Sandwich column buckling – A hyperelastic formulation

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\begin{abstract}

The macro-buckling equations for a sandwich column are developed. A layer-wise Timoshenko beam displacement approximation is assumed. The constitutive relationships and equilibrium equations for the core and face sheets are derived using a consistent hyperelastic neo-Hookean formulation. The derivations in this paper are consistent with that of Haringx's and Reissner's proposal for beam actions. The buckling formulation includes the axial deformation prior to buckling and the transverse shear deformation of the core and face sheets. The buckling equations derived agree with the equation of [Allen, H.G., 1969. Analysis and Design of Structural Sandwich Panels, Pergamon, Oxford] for thick faces but are also applicable to any ratio of face sheet to core thickness and material properties. The formulation is compared to experimental results for sandwich columns and shows good comparison except for very short columns. The formulation is also compared to the buckling experimental results for short rubber rods and also compared well. The formulation does not predict a shear buckling mode.

\end{abstract}

1. Introduction

Estimating the elastic column buckling load for helical springs, elastomeric bearings, sandwich plates and, built-up and laced columns requires the correct inclusion of shear deformations (see Attard, 2003; Bazant, 2003; Bazant and Beghini, 2004, 2006; Engesser, 1889, 1891; Gjelsvik, 1991; Haringx, 1948, 1949, 1942; Kardomateas and Dancila, 1997; Reissner, 1972, 1982; Simo et al., 1984; Simo and Kelly, 1984; Timoshenko and Gere, 1963; Zielger, 1982). The inclusion of shear deformations are also important in the analysis of the compressive strength of fiber composites where fiber microbuckling models have been postulated, and sandwich columns (see Budiansky and Fleck, 1994; Fleck, 1997; Niu and Talreja, 2000). Euler’s column buckling formula was first modified to include shear deformations by Engesser (1889, 1891). For a prismatic straight column Engesser formula is

$$\frac{1}{P_{cr}} = \frac{1}{P_{euler}} + \frac{1}{P_{S}}$$

$$\frac{P_{cr}}{P_{S}} = \frac{P_{euler}}{P_{S}} \left(1 + \frac{P_{euler}}{P_{S}}\right)$$

where $P_{cr}$ is the elastic critical load, $P_{euler} = \frac{\pi^2 EI}{L^2}$ is the Euler buckling load and $P_{S} = GA$ is a so-called localized “shear buckling load” ($E$ is the elastic modulus, $G$ the shear modulus, $I$ the second moment of area and $A$ the cross-sectional area). The Engesser's buckling load has an upper limit of $GA$ as the slenderness is reduced which is associated by some with a shear buckling failure mode. Shear buckling is sometimes referred to as “shear crimping” and is illustrated in Fig. 1. Rosen (1965) derived a similar shear microbuckling load limit for composites taken for very large buckling wavelengths, defined by $\frac{P_{S}}{P_{euler}}$ ($G_b$ is the binder shear modulus, $A_b$ is the binder cross-sectional area and $v_f$ is the fiber volume fraction). Rosen (1965) describes shear microbuckling in composites as when “…adjacent fibers buckling in the same wavelength and in

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\begin{thebibliography}
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phase with one another, so that the deformation of the matrix material between adjacent fibers is primarily a shear deformation.” Shear crimping, on the other hand is a localized failure. It is often initiated by a localized material failure. Vadakke and Carlsson (2004) proposed that shear crimping is a form of face wrinkling or a localized postbuckling mode.

Haringx (1942) developed an alternate column buckling formula which unlike Eq. (1) predicted an infinite buckling load as the slenderness approached zero. Haringx’s formula compared well with experimental buckling results for short rubber rods and helical springs (see Attard and Hunt, 2008). Attard and Hunt (2008) detailed a hyperelastic finite strain derivation for the buckling of straight prismatic isotropic columns and concluded that the notion of localised shear buckle as an column elastic buckling concept was not valid.

The theoretical arguments about whether Engesser’s or Haringx’s approach is correct have been reviewed by Bazant and Beghini (2004, 2006) when investigating the buckling of sandwich columns. Bazant and Beghini (2004, 2006) found that Engesser’s column buckling formula Eq. (1) was correct for sandwich columns with weak cores. Haringx’s column formula did not provide a reasonable fit to the sandwich column buckling results of Fleck and Sridhar (2002). Sandwich panels essentially consist of two thin load-bearing face-sheets bonded to a lightweight core such as foam, cellular cores or aluminium honeycomb which are often soft in shear (see Fig. 2). The thickness of the face sheets is here denoted by \( t \) while the core depth is given by \( c \). The width of a sandwich section is taken as \( b \). The face sheet elastic modulus and shear modulus are denoted by \( E_f \) and \( G_f \), respectively, while for the core they are denoted by \( E_c \) and \( G_c \), respectively. The bending rigidity is related to the distance between the face-sheets while shear is taken predominately through the core. Fleck and Sridhar tested 15 sandwich columns with soft cores in compression with 9 columns observed to fail by core shear macro-buckling. The test specimens were constructed from Divinycell H30, H100 and H200 foam for the core and face sheets made from four layers of eight harness satin weave 7781 E-glass fibres.

Allen’s text (Allen, 1969) is one of the classic references on sandwich panels. Allen gives two buckling formulas quoted widely in the literature, for thin or thick face sheets:

Thin faces:

\[
\frac{P_{cr, Allen}}{G_f A_m} = \frac{P_{euler}}{G_f A_m} \left( 1 + \frac{P_{cr, Allen}}{P_{euler}} \frac{G_f}{G_c} \right) A_m = A_c \frac{(c + t)^2}{c^2} A_c = cb
\]

(2)

Thick faces:

\[
\frac{P_{cr, Allen}}{G_c A_m} = \frac{P_{face}}{G_c A_m} \left( 1 + \frac{P_{cr, Allen}}{P_{face}} \frac{G_c}{G_f} \right)
\]

(3)

Fig. 1. Failure modes (a) shear crimping and (b) Euler buckling.

Fig. 2. Sandwich column dimensions (the width of the section is denoted by “b”).
In the above, \( A_m \) is the effective core area in shear, \( P_{\text{face}} \) is the buckling capacity of the face sheets as independent struts and \( P_{\text{euler}} \) is the Euler buckling load for the composite sandwich section (note the contribution of the core to the flexural stiffness is included here). The formula for thick faces Eq. (3) reduces to the formula for thin faces Eq. (2) as \( P_{\text{face}} \) approaches zero for large slenderness. We can see that Allen’s formula for thin face sheets Eq. (2) is essentially the same as Engesser’s formula Eq. (1). However, as observed by Allen, for a core weak in shear, as the slenderness is reduced, Eq. (3) rather than Eq. (2) is applicable for both thin face sheets as well as thick face sheets, as when the core ceases to provide effective connection between the faces, the face sheets buckle as independent struts. Therefore, we can conclude that there is no shear buckling upper limit for the critical load of sandwich columns as the slenderness is reduced, since in the limit \( P_{\text{cr}} \) approaches \( P_{\text{face}} \). At least this is the conclusion derived from Eq. (3).

Fig. 3 contains plots of normalized experimental buckling loads for sandwich columns tested by Fleck and Sridhar (2002), Hoff and Mautner (1948). Hoff and Mautner (1948) tested 64 sandwich columns with either balsa material or cellular cellulose acetate cores and Alcad aluminum face sheets. Hoff and Mautner’s results for the cellular cellulose acetate cores assuming a shear modulus of 17.25 MPa. The results show a large scatter, in part due to the variability of the cellular cores. Fig. 3 does not display an upper limit of \( G_A m \) as the slenderness becomes very small (this is because the face sheets still retain a buckling capacity even if the core carries no load unless there is a material failure or delamination).

A strain energy density for isotropic hyperelastic materials under finite strain was proposed in Attard (2003), Attard and Hunt (2004) and used to derive constitutive relationships for problems involving shear deformations. The hyperelastic formulation in Attard (2003), Attard and Hunt (2008), Attard and Hunt (2004), Attard (2003) when applied to the problem of column buckling was shown to be consistent with Haringx approach and Reissner’s proposal for beam actions (Attard, 2003; Reissner, 1972). In this paper, the proposed hyperelastic formulation in Attard (2003), Attard and Hunt (2008), Attard and Hunt (2004), Attard (2003) and hence Haringx’s and Reissner’s beam approach is used to derive the column buckling equations for sandwich columns incorporating transverse shear deformations within the face sheet and core, as well as the axial deformation prior to buckling. Wrinkling and face sheet interaction buckling (see Hunt and Wadee, 1998) are not considered in this paper.

2. Displacement model of a sandwich column under bending, shear and axial deformation

Consider a straight sandwich column. The longitudinal axis of centroids of the undeformed column is taken as the \( x \) or 1 axis (see Fig. 2). The principal axes in the plane of the cross-section are taken as the \( y \) or 2 axis and the \( z \) or 3 axis. The initial axis system chosen is a Cartesian rectangular system. The initial material lines within the column are assumed to be parallel to the Cartesian coordinate system and therefore the initial tangent base vectors in the undeformed state are aligned with the axis of the column and the principal axes. The deflected shape of the column cross-section will be characterized by a zig-zag pattern of deflections through the depth of cross-sectional plane (see Fig. 4). The planes of the core and face sheets remain plane but are rotated by different amounts (layer-wise Timoshenko beam or first order shear deformation laminate theory see Ghugal and Shimpi (2001)). The plane of the face sheet is not perpendicular to the centroidal axis during deformation but is assumed to undergo a shear deformation. The bending of the face sheet plane is denoted by the angle \( \phi_f \) while the shear angle of the face sheet is denoted by \( \phi_f \) (see Fig. 4). The rotation of the mid-plane of the face sheet is the sum of the rotations \( \phi_f + \phi_l \) (shown anti-clockwise in Fig. 4). The core cross-sectional plane rotates \( \theta_c \) (shown with a clockwise rotation in Fig. 4) from the vertical while the shear of the core is defined by

![Fig. 3. Comparison of sandwich column buckling test results (Fleck and Sridhar, 2002; Hoff and Mautner, 1948) with shear buckling formula and Eq. (2).](image-url)
The bending rotations $h_f, h_c$ and shear angles $u_c, u_f$ are all assumed to be functions of the longitudinal coordinate $x$, only. For fully composite bending with no shearing across the interface we would have $h_c = -h_f$ or $u_c = u_f$.

At the centroidal axis ($y = 0$) we define the displacements

$$ u_1(x, 0) = u_o, \quad u_2(x, 0) = v $$

where $u_1$ and $u_2$ are displacement components in the $x$ and $y$ directions, respectively, and where $u_o$ and $v$ are the longitudinal (in direction 1 or $x$) and transverse (in direction 2 or $y$) displacements of the centroidal axis, respectively. Displacements are assumed to occur only in the plane of the cross-section (this is essentially a plane strain assumption). Assuming there is no dilation of the core or face sheets, the displacement functions in the $x$ and $y$ directions for the face sheets and core can be written as

**Top bottom face sheets:**

$$ u_1 = u_o + \frac{c}{2} \sin \theta_c - \left( y + \frac{c}{2} \right) \sin \theta_t, \quad u_2 = v + \frac{c}{2} (\cos \theta_c - 1) + \left( y + \frac{c}{2} \right) (\cos \theta_t - 1) $$

**Core:**

$$ u_1 = u_o + y \sin \theta_c, \quad u_2 = v + y (\cos \theta_c - 1) $$

With compatibility of displacements satisfied at the core–face sheet interfaces.

The constitutive law for the physical Lagrangian stresses normal $S_{11}^R$ and tangential $S_{12}^R$ to the beam cross-section as derived in Appendix A, Eq. (A18) is given by

$$ S_{11}^R = E (\lambda_{n1} - 1), \quad S_{12}^R = G \lambda_{s1} $$

where $\lambda_{n1}$ and $\lambda_{s1}$ are the normal and tangential components of the longitudinal stretch $\lambda_1$, respectively, $E = 2G + A$, $A = \frac{2G}{1-2\nu}$ is the Lamé constant and $\nu$ is the Poisson’s ratio. The subscript ‘R’ used in the above notation of stresses is to indicate that these stresses are in agreement with Reissner’s proposal for beam actions (see Appendix A and Attard, 2003; Reissner, 1972). The material parameter governing the normal stress is not the elastic modulus $E$ as would be expected for a uniaxial stress state. This is because the assumed two dimensional displacements restrain the dilation of the cross-section shape which is associated with lateral stresses not be present under a uniaxial stress state (see Attard, 2003 and Attard and Hunt, 2008). A further approximation in beam theory is to replace $E$ by $E$ in Eq. (8). For a core constructed from foam with a Poisson’s ratio almost zero $E = 2G = E$. The constitutive relationships for the internal actions can be determined by defining the internal actions as the stress resultants over the cross-section (see Attard, 2003 and Attard and Hunt, 2008) thus

$$ N = \int_A S_{11}^R \, dA, \quad Q = \int_A S_{12}^R \, dA, \quad M = \int_A y S_{11}^R \, dA $$

Here, $N$ is the axial force defined perpendicular to the cross-sectional plane, $Q$ is the shear force within the cross-sectional plane and $M$ is the bending moment defined by the stresses perpendicular to the cross-sectional plane (see Fig. 5).
To determine the constitutive relationships from Eqs. (8) and (9) we need the normal and shear components of stretch for the face sheets and core. Appendix A contains the relationships between the components of the longitudinal stretch and the displacements of the cross-section for a two dimensional problem. Using Eqs. (6) and (A5), we have for the face sheet deformations:

\[
\begin{align*}
\lambda_1 \cos \varphi &= \lambda_{10} \cos \varphi_t + \frac{c}{2} \cos(\varphi_c - \varphi_t) \partial_{c,x} - \left( \frac{v_c}{2} \right) \partial_{t,x} \\
\lambda_1 \sin \varphi &= \lambda_{10} \sin \varphi_t + \frac{c}{2} \sin(\varphi_c - \varphi_t) \partial_{c,x}
\end{align*}
\]  

(10)

In which

\[
\begin{align*}
\lambda_{10} \cos \varphi_t &= (1 + u_{0,x}) \cos \theta_t + v_x \sin \theta_t \\
\lambda_{10} \sin \varphi_t &= -(1 + u_{0,x}) \sin \theta_t + v_x \cos \theta_t
\end{align*}
\]  

(11)

\(\varphi\) is the shear angle as defined in Appendix A and \(\lambda_{10}\) defines the longitudinal stretch measured at the centroid of a sandwich section given by

\[
\lambda_{10} = \sqrt{(1 + u_{0,x})^2 + v_x^2}
\]  

(12)

Differentiation is indicated by a comma subscript such as \(\partial_{t,x}\). Manipulating Eqs. (11), we have for the rotation of the centroidal axis (see Fig. 4):

\[
\tan(\theta_t + \varphi_t) = \frac{v_x}{1 + u_{0,x}}
\]  

(13)

Similarly for the core, using Eqs. (7) and (A5) we have

\[
\begin{align*}
\lambda_1 \cos \varphi &= \lambda_{10} \cos \varphi_c + y \theta_{c,x} \\
\lambda_1 \sin \varphi &= \lambda_{10} \sin \varphi_c
\end{align*}
\]  

(14)

With

\[
\begin{align*}
\lambda_{10} \cos \varphi_c &= (1 + u_{0,x}) \cos \theta_c - v_x \sin \theta_c \\
\lambda_{10} \sin \varphi_c &= (1 + u_{0,x}) \sin \theta_c + v_x \cos \theta_c
\end{align*}
\]  

(15)

Using Eq. (4) and the above, we can determine that

\[
\begin{align*}
1 + u_{0,x} &= \lambda_{10} \cos(\theta_t + \varphi_t) \\
v_x &= \lambda_{10} \sin(\theta_t + \varphi_t)
\end{align*}
\]  

(16)

Substituting Eqs. (10) and (14) into the constitutive relationships Eqs. (8) and (9), gives for the internal actions defined in Fig. 5:

- Top face:

\[
\begin{align*}
N_t &= E_f A_t \left( \lambda_{10} \cos \varphi_t - 1 \right) - E_f A_t \frac{c}{2} \partial_{t,x} + E_f A_t \frac{c}{2} \cos(\varphi_c - \varphi_t) \partial_{c,x} \\
Q_t &= G_f A_t \lambda_{10} \sin \varphi_t - G_f A_t \frac{c}{2} \sin(\varphi_c - \varphi_t) \partial_{c,x}, \\
M_t &= E_f I_t \partial_{t,x}
\end{align*}
\]  

(17)
• Bottom face:

\[
N_{b} = E_{b}A_{f}(\lambda_{10} \cos \phi_{t} - 1) + E_{f}A_{f} \frac{t}{2} \theta_{tx} - E_{f}A_{f} \frac{c}{2} \cos(\varphi_{c} - \varphi_{t}) \theta_{cx},
\]

\[
Q_{b} = G_{f}A_{f}(\lambda_{10} \sin \phi_{t} + G_{f}A_{f} \frac{c}{2} \sin(\varphi_{c} - \varphi_{t}) \theta_{cx}, M_{b} = E_{f}I_{f} \theta_{tx}
\]

(18)

• Core:

\[
N_{c} = E_{c}A_{c}(\lambda_{10} \cos \varphi_{c} - 1), Q_{c} = G_{c}A_{c}(\lambda_{10} \sin \varphi_{c}, M_{c} = -E_{c}I_{c} \theta_{cx}
\]

(19)

In the above:

\[
A_{f} = \text{t}_{b}, l_{f} = \frac{b t^{3}}{12}, A_{c} = cb, l_{c} = \frac{bc^{3}}{12}
\]

(20)

For fully composite bending action (\( \theta_{c} = -\theta_{t} \) and \( \varphi_{c} = \varphi_{t} \)), we see that Eqs. (17)–(19) would be consistent with Reissner beam theory (Reissner, 1972). The constitutive relationships for the internal actions to second order terms are therefore

• Top face:

\[
N_{b} \approx E_{f}A_{f}\left(u_{o,x} + \frac{1}{2} v_{x}^{2} - \frac{1}{2} \varphi_{t}^{2}\right) - E_{f}A_{f}\left(\frac{c}{2} + \frac{t}{2}\right) \theta_{tx} + E_{f}A_{f} \frac{c}{2}\left(\varphi_{cx} - \varphi_{tx}\right)
\]

\[
Q_{b} \approx G_{f}A_{f}(1 + u_{o,x})\varphi_{t} - G_{f}A_{f} \frac{c}{2}(\varphi_{c} - \varphi_{t}) \theta_{cx}, M_{b} = E_{f}I_{f} \theta_{tx}
\]

(21)

• Bottom face:

\[
N_{b} \approx E_{f}A_{f}\left(u_{o,x} + \frac{1}{2} v_{x}^{2} - \frac{1}{2} \varphi_{t}^{2}\right) + E_{f}A_{f}\left(\frac{c}{2} + \frac{t}{2}\right) \theta_{tx} - E_{f}A_{f} \frac{c}{2}\left(\varphi_{cx} - \varphi_{tx}\right)
\]

\[
Q_{b} \approx G_{f}A_{f}(1 + u_{o,x})\varphi_{t} + G_{f}A_{f} \frac{c}{2}(\varphi_{c} - \varphi_{t}) \theta_{cx}, M_{b} = E_{f}I_{f} \theta_{tx}
\]

(22)

• Core:

\[
N_{c} \approx E_{c}A_{c}\left(u_{o,x} + \frac{1}{2} v_{x}^{2} - \frac{1}{2} \varphi_{c}^{2}\right) Q_{c} \approx G_{c}A_{c}(1 + u_{o,x})\varphi_{c}, M_{c} = -E_{c}I_{c} \theta_{cx}
\]

(23)

3. Virtual work

Appendix A contains the derivation for the virtual work \( \delta W \), in terms of the Reissner stresses and for the case when the cross-sectional shape remains unchanged (\( \lambda_{2} = 1 \)). That is from Eq. (A22):

\[
\delta W = \int_{V} S_{11}^{\delta}(\lambda \cos \varphi) + S_{22}^{\delta}(\lambda \sin \varphi) \right) dV - \int_{S} \mathbf{p} \cdot \mathbf{u} dS
\]

(24)

With \( V \) being the volume in the undeformed state, \( S \) the surface where the externally applied traction vector \( \mathbf{p} \) acts, kinematically admisible variations denoted by the symbol \( \delta \) and displacement vector \( \mathbf{u} \). Equation (24) can be applied to the face sheets and core separately. Integrating over the cross-section segments and making use of Eqs. (9)–(11), (14), (15) results in

\[
\int_{0}^{L} \left[ P_{x} \delta u_{x} + P_{y} \delta v_{x} - \lambda_{10}(Q_{tx,zn} + Q_{bx,zn}) \delta \theta_{t} + M_{t} \delta \theta_{tx} + \lambda_{10} Q_{c,zn} \delta \theta_{c} - M_{c} \delta \theta_{cx} + w_{c}(\delta \theta_{c} + \theta_{c}) \right] dx = \int_{S} \mathbf{p} \cdot \mathbf{u} dS
\]

(25)

In which

\[
P_{x} = [N_{b} + N_{b,0}] \cos \theta_{t} - [Q_{b} + Q_{b,0}] \sin \theta_{t} + N_{c} \cos \theta_{c} + Q_{c} \sin \theta_{c}
\]

(26)

\[
P_{y} = [N_{b} + N_{b,0}] \sin \theta_{t} + [Q_{b} + Q_{b,0}] \cos \theta_{t} - N_{c} \sin \theta_{c} + Q_{c} \cos \theta_{c}
\]

(27)

\[
M_{t} = M_{b} + M_{b,0} + [N_{b} - N_{b,0}]
\]

(28)

\[
M_{c} = M_{c} + \frac{c}{2} ([N_{b} - N_{b,0}] \cos(\varphi_{c} - \varphi_{t}) + [Q_{b} - Q_{b,0}] \sin(\varphi_{c} - \varphi_{t}))
\]

(29)

\[
\lambda_{10}(Q_{tx,zn} + Q_{tx,zn}) = -[N_{b} + N_{b,0}] \lambda_{10} \sin \varphi_{t} + [Q_{b} + Q_{b,0}] \lambda_{10} \cos \varphi_{t}
\]

(30)

\[
\lambda_{10}(Q_{c,zn} = -N_{c} \lambda_{10} \sin \varphi_{c} + Q_{c} \lambda_{10} \cos \varphi_{c}
\]

(31)

\[
w_{c} = ([N_{b} - N_{b,0}] \sin(\varphi_{c} - \varphi_{t}) + [Q_{b} - Q_{b,0}] \cos(\varphi_{c} - \varphi_{t})) \frac{c}{2} \theta_{cx}
\]

(32)
In the above equations the following notation has been used:

\[
\begin{align*}
\frac{dP_x}{dx} &= 0, \quad \frac{dP_y}{dx} = 0 \quad (33) \\
\frac{d(M_t + M_c)}{dx} &= -\lambda_10(Q_{nt,tn} + Q_{cb,tn} + Q_{ct,tn}) \quad (34)
\end{align*}
\]

In the core:

\[
\frac{dM_c}{dx} = -\lambda_10Q_{ct,tn} - W_0 \quad (35)
\]

It is easy to verify Eqs. (34) and (35) by applying the principles of equilibrium to a freebody of a segment of a sandwich column of length \(dx = \lambda_1 \, dx\) measured parallel to the deformed centroidal axis. In addition (using Eqs. (26) and (27)), we can express the relationship between the tangential shear and axial force with the internal force resultants \(P_x\) and \(P_y\) thus

\[
\begin{align*}
-P_x \sin(\theta_1 + \varphi_t) + P_y \cos(\theta_1 + \varphi_t) &= (Q_{nt,tn} + Q_{nt,tn} + Q_{ct,tn}) \\
&= -[N_{nt} + N_{ct}] \sin \varphi_t + [Q_{nt} + Q_{ct}] \cos \varphi_t - N_c \sin \varphi_c + Q_c \cos \varphi_c \\
-P_x \cos(\theta_1 + \varphi_t) + P_y \sin(\theta_1 + \varphi_t) &= [N_{nt} + N_{ct}] \cos \varphi_t + [Q_{nt} + Q_{ct}] \sin \varphi_t + N_c \cos \varphi_c + Q_c \sin \varphi_c
\end{align*}
\]

4. Column buckling

Consider a straight prismatic column under initial compressive axial stress such that the internal resultant in the \(x\) direction is \(P_x = -P\). To ascertain the buckling load, we apply small kinematically admissible variations denoted by the symbol \(\delta\) of the displacement field. Initially, for the straight column before perturbations, we have for the axial force resultants in the face sheet and core:

\[
\begin{align*}
-P &= [E_cA_c + 2EI_c][\lambda_10 - 1] = N_{nt} + N_{ct} + N_c \\
\therefore \lambda_10 &= 1 - \frac{P}{EA_{tot}} = 1 - \bar{P}, \quad EA_{tot} = EcA_c + 2EI_c \\
N_c &= -\frac{EcA_c}{EA_{tot}} P = -m_ecP, \quad N_{nt} + N_{ct} = -2EI_c \bar{P} = -m_{ec}P
\end{align*}
\]

Applying the perturbations to the constitutive relationships for the internal actions we have for the overall bending moment:

\[
\begin{align*}
M_f + M_c &= M_{nt} + M_{ct} + M_c + \left(\frac{c + f}{2}\right)[N_{ct} - N_{nt}] \\
&= EI_{tot} \delta \theta_{tx} - \left[E_cI_c + 2EI_c\left(\frac{c}{2}\right)\left(\frac{c + f}{2}\right)\right]\left(\delta \varphi_{cx} - \delta \varphi_{tx}\right) \\
&= EA_{tot} \left[r^2 \delta \theta_{tx} - r^2_c (\delta \varphi_{cx} - \delta \varphi_{tx})\right] \quad (39)
\end{align*}
\]

and for the core bending moment:

\[
\begin{align*}
M_c &= Ec + \frac{c}{2}[N_{ct} - N_{nt}] \\
&= \left[E_cI_c + 2EI_c\left(\frac{c}{2}\right)\left(\frac{c}{2}\right)\right] \delta \theta_{tx} - \left[E_cI_c + 2EI_c\left(\frac{c}{2}\right)\right]^2 \left(\delta \varphi_{cx} - \delta \varphi_{tx}\right) \\
&= EA_{tot} \left[r^2 \delta \theta_{tx} - r^2_c (\delta \varphi_{cx} - \delta \varphi_{tx})\right] \quad (40)
\end{align*}
\]

In the above equations the following notation has been used:

\[
\begin{align*}
EI_{tot} &= 2EI_c + E_cI_c + 2EI_c\left(\frac{c + f}{2}\right)^2, \quad r^2 = \frac{EI_{tot}}{EA_{tot}} \\
r^2_c &= \frac{E_cI_c + 2EI_c(\frac{c}{2})^2}{EA_{tot}} \quad (41)
\end{align*}
\]

To establish the buckling load we can either look at the equilibrium Eqs. (33)–(37) under small perturbations about the axially loaded state or we can look at the second variation of the virtual work Eq. (25) as detailed in Appendix B and discussed in reference (Attard and Hunt, 2008). Here, we will establish the buckling load by looking at the perturbations of the equilib-
rium equations about the axially loaded state. Differentiating the equilibrium equation for the whole cross-section Eq. (34), assuming that $\lambda_{10,x} = 0$, using Eq. (39) and for the shear force Eq. (36), we can write

$$\frac{d^2(M_f + M_c)}{dx^2} = -\lambda_{10}(Q_{b,x} + Q_{f,x} + Q_{c,x}) \frac{d\delta\varphi_{c,x}}{dx}$$

(42)

$$r^2\delta\theta_{xx} - r^2(\delta\varphi_{c,xx} - \delta\varphi_{f,xx}) = -P(1-P)(\delta\theta_{tx} + \delta\varphi_{f})$$

While for the core we have the equation:

$$\frac{dM_c}{dx} = -\lambda_{10}Q_{c,xx}$$

(43)

$$r^2\delta\theta_{xx} - r^2(\delta\varphi_{c,xx} - \delta\varphi_{f,xx}) = (1-P)(N_c - G_c A_c (1-P)) \frac{\delta\varphi_c}{EA_{tot}}$$

$$r^2\delta\theta_{xx} - r^2(\delta\varphi_{c,xx} - \delta\varphi_{f,xx}) = -(1-P)m_1\delta\varphi_c$$

In which:

$$m_1 = m_{gc}(1-P) + Pm_{ec}, \quad m_{gc} = \frac{G_c A_c}{EA_{tot}}, \quad m_{ec} = \frac{E_c A_c}{EA_{tot}}$$

(44)

The shear angle in the face sheet can be expressed in terms of $\varphi_{c}$ by differentiating Eq. (36), that is

$$P(\delta\theta_{tx} + \delta\varphi_{f}) = (2G_c A_t (1-P) + Pm_{ct})\delta\varphi_{f} + (G_c A_c (1-P) + Pm_{ec})\delta\varphi_{c}$$

(45)

$$-m_2\delta\varphi_{f} = P\delta\theta_{tx} + m_1\delta\varphi_{c}$$

where

$$m_2 = m_{gf}(1-P) - Pm_{ec} m_{sf} = \frac{2G_c A_f}{EA_{tot}}$$

(46)

We get the same result from Eq. (33), that is from $\frac{\partial \varphi}{\partial x} = 0$. Solving Eq. (45) gives for shear angle in the face sheets as

$$\varphi_{f} = \frac{P}{m_2}\varphi_{1} - \frac{m_1}{m_2}\varphi_{c} + C_6$$

(47)

with $C_6$ an unknown constant. Substituting Eq. (47) into Eqs. (42) and (43), we can now write for the general solution to the deformation variables:

$$\varphi_{1} = C_1 + C_2 \cos(\varphi_{1}) + C_3 \sin(\varphi_{1}) + C_4 \cosh(\varphi_{1}) + C_5 \sinh(\varphi_{1})$$

(48)

$$\varphi_{c} = \beta_1(C_2 \cos(\varphi_{1}) + C_3 \sin(\varphi_{1}) + \beta_2(C_4 \cosh(\varphi_{1}) + C_5 \sinh(\varphi_{1}))$$

(49)

$$\varphi_{f} = \gamma_1(C_2 \cos(\varphi_{1}) + C_3 \sin(\varphi_{1})) + \gamma_2(C_4 \cosh(\varphi_{1}) + C_5 \sinh(\varphi_{1})) + C_2$$

(50)

Involving the unknown constants $C_1, C_2, \cdots, C_6$ and

$$C_7 = \frac{P}{m_2}C_1 + C_6, \quad \gamma_1 = \frac{P}{m_2} - \frac{m_1}{m_2}\beta_1, \quad \gamma_2 = \frac{P}{m_2} - \frac{m_1}{m_2}\beta_2$$

(51)

$$\beta_1 = \frac{x_1^2}{(1-P)} \left\{ \frac{2x_1^2(r_1^2 - r_1^2)}{P(1-P)m_1} \right\}, \quad \beta_2 = \frac{x_2^2}{(1-P)} \left\{ \frac{2x_2^2(r_2^2 - r_2^2)}{P(1-P)m_2} \right\}$$

(52)

$$\beta_1 = \frac{x_1^2}{(1-P)} \left\{ \frac{2x_1^2(r_1^2 - r_1^2)}{P(1-P)m_1} \right\}, \quad \beta_2 = \frac{x_2^2}{(1-P)} \left\{ \frac{2x_2^2(r_2^2 - r_2^2)}{P(1-P)m_2} \right\}$$

(53)

It is more compact to write explicit expressions for $x_1$ and $x_2$ thus

$$\left(\frac{x_1^2}{1-P}\right)^2 (m_1 + m_2)(r_2^2 - r_1^2) - \left(\frac{x_2^2}{1-P}\right)^2 (r_1^2 P + m_1 + m_2) - m_1(2r_2^2 P + m_2 P) - Pm_1(P + m_2) = 0$$

(54)

$$\left(\frac{x_2^2}{1-P}\right)^2 (m_1 + m_2)(r_2^2 - r_1^2) + \left(\frac{x_1^2}{1-P}\right)^2 (r_1^2 P + m_1 + m_2) - m_1(2r_2^2 P + m_2 P) - Pm_1(P + m_2) = 0$$

(55)

Let’s consider a column which is fully fixed at the end boundaries but allows axial deformation as depicted in Fig. 6. This is the boundary condition for the most common configuration for a sandwich column buckling test (see ASTM C 364). The boundary conditions would be

$$\text{at } x = 0: \delta\varphi = 0, \delta\varphi_{tx} = 0, \delta\varphi_{tx} = 0$$

$$\text{at } x = L: \delta\varphi = 0, \delta\varphi_{tx} = 0, \delta\varphi_{tx} = 0$$

(56)
Eq. (13) can be used to establish expressions for the vertical deflection $\delta$ v. Applying the boundary conditions (see Appendix C), the determinant of the system of equations is

$$
\begin{align*}
\beta_1(1 + \gamma_2) - \beta_2(1 + \gamma_1) & = -\frac{2m}{9L} \sin(\alpha L)(1 + \gamma_2)(\cosh(\alpha L) - 1) \\
-\frac{2m}{9} \sin(\alpha L)(1 + \gamma_2) & = -\frac{2m}{9L} \sin(\alpha L)(1 + \gamma_1)(\cos(\alpha L) - 1)
\end{align*}
$$

(57)

Therefore non-zero configurations for the deformation perturbations exist if

$$
\alpha = \frac{2n\pi}{L}, \quad \beta = \frac{-m_{\text{eff}}}{(1 - m_{\text{ec}} - m_{\text{eff}})}, \quad \beta_1 = \beta_2
$$

(58)

where $n$ is an integer (represents the buckling mode number). The second solution is a tensile limit. Substituting the solution for $\alpha$ in Eq. (58) into Eq. (54) gives the equation for the critical buckling load. Eq. (54) is a fifth order polynomial in $P$ which can be solved numerically. Simplified solutions can also be obtained by incorporating various approximations.

Firstly, let’s look at the situation where there is a weak core, no account is taken of shear deformations in the face sheets and the axial deformation prior to buckling is ignored. Hence only Eqs. (42) and (43) need to be considered with $\delta = 0$ and $N_c = 0$. The resulting buckling formula is

$$
P_{\text{cr}} \frac{G_cA_c}{G_cA_c} = \frac{P_{\text{euler}}}{G_cA_c} \left( 1 - \frac{\frac{P_{\text{euler}}}{G_cA_c} r_c^2}{1 + \frac{P_{\text{euler}}}{G_cA_c} r_c^2} \right)
$$

(59)

where $P_{\text{euler}} = \frac{E^2r_c^2}{12}la$ and $n = 2$ for the fully fixed end boundaries. If we incorporate the shear deformations in the face sheets, the resulting buckling formula from Eqs. (54) and (58) is

$$
P_{\text{cr}} \frac{G_cA_c}{G_cA_c} = -\frac{1}{2} \left( \frac{P_{\text{euler}}}{G_cA_c} \left( \frac{r_c^2 - 2L^2}{r^2} - a_1 \right) - \frac{m_{\text{eff}}}{m_{\text{ec}}} \right)
$$

(60)

$$
\left[ \frac{P_{\text{euler}}}{G_cA_c} \left( 4 + \frac{r_c^2}{r^2} - 4 \frac{r_c^2}{r^2} \right) + \frac{P_{\text{euler}}}{G_cA_c} \frac{m_{\text{eff}}}{m_{\text{ec}}} + 4 \frac{P_{\text{euler}}}{G_cA_c} \right]^2 \frac{r_c^2}{r^2} - 4 \frac{r_c^2}{r^2} + \frac{m_{\text{eff}}}{m_{\text{ec}}}
$$

Since $\frac{P_{\text{euler}}}{G_cA_c} \ll 1$ Allen’s formula for thick faces Eq. (3) and the proposed new formula Eq. (59) can both be approximated by

$$
P_{\text{cr}} \frac{G_cA_c}{G_cA_c} = \frac{P_{\text{euler}}}{G_cA_c} \left( 1 + \frac{P_{\text{euler}}}{G_cA_c} \right)
$$

(61)

When the effective length of the column is relatively long $\frac{P_{\text{euler}}}{G_cA_c}$ would be small and the above equation matches Eqs. (1) and (2), essentially Engesser’s solution. However, if the slenderness is made very small, the buckling load of the face sheets acting independently would dominate. For most practical sandwich column configurations with thin face sheets and weak cores, Eqs. (3),(59) and (60) give solutions very close to those derived using formula Eq. (61). Eqs. (59) and (60), however, have importantly been derived in a manner consistent with the approaches of Haringx and Reissner. Although it is correct that Haringx’s “column” buckling formula does not work for sandwich columns because it does not take account of the shearing between the core and face sheets as shown in Fig. 4, Haringx’s “approach” is still valid. Figs. 7 and 8 compare the experimental results of Fleck and Sridhar (2002), Hoff and Mautner (1948) for sandwich columns with weak cores and relatively thin face sheets with the predictions of Eq. (59). The approximate formula in Eq. (61) is also plotted in Fig. 7 and plots as a continuous line as the experiments of Fleck and Sridhar (2002) are for sections with the same core depth and face sheet thickness. The sandwich column properties and the experimental buckling loads are listed in Tables 1 and 2. We see that the only major discrepancy between the predicted results and the available experimental data is when the column length is very small (20–50 mm for the experiments of Fleck and Sridhar (2002) and 127 mm for the experiments of Hoff and Mautner (1948)).

Haringx’s buckling formula (see Attard and Hunt, 2008) agreed well with the experimental results for short rubber rods tested by Haringx in 1949. To check the versatility of the proposed formulation for the extreme case when the core and face
sheets are of the same material, the experimental results of Haringx on short rubber rods was investigated with two values of $c/t$ of 1 and 10, corresponding to very thick and thin face sheets, respectively. Fig. 9 shows the experimental results of Haringx as well as predictions made with Haringx’s buckling formula. These results are important for the design of elastomeric bearings where Haringx’s buckling formula is commonly used. Also shown in Fig. 9 are the predictions of Allen’s formula for thick faces Eqn (3) and numerical solutions of Eqn (54) which incorporates shear deformations of the core and face sheets and the initial axial deformation prior to buckling. The numerical solution provides reasonable predictions for both cases of core to face sheet ratios while Allen’s formula provides a reasonable comparison only when the face sheets are extremely thick, $c/t = 1$.

5. Summary

Bending and shear displacements have been derived for a sandwich column assuming a layer-wise Timoshenko beam zigzag displacement approximation through the depth of the column cross-section. The constitutive relationships for the core and the face sheets were derived using a consistent hyperelastic neo-Hookean formulation. The internal actions in the face
Table 1
Experimental results of Fleck and Sridhar (2002)

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<th>t (mm)</th>
<th>b (mm)</th>
<th>Ec (MPa)</th>
<th>Gc (MPa)</th>
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Table 2
Experimental results of Hoff and Mautner (1948)

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sheets and core were consistent with that of Haringx’s and Reissner’s proposal for beam actions, with the axial force taken perpendicular to either the core or face sheet cross-section while the shear taken parallel to the core or face sheet cross-sectional plane. The buckling formulation included both the axial deformation prior to buckling and the transverse shear deformations of both the core and face sheets. The formulation was comparable to the equation of Allen (1969) for thick faces. The formulation was also applicable to any ratio of face sheet to core thickness and material properties. There are very few sets of experimental data on the buckling of sandwich columns with isotropic face sheets. The experimental data provided in Fleck and Sridhar (2002) and Hoff and Mautner (1948) for sandwich columns with relatively thin face sheets and weak cores was used to compare the proposed formulation and showed very good comparison except for very short columns. The proposed formulation was also compared to the buckling experimental results for short rubber rods and also compared well. The formulas in this paper, Eqs. (59)–(61), however, have importantly been derived in a manner consistent with those of Haringx. Although Haringx’s column buckling formula cannot be used for sandwich columns as it does not take account of the shearing between the core and face sheets, Haringx’s approach as opposed to Engesser’s is still valid. The developed formulation in this paper is limited to macro Euler type buckling. For very low column slenderness the derived formulation did not predict a shear buckling mode of failure in sandwich columns.

Appendix A. Two dimensional hyperelastic mechanics

Consider a two-dimensional plane continuum with an initial Cartesian coordinate system. A rectangular element which after deformation becomes a parallelogram is shown in Fig. 10 and has base vectors in the deformed state given by

\[
\begin{align*}
\hat{g}_1 &= \lambda_1 (\cos \beta \hat{e}_1 + \sin \beta \hat{e}_2), \\
\hat{g}_2 &= \lambda_2 (\sin \alpha \hat{e}_1 + \cos \alpha \hat{e}_2),
\end{align*}
\]

\[
\begin{align*}
\gamma^1 &= \frac{\cos \alpha \hat{e}_1 - \sin \alpha \hat{e}_2}{\lambda_1 \cos \varphi}, \\
\gamma^2 &= \frac{-\sin \beta \hat{e}_1 + \cos \beta \hat{e}_2}{\lambda_2 \cos \varphi},
\end{align*}
\]

\[
\varphi = \alpha + \beta
\]

\[
\begin{align*}
\hat{g}_1 &= e_1, \\
\hat{g}_2 &= e_2
\end{align*}
\]

Fig. 10. Two dimensional deformed parallelogram with initial Cartesian coordinates.
in which $\mathbf{g}_1$ and $\mathbf{g}_2$ are the covariant and contravariant tangent base vectors in the deformed state, respectively, $\mathbf{g}_1$ and $\mathbf{g}_2$ are the covariant and contravariant initial base vectors in the undeformed state, respectively, angles $\lambda$ and $\phi$ are defined in Fig. 10. $\lambda_1$ and $\lambda_2$ are relative stretches ($\lambda_i = \sqrt{g_{ii}}$) and, $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors in the directions 1 and 2, respectively. The angle $\phi$ is the shear angle. The normal and shear components of the relative stretches are therefore (refer to Fig. 11):

$$
\begin{align*}
\lambda_1 &= \lambda_1 \cos \phi, \\
\lambda_2 &= \lambda_1 \sin \phi, \\
\lambda_{a1} &= \lambda_1 \cos \phi, \\
\lambda_{a2} &= \lambda_1 \sin \phi
\end{align*}
$$

The relationships between the stretches and angles $\lambda$ and $\phi$ with the displacement gradients can be determined and are

$$
\begin{align*}
\lambda_1 \sin (\phi - \lambda) &= u_{21}, \\
\lambda_1 \cos (\phi - \lambda) &= 1 + u_{11} \\
\lambda_2 \sin \lambda &= u_{12}, \\
\lambda_2 \cos \lambda &= 1 + u_{22}
\end{align*}
$$

where

$$
\begin{align*}
\lambda_1 &= \sqrt{(1 + u_{11})^2 + u_{21}^2}, \\
\lambda_2 &= \sqrt{(1 + u_{22})^2 + u_{12}^2} \\
\tan (\phi - \lambda) &= \frac{u_{21}}{1 + u_{11}}, \\
\tan \lambda &= \frac{u_{12}}{1 + u_{22}}
\end{align*}
$$

Manipulating Eqs. (A3) we can write for the normal and shear components of stretch:

$$
\begin{align*}
\lambda_1 \cos \phi &= (1 + u_{11}) \cos \lambda - u_{21} \sin \lambda \\
\lambda_1 \sin \phi &= (1 + u_{11}) \sin \lambda + u_{21} \cos \lambda \\
\lambda_2 \cos \phi &= (1 + u_{22}) \cos (\phi - \lambda) - u_{12} \sin (\phi - \lambda) \\
\lambda_2 \sin \phi &= (1 + u_{22}) \sin (\phi - \lambda) + u_{12} \cos (\phi - \lambda)
\end{align*}
$$

The deformation of the material can be characterized by the deformation gradient tensor $\mathbf{F}$ which defines a linear mapping of the initial line differential $\mathrm{ds}$ in the undeformed state to that in the deformed state $\mathrm{ds}$ (points $P$ and $Q$ in Fig. 10) associated with a displacement vector $\mathbf{u}$ (assumed to be smooth and differentiable), such that

$$
\mathrm{ds} = \mathbf{F} \cdot \mathrm{ds}
$$

in which, $\mathbf{g}_i = (\delta_i^j + u_i^j)\mathbf{g}_j$ are the covariant tangent base vectors in the deformed state (see Fig. 10), $\delta_i^j$ is the kronecker delta, $\mathbf{I} = \mathbf{g}_i \otimes \mathbf{g}_i$ is the identity tensor, $\nabla \otimes \mathbf{u} = u_i^j \mathbf{g}_j \otimes \mathbf{g}_i$ is the grad of the displacement vector, and $u_i^j$ represents the covariant derivatives of the $u_i$ vector component with respect to the coordinate corresponding to the index $i$. Using Eqs. (A1) and (A6), the components of the associated deformation gradient tensor are therefore

$$
\mathbf{F} = \begin{bmatrix} 
\lambda_1 \cos (\phi - \lambda) & \lambda_2 \sin \lambda & 0 \\
\lambda_1 \sin (\phi - \lambda) & \lambda_2 \cos \lambda & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

With the volume invariant $J = \det \mathbf{F}$ given by

$$
J = \lambda_1 \lambda_2 \cos \phi
$$
The right Cauchy–Green deformation tensor $$\mathbf{C} = \mathbf{F}^T \mathbf{F}$$ for the essentially two dimensional deformation defined in Eq. (A7), is therefore

$$\mathbf{C} = \begin{bmatrix} (\lambda_1)^2 & \lambda_1 \lambda_2 \sin \varphi & 0 \\ \lambda_1 \lambda_2 \sin \varphi & \lambda_2^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$  \hspace{1cm} (A9)$$

The strain energy density function $$U$$ for a compressible isotropic neo-Hookean material (see Attard and Hunt, 2004) is given by

$$U = \frac{1}{2} \mathcal{E} (\mathbf{C} - \mathbf{I}) - 2 \ln J + \frac{1}{2} \mathcal{A} (\ln J)^2$$  \hspace{1cm} (A10)$$

where $$\mathcal{E} = \frac{E}{2(1+\nu)}$$ is the shear modulus, $$\mathcal{A} = \frac{E \nu}{(1+\nu)(1-2\nu)}$$ is the Lamé constant, $$E$$ is the elastic modulus, $$\nu$$ is the Poisson’s ratio and $$\ln$$ symbolize the trace of a tensor. The constitutive relationship for a hyperelastic material can be established for the second Piola Kirchhoff stress tensor $$\mathbf{P} = \mathbf{P}^0 \mathbf{g} \otimes \mathbf{g}$$ by (see Attard and Hunt, 2004)

$$\mathbf{P} = 2 \frac{\partial U}{\partial \mathbf{C}} = \mathbf{G} - p_h \mathbf{C}^{-1}$$  \hspace{1cm} (A11)$$

With

$$p_h = G - A \ln J$$  \hspace{1cm} (A12)$$

In the above, $$p_h$$ represents a hydrostatic stress. For an initial rectangular coordinate system in the undeformed state, the physical counterpart of the second Piola–Kirchhoff stress tensor, the Lagrangian stress (engineering stress) $$\mathbf{S}^0$$, is given by

$$\mathbf{S}^0 = \mathbf{P}^0 \lambda_{ij}$$  \hspace{1cm} (A13)$$

Using Eqs. (A9) and (A10) the compliant second Piola–Kirchhoff stress tensor for an essentially two dimensional deformation is

$$\mathbf{P} = \begin{bmatrix} G - \frac{p_h}{\lambda_1 \cos \varphi} & \frac{p_h \tan \varphi}{\lambda_1 \cos \varphi} & 0 \\ \frac{p_h \tan \varphi}{\lambda_1 \cos \varphi} & G - \frac{p_h}{\lambda_2 \cos \varphi} & 0 \\ 0 & 0 & G - p_h \end{bmatrix}$$  \hspace{1cm} (A14)$$

where

$$p_h = G - A \ln(\lambda_1 \lambda_2 \cos \varphi)$$  \hspace{1cm} (A15)$$

Stress tensors have stress components on any of the surfaces of the deformed elemental parallelogram/parallelepiped which are aligned with a fixed axis system defined either in the initial or deformed state. Alternatively, we can deal with physical stress components which have different orientations on each of the faces/surfaces of the deformed elemental parallelogram/parallelepiped. These stresses are no longer second order tensors but still have vectorial properties. Reissner developed a planar model for beam actions in which the axial force was defined as perpendicular to the cross-sectional plane and the shear force within the cross-sectional plane. Here we extend this orientation for the description of stresses. The chosen physical Lagrangian stress system has normal components which are normal to the surfaces on which they act and the shears tangential to the surface on which they act. These stresses will be called Reissner stresses and denoted by $$\mathbf{S}^R$$ (see Fig. 12). The Reissner stress components form an orthogonal system on each of the surfaces on which we act. We will now derive the internal virtual work equation for this stress system and obtain the stress deformation constitutive laws consistent with the assumed strain energy density in Eq. (A10). Firstly, by considering equilibrium on the surfaces of the deformed elemental parallelogram/parallelepiped (see Fig. 10), we can derive the following relationships:

$$\begin{align*}
\mathbf{S}^R_{11} &= P_{11} \lambda_1 \sin \varphi + P_{12} \lambda_2, \\
\mathbf{S}^R_{12} &= P_{21} \lambda_1 \cos \varphi, \\
\mathbf{S}^R_{22} &= P_{22} \lambda_2 \cos \varphi \\
\mathbf{S}^R_{11} \lambda_1 \sin \varphi - \mathbf{S}^R_{12} \lambda_2 \cos \varphi &= \mathbf{S}^R_{22} \lambda_1 \cos \varphi - \mathbf{S}^R_{22} \lambda_2 \sin \varphi
\end{align*}$$  \hspace{1cm} (A16)$$

where the terms $$M_{12}$$ & $$M_{21}$$ are moment stresses about the centre of the deformed parallelogram depicted in Fig. 11. The constitutive laws for the Reissner stresses can be derived using Eqs. (A14) and (A16) and are

$$\begin{align*}
\mathbf{S}^R_{11} &= G \lambda_1 \cos \varphi - \frac{p_h}{\lambda_1 \cos \varphi}, \\
\mathbf{S}^R_{12} &= G \lambda_1 \sin \varphi \\
\mathbf{S}^R_{22} &= G \lambda_2 \cos \varphi - \frac{p_h}{\lambda_2 \cos \varphi}, \\
\mathbf{S}^R_{21} &= G \lambda_2 \sin \varphi
\end{align*}$$  \hspace{1cm} (A17)$$
For the case when \( J \lambda_1 \cos \varphi \) and \( \lambda_2 \cos \varphi \) are all close to unity (small strain), we have
\[
\begin{align*}
S_{11}^1 &= (2G + A)(\lambda_1 \cos \varphi - 1) = (2G + A)(\lambda_{11} - 1) \\
S_{12}^1 &= G \lambda_1 \sin \varphi = G \lambda_{11} \\
S_{22}^1 &= (2G + A)(\lambda_2 \cos \varphi - 1) = (2G + A)(\lambda_{22} - 1) \\
S_{21}^1 &= G \lambda_2 \sin \varphi = G \lambda_{22}
\end{align*}
\] (A18)

For kinematically admissible variations denoted by the symbol \( \delta \), the Lagrangian first variation of work \( \delta W \) based on virtual displacements can be written as
\[
\delta W = \int \int \int V \delta UdV - \int \int \int S \cdot \delta \mathbf{u} dS = \int \int \int \frac{1}{2} \text{tr}(\Pi \delta \mathbf{C}) dV - \int \int S \cdot \delta \mathbf{u} dS = 0
\] (A19)

With \( V \) being the volume in the undeformed state, \( S \) the surface where the externally applied traction vector \( \mathbf{p} \) acts. The variation of the right Cauchy–Green deformation tensor is taken from Eq. (A9). The first variation of work is then
\[
\delta W = \int \int \int \frac{1}{2} \text{tr}(\Pi \delta \mathbf{C}) dV - \int \int S \cdot \delta \mathbf{u} dS =
\int \int \left( (\Pi_{11}^1 \lambda_1 + \Pi_{12}^1 \lambda_2 \sin \varphi) \delta \lambda_1 + (\Pi_{22}^1 \lambda_2 + \Pi_{21}^1 \lambda_1 \sin \varphi) \delta \lambda_2 \right)
+ \frac{1}{2} \lambda_1 \lambda_2 \cos \varphi \Pi_{12}^1 + \Pi_{21}^1 \delta \varphi dV - \int \int S \cdot \delta \mathbf{u} dS = 0
\] (A20)

Substituting the relationships in Eq. (A16), the variation in work can be written in terms of the Reissner stresses as
\[
\delta W = \int \int \left[ S_{11}^2 \delta (\lambda_1 \cos \varphi) + S_{22}^2 \delta (\lambda_2 \cos \varphi) + \frac{1}{2} (M_{12}^2 + M_{21}^2) \delta \varphi \right] dV - \int \int S \cdot \delta \mathbf{u} dS
\] (A21)

For the case when \( \lambda_2 = 1 \), we have
\[
\delta W = \int \int \left[ S_{11}^2 \delta (\lambda_1 \cos \varphi) + S_{22}^2 \delta (\lambda_1 \sin \varphi) \right] dV - \int \int S \cdot \delta \mathbf{u} dS
\] (A22)

**Appendix B. Second variation of work**

The second variation of work for the sandwich column can be derived from Eq. (25), thus
\[
\delta^2 W = \frac{1}{2} \int \left[ \begin{aligned}
&\frac{E A_0}{2} (\delta \mathbf{u}_x)_x^2 + E I_0 (\delta \theta_{P})_x^2 + r^2_2 E A_0 (\delta \phi_x - \delta \phi_s)_x^2 \\
&- \frac{2r_2^2 E A_0 (\delta \phi_x - \delta \phi_s)_x (\delta \theta_{P})_x - P(1 - P)(\delta \theta_{P})_x + 2(\delta \phi_s)_x \delta \theta_{P}} + \{2G A_t(1 - P) + N_s(1 - P)(\delta \phi_s)_x \}
\end{aligned} \right] dV - \int \int S \cdot \delta^2 \mathbf{u} dS
\] (B1)

For the case when both the shear deformation in the face sheet and the axial deformation prior to buckling are ignored, the second variation of work simplifies to
\[ d^2W = \frac{1}{2} \int_0^L \left[ \frac{EA_{tot}(\partial u_x)^2}{1 + \gamma_1^2} + \frac{EL_{tot}(\partial \theta_x)^2}{1 + \gamma_1^2} + \frac{r_{12}^2EA_{tot}(\partial \varphi_{xx})^2}{1 + \gamma_1^2} - 2r_{12}^2EA_{tot}(\partial \varphi_{xx} \partial \theta_x - P(\partial \theta_x)^2 + (GcA_c - N_c)(\partial \varphi_{xx})^2 \right] dx - \int_0^L \mathbf{p} \cdot d^2u ds \] \quad \text{(B2)}

### Appendix C. Column boundary conditions

Substituting the boundary conditions Eq. (56) into Eqs. (48)-(50) and using Eq. (13) to determine the vertical deflection

\[ \delta v = (1 - P) \left[ \int_0^L \left( \partial \theta_x + \delta \varphi_x \right) dx + C_8 \right] \] \quad \text{(B3)}

gives for the following set of equations for the undetermined constants:

- \[ at \ x = 0 : \delta v = 0, \quad -C_3 x_2 (1 + \gamma_1) + C_5 x_1 (1 + \gamma_2) + C_8 = 0 \] \quad \text{(B4)}
- \[ at \ x = 0 : \partial \theta_x = 0, \quad C_2 + C_4 + C_1 = 0 \] \quad \text{(B5)}
- \[ at \ x = 0 : \partial \varphi_x = 0, \quad \gamma_1 C_2 + \gamma_2 C_4 + C_7 = 0 \] \quad \text{(B6)}
- \[ at \ x = 0 : \delta \varphi_x = 0, \quad C_2 \beta_1 + C_4 \beta_2 = 0 \] \quad \text{(B7)}
- \[ at \ x = L : \delta v = 0, \quad C_1 L + \left( \frac{1 + \gamma_1}{x_1} \right) (C_2 \sin(x_1 L) - C_3 \cos(x_1 L)) \] \quad \text{(B8)}
- \[ + \left( \frac{1 + \gamma_1}{x_1} \right) (C_4 \sinh(x_2 L) + C_6 \cosh(x_2 L)) + C_7 L + C_8 \]
- \[ at \ x = L : \partial \theta_x = 0, \partial \varphi_x = 0, \quad \delta \varphi_x = 0, \quad \gamma_1 C_2 \sin(x_1 L) + C_3 \sin(x_1 L) \] \quad \text{(B9)}
- \[ + (1 + \gamma_2) (C_4 \cosh(x_2 L) + C_5 \sinh(x_2 L)) \]
- \[ at \ x = L : \delta \theta_x = 0, \delta \varphi_x = 0 \]
- \[ \beta_1 (C_2 \cos(x_1 L) + C_3 \sin(x_1 L)) + \beta_2 (C_4 \cosh(x_2 L) + C_5 \sinh(x_2 L)) \] \quad \text{(B10)}

### References