \rho^{-n}\langle v,.\rangle} \searrow \quad \swarrow {\scriptstyle \rho^{-m}\langle v,.\rangle} & \\ & \mathbb{R}. & \end{array}$$

The code of $\bar{x}$ is $(p, b_1, \ldots, b_n)$ and the one of $\bar{y}$ is $(q, v_1, \ldots, v_m)$. In other words,

$$\bar{x} = M^n p + M^{n-1} b_1 + \cdots + M b_{n-1} + b_n$$

and

$$\bar{y} = M^m q + M^{m-1} v_1 + \cdots + M v_{m-1} + v_m.$$

As

$$\frac{1}{\rho^m}\langle v, \bar{y}\rangle = \frac{1}{\rho^n}\langle v, \bar{x}\rangle = \frac{\rho^{m-n}}{\rho^m}\langle v, \bar{x}\rangle = \frac{1}{\rho^m}\big\langle v, M^{m-n}\bar{x}\big\rangle$$

then, taking the fact that the mapping $\rho^{-m}\langle v, .\rangle$ is one-to-one into consideration, we get $M^{m-n}\bar{x} = \bar{y}$, so

$$\begin{aligned} M^{m-n}\bar{x} &= M^m p + M^{m-1} b_1 + \cdots + M^{m-n+1} b_{n-1} + M^{m-n} b_n \\ &= M^m q + M^{m-1} v_1 + \cdots + M v_{m-1} + v_m. \end{aligned}$$

This is equivalent to

$$(q, v_1, \ldots, v_{m-1}, v_m) = (p, b_1, \ldots, b_n, 0, \ldots, 0),$$

i.e., $p = q$, $v_1 = b_1, \ldots, v_n = b_n$ and $v_{n+1} = \cdots = v_m = 0$. In conclusion, we get:

Proposition 3. *Under the hypothesis (H_0), the mapping $\mathbb{Z}^d \times B^{(\mathbb{N}^*)} \to \mathbb{R}$ which sends the code $(p, b_1, \ldots, b_n, 0, \ldots)$ to the real number*

$$\langle v, p\rangle + \frac{1}{\rho}\langle v, b_1\rangle + \cdots + \frac{1}{\rho^n}\langle v, b_n\rangle$$

is one-to-one and its range is a totally ordered dimension group.

4. Completion

We intend to prove, under some hypotheses which we will formulate, the following result: the mapping: $\mathbb{Z}^d \times B^{\mathbb{N}^*} \to \mathbb{R}$ that sends $(p, b_1, b_2, \ldots)$ to the real number

$$\langle v, p\rangle + \sum_{n\geqslant 1} \frac{1}{\rho^n}\langle v, b_n\rangle$$

is surjective.

We consider the following set:

$$S = \{(q_n)_{n\geqslant 0} \mid q_n \in \mathbb{Z}^d \text{ and } q_n \in Mq_{n-1} + B\}.$$

Given the sequence $(q_n)_{n\geqslant 0}$, we set $q_n = Mq_{n-1} + r_n$ which gives a sequence $(q_0, r_1, r_2, \ldots)$. Thus, we define a bijection between S and $\mathbb{Z}^d \times B^{\mathbb{N}^*}$. The sequence $(q_0, r_1, r_2, \ldots)$ is an infinite code which generalizes the finite codes that appear in the dimension group. In the case $d = 1$, it is a question of the usual coding of real numbers: the matrix M is an integer $b \geqslant 2$, the sequence $(q_0, q_1, q_2, \ldots)$ is given by $q_n = [b^n x]$ and

$$\frac{q_n}{b^n} = q_0 + \frac{r_1}{b} + \frac{r_2}{b^2} + \cdots + \frac{r_n}{b^n}.$$

We intend to generalize this situation. Let us take again the system

$$\mathbb{Z}^d \xrightarrow{M} \mathbb{Z}^d \xrightarrow{M} \mathbb{Z}^d \xrightarrow{M} \cdots.$$

Every lattice is indexed by an integer $0, 1, \ldots$ which can be called the level. The transition from one level to the next one is performed by M. So, in the sequence $(q_0, q_1, \ldots) \in S$, we must consider $q_n \in \mathbb{Z}^d$ as a term of level n of the inductive system. It is a question of associating to each real number x an element $(p, b_1, b_2, \ldots)$ from the set $\mathbb{Z}^d \times B^{\mathbb{N}^*}$ such that

$$x = \langle v, p\rangle + \frac{1}{\rho}\langle v, b_1\rangle + \frac{1}{\rho^2}\langle v, b_2\rangle + \cdots.$$

We define a total order on $\mathbb{Z}^d$ by setting $z \preccurlyeq z'$ if $\langle v, z\rangle \leqslant \langle v, z'\rangle$ and we denote by b^1 (respectively b^k) where $k = |\det M| - 1$ the lowest non-zero (respectively the greatest) element in B. Let us make the following hypotheses:

(H_1) $b^1 = \mathbf{1} = Mx_1$, $x_1 \in]0, 1[^d$;
(H_2) For each i, $1 \leqslant i \leqslant k$, we have $b^{i+1} - b^i \preccurlyeq \mathbf{1}$,

where $\mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

One notices at once the equality $b^k + b^1 = Mb^1$. Indeed, we observe that the elements of B are Mx such that $x \in [0, 1[^d$ and $Mx \in \mathbb{Z}^d$, so if x_1 satisfies $Mx_1 = b^1$, then $\langle v, Mx_1\rangle \leqslant \langle v, My\rangle$ for each $y \in [0, 1[^d \backslash \{0\}$ such that $My \in \mathbb{Z}^d$. Let

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix} \quad \text{in } [0, 1[^d \backslash \{0\};$$

we denote by A the subset of $\{1, \dots, d\}$ such that $i \in A$ if $y_i \neq 0$ and set $e_A = (\mathbf{1}_A(i))$. Then $M(e_A - y) \in B$ and

$$\langle v, Mx_1\rangle \leqslant \bigl\langle v, M(e_A - y)\bigr\rangle \leqslant \bigl\langle v, M(\mathbf{1} - y)\bigr\rangle.$$

Thus, $M(\mathbf{1} - x_1)$ is greater than all elements My such that $y \in [0, 1[^d$ and $My \in \mathbb{Z}^d$.

Let x be a real number; there exists $p \in \mathbb{Z}^d$ such that $\langle v, p\rangle \leqslant x < \langle v, p + \mathbf{1}\rangle$, whence

$$0 \leqslant \rho\bigl(x - \langle v, p\rangle\bigr) < \langle v, M\mathbf{1}\rangle.$$

The subdivision that results from the inequalities

$$0 = b^0 \prec b^1 \prec b^2 \prec \cdots \prec b^k \prec M\mathbf{1}$$

produces in $\mathbb{R}$ the inequalities

$$0 < \bigl\langle v, b^1\bigr\rangle < \bigl\langle v, b^2\bigr\rangle < \cdots < \bigl\langle v, b^k\bigr\rangle < \langle v, M\mathbf{1}\rangle,$$

and we have two possibilities.

The first possibility is that $\langle v, b^i\rangle \leqslant \rho(x - \langle v, p\rangle) < \langle v, b^{i+1}\rangle$. In this case we get

$$\rho\bigl(x - \langle v, p\rangle\bigr) = \bigl\langle v, b^i\bigr\rangle + r_1 \quad \text{and} \quad 0 \leqslant r_1 < \bigl\langle v, b^{i+1} - b^i\bigr\rangle.$$

Consequently,

$$x = \langle v, p\rangle + \frac{1}{\rho}\bigl\langle v, b^i\bigr\rangle + \frac{r_1}{\rho},$$

and we conclude, by taking the inequalities $0 \leqslant r_1 < \langle v, b^{i+1} - b^i\rangle$ into consideration, that

$$0 \leqslant \rho r_1 < \bigl\langle v, \rho\bigl(b^{i+1} - b^i\bigr)\bigr\rangle \leqslant \langle v, M\mathbf{1}\rangle,$$

which enables us to iterate with ρr_1.

In the second case we have $\langle v, b^k\rangle \leqslant \rho(x - \langle v, p\rangle) < \langle v, M\mathbf{1}\rangle$, which gives

$$\rho\bigl(x - \langle v, p\rangle\bigr) = \bigl\langle v, b^k\bigr\rangle + r_1', \quad 0 \leqslant r_1' < \bigl\langle v, M\mathbf{1} - b^k\bigr\rangle.$$

Since $b^k + b^1 = M\mathbf{1}$ and $b^1 = \mathbf{1}$, we have

$$x = \langle v, p\rangle + \frac{1}{\rho}\langle v, b^k\rangle + \frac{r_1'}{\rho},$$

and we conclude that $0 \leqslant \rho r_1' < \rho\langle v, \mathbf{1}\rangle$. Then, we get $0 \leqslant \rho r_1' < \langle v, M\mathbf{1}\rangle$ and we can iterate with $\rho r_1'$.

Finally, we have proved:

Proposition 4. *Under the hypotheses* (H_0), (H_1) *and* (H_2) *given above, the mapping* $\mathbb{Z}^d \times B^{\mathbb{N}^*} \to \mathbb{R}$ *which sends the element* $(p, b_1, b_2, \ldots)$ *to the real number*

$$\langle v, p\rangle + \sum_{n\geqslant 1} \frac{1}{\rho^n}\langle v, b_n\rangle$$

is surjective.

Acknowledgments

This article was written while the author visited the Université de Bourgogne. He would like to thank the Institut de Mathématiques de Bourgogne and particularly G. Michon for their hospitality.

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