Quadratic algebras of skew type and the underlying monoids

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Abstract

We consider algebras over a field $K$ defined by a presentation $K\langle x_1, \ldots, x_n \mid R \rangle$, where $R$ consists of $\binom{n}{2}$ square-free relations of the form $x_i x_j = x_k x_l$ with every monomial $x_i x_j$, $i \neq j$, appearing in one of the relations. Certain sufficient conditions for the algebra to be noetherian and PI are determined. For this, we prove more generally that right noetherian algebras of finite Gelfand–Kirillov dimension defined by homogeneous semigroup relations satisfy a polynomial identity. The structure of the underlying monoid, defined by the same presentation, is described. This is used to derive information on the prime radical and minimal prime ideals. Some examples are described in detail. Earlier, Gateva-Ivanova and van den Bergh, and Jespers and Okniński considered special classes of such algebras in the contexts of noetherian algebras, Gröbner bases, finitely generated solvable groups, semigroup algebras, and set theoretic solutions of the Yang–Baxter equation.

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1. Introduction

We consider finitely generated monoids with a monoid presentation of the form

\[ S = \langle x_1, x_2, \ldots, x_n \mid x_ix_j = x_kx_l \rangle \]

with \( \binom{n}{2} \) relations, where \( i \neq j, k \neq l \), and every product \( x_px_q \) with \( p \neq q \) appears in one of the relations. So each \( x_px_q \) appears in exactly one relation. We call such an \( S \) a semigroup of skew type. Special classes of monoids of this type, and algebras defined by the same presentations, arise in a natural way from the study of set-theoretic solutions of the Yang–Baxter equation and independently from certain problems in the theory of associative algebras [2,5,7,11]. These algebras turn out to have very nice properties. In particular, they are noetherian domains of finite global dimension, satisfy a polynomial identity, are Koszul, Auslander–Gorenstein, and Cohen–Macaulay [7]. Reasons and tools for dealing with these properties came from the study of homological properties of Sklyanin algebras by Tate and Van den Bergh [20] and from the work of Gateva-Ivanova on so-called skew-polynomial rings with binomial relations [5].

The above mentioned special classes of semigroups surprisingly define submonoids of torsion-free abelian-by-finite groups. Under the additional assumption that \( i > j, k < l \), \( i > k, j < l \) for each of the defining relations \( x_ix_j = x_kx_l \), every element of \( S \) can be written uniquely in the form \( x_1^{k_1} \cdots x_n^{k_n} \) for some non-negative integers \( k_i \). In particular, the Gelfand–Kirillov dimension of \( K[S] \) [14,15], denoted by \( \text{GK}(K[S]) \), is equal to \( n \). We note that some other (but related) types of algebras defined by quadratic relations have been investigated, see, for example, [5,13].

Our aim is to study the noetherian property of algebras \( K[S] \) of skew type, its relation to the growth and the PI-property, and the role of the minimal prime ideals with respect to the least cancellative congruence on \( S \). This is motivated by the results on algebras of binomial semigroups, where the height one primes turned out to be crucial for the properties of the algebra [11].

Our main result asserts that \( K[S] \) is a noetherian PI algebra for a wide class of semigroups of skew type. A combinatorial approach allows us to derive a rich structural information on \( S \). This is of independent interest and becomes the main tool in the proof. As an intermediate step, we prove the following general result. Suppose \( A \) is a unitary \( K \)-algebra defined via a presentation \( K\langle x_1, \ldots, x_n \mid R \rangle \), where \( R \) consists of relations of the type \( u = v \) with \( u \) and \( v \) words of equal length in the generators. If \( A \) is right noetherian and of finite Gelfand–Kirillov dimension, then \( A \) satisfies a polynomial identity.

2. Cyclic condition

We start with a combinatorial condition that allows us to build several examples of noetherian PI-algebras \( K[S] \). If \( S \) is a monoid and \( Z \subseteq S \), then we denote by \( (Z) \) the submonoid generated by \( Z \).

We say that a monoid \( S \) generated by a finite set \( X \) satisfies the cyclic condition (C) if for every pair \( x, y \in X \) there exist elements \( x = x_1, x_2, \ldots, x_s, y' \in X \) such that
If $S$ is a monoid of skew type, then it follows that for every $x, y \in X$, $x \neq y$, there exist $x', y' \in X$ such that $yx' = xy'$. Since for a given $y \in X$ there are $|X| - 1$ words of the form $yz$, where $z \in X$ and $z \neq y$, and because every such word is in exactly one defining relation, it follows that there are no relations of the type $yx = yz$ for $x, z \in X$, $x \neq z$.

**Lemma 2.1.** Let $S$ be a monoid of skew type. Assume that, if $yx = x_2y'$ is one of the defining relations in $S$ then there also is a relation of the form $y_2x = x_3y'$ for some generator $x_3$. Then $S$ satisfies the cyclic condition and elements $x_2, x_3, \ldots, x_k$ in condition (C) can be chosen distinct.

**Proof.** We know that $y \neq x_2$. Put $x_1 = x$. Applying the assumption several times, we get

$$yx = x_2y', \quad yx_2 = x_3y', \quad \ldots, \quad yx_k = x_{k+1}y'. $$

where $x_i \neq y$ for every $i$ and $x_{k+1} = x_r$ for some $r < k$. Let $r$ be a minimal integer with this property. If $r \geq 2$ then $y_{x_{r-1}} = x_r y'$ and $y_{x_k} = x_{k+1} y'$ imply that $k = r - 1$, contradicting the minimality of $r$. Hence, $r = 1$ and we get $x = x_{k+1}$, as desired. The last assertion is an immediate consequence of the proof. $\square$

It is shown in [5] (see also [11]) that binomial monoids satisfy the cyclic condition. More generally, monoids of skew type that provide set theoretical solutions of the quantum Yang–Baxter equation and that are non-degenerate (in the sense of Section 4) also satisfy the cyclic condition [6]. We show that the cyclic condition is symmetric.

**Proposition 2.2.** Let $S = (X; R)$ be a semigroup of skew type. Assume $S$ satisfies the cyclic condition. Then the full cyclic condition (FC) holds in $S$, that is, for any pair $x, y \in X$, there exist two sequences $x = x_1, x_2, \ldots, x_k$ and $y = y_1, y_2, \ldots, y_p$ in $X$ such that

$$y_1x_1 = x_2y_2, \quad y_1x_2 = x_3y_2, \quad \ldots, \quad y_1x_k = x_1y_2,$$

$$y_2x_1 = x_2y_3, \quad y_2x_2 = x_3y_3, \quad \ldots, \quad y_2x_k = x_1y_3,$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_px_1 = x_2y_1, \quad y_px_2 = x_3y_1, \quad \ldots, \quad y_px_k = x_1y_1.$$

We call this a cycle of type $k \times p$.

**Lemma 2.3.** Under the hypothesis of Proposition 2.2, let $ax_1 = x_2b$ for some $a, b, x_1, x_2 \in X$. Then

(1) there exist $c, x_3, x_0 \in X$ such that: (a) $ax_2 = x_3b$, (b) $ax_0 = x_1b$, and (c) $cx_1 = x_2a$;

(2) if (i) $ax_1 = x_2b$, (ii) $ax_2 = x_3b$, and (iii) $cx_1 = x_2a$, then $cx_2 = x_3a$. 


Proof. (1a) and (1b) follow immediately from condition (C). Indeed, (C) applied to $ax_1 = x_2b$ implies $a* = x_1b$ and $ax_2 = x_2b$, with “∗” meaning an element of $X$. In general, the letter * is different in the first and the second equality. For (c) consider $x_2b = ax_1$.

Applying (a) we get $x_2a = cx_1$ for some $c \in X$.

(2) Assume (i), (ii), and (iii) hold. Applying (1c) to $ax_2 = x_3b$ yields $t \in X$ such that $tx_2 = x_3a$. But then, applying (1b), we get $ts = x_2a$ for some $s \in X$. Since $S$ is of skew type, comparing the latter with $cx_1 = x_3a$, we obtain $t = c$. Hence, $cx_2 = x_3a$. □

Now the statement of the proposition can be derived from the lemma as follows.

Let $x, y \in X$. From (C) it follows that the sequence for the “internal cycle” $x = x_1, \ldots, x_k$ exist, so that $yx_1 = x_{i+1}z$ and $yx_k = x_1z$, for some $z \in X$ and all $i = 1, \ldots, k-1$.

By (1c) there exists $y^{(1)} \in X$, such that

$$y^{(1)}x_1 = x_2y.$$  

Hence, $yx_1 = x_2z$, $yx_2 = x_3z$, and $y^{(1)}x_1 = x_2y$ (if $k = 1$ then we put $x_2 = x_3 = x_1$ and if $k = 2$ then we put $x_3 = x_1$). So because of (2), we get $y^{(1)}x_2 = x_3y$. It follows by an induction procedure, that $y^{(1)}$ is compatible with the whole cycle $x_1, \ldots, x_k$, that is

$$y^{(1)}x_1 = x_2y, \quad y^{(1)}x_2 = x_3y, \quad \ldots, \quad y^{(1)}x_k = x_1y.$$  

Applying the same procedure to (iv), we obtain a $y^{(2)} \in X$ such that

$$y^{(2)}x_1 = x_2y^{(1)}, \quad y^{(2)}x_2 = x_3y^{(1)}, \quad \ldots, \quad y^{(2)}x_k = x_1y^{(1)}.$$  

Condition (C) applied to $x_2y$ implies that, after finitely many such steps, we shall close the cycle for y’s, that is, we obtain a sequence of generators $y^{(1)}, \ldots, y^{(p-1)}$ such that $x_2y^{(i)} = y^{(i+1)}x_1$, for $i = 1, \ldots, p - 2$ and $x_2y^{(p-1)} = yx_1$. Since $yx_1 = x_2z$, we get $y^{(p-1)} = z$. Also

$$y^{(i+1)}x_1 = x_2y^{(i)}, \quad y^{(i+1)}x_2 = x_3y^{(i)}, \quad \ldots, \quad y^{(i+1)}x_k = x_1y^{(i)}$$  

for $i = 1, \ldots, p - 2$. The assertion follows by reindexing the elements $y^{(1)}, \ldots, y^{(p-1)}$. □

The following result allows to construct many examples of noetherian PI algebras from semigroups of skew type.

Proposition 2.4. Assume that $S = \{x_1, \ldots, x_n\}$ is a semigroup of skew type that satisfies the cyclic condition and $S = \{x_1^{a_1}, \ldots, x_n^{a_n} \mid a_i \geq 0\}$. Then $K[S]$ is a finite left and right module over a commutative subring of the form $K[A]$, where $A = \{x_1^p, \ldots, x_n^p\}$ for some $p \geq 1$.

Namely $S = \bigcup_{c \in C} cA$ with $C = \{x_1^{i_1}, \ldots, x_n^{i_n} \mid i_j < p\}$ and $cA = Ac$ for every $c \in C$. In particular, $K[S]$ is a right and left noetherian PI algebra.

Proof. Let $x, y_1 \in X = \{x_1, \ldots, x_n\}$. Then, for some $t, y_1, \ldots, y_s \in X$, we have

$$xy_1 = y_2t, \quad xy_2 = y_3t, \quad \ldots, \quad xy_s = y_1t.$$  

(1)
This easily implies that \( x^i y_i = y_i t^s \) for all \( i = 1, \ldots, s \). Hence, for every distinct \( x, y \in X \) there exists \( t \in X \) such that \( x^r y = y t^r \), where \( r \) is the least common multiple of the lengths of all cycles in \( S \). Since \( y \) acts as a bijection on \( X \) by mapping \( x \) to \( t \) if \( xy = y' t \) (this follows by the symmetry of the cyclic condition and by the comment preceding Lemma 2.1), we obtain that \( y \) also acts as a bijection on the set \( \{ x^1, \ldots, x^n \} \). Hence, for every distinct \( x, y \in X \) there exists \( t \in X \) such that \( x r y = y t r \), where \( r \) is the least common multiple of the lengths of all cycles in \( S \). Since \( y \) acts as a bijection on \( X \) by mapping \( x \) to \( t \) if \( xy = y' t \) (this follows by the symmetry of the cyclic condition and by the comment preceding Lemma 2.1), we obtain that \( y \) also acts as a bijection on the set \( \{ x^1, \ldots, x^n \} \). Hence, there is a multiple \( p \) of \( r \) such that \( x^p y_i = x^p y_i \) for all \( i, j \). Since \( S = \{ x_1^{a_1} \cdots x_n^{a_n} | a_i \geq 0 \} \), it now follows that \( S = CA \). Moreover, \( cA = Ac \) for \( c \in C \) because \( c \) acts as a bijection on the set of generators of \( A \). ✷

We note, that the previous proof still works if \( S \) is a semigroup of skew type that satisfies the cyclic condition and \( S \) is the union of sets of the form \( \{ y_1^{a_1} \cdots y_k^{a_k} | a_i \geq 0 \} \) where \( y_1, \ldots, y_k \in X \) and \( k \leq n \). In Theorem 4.5 we will prove that the latter is a consequence of the cyclic condition. Moreover (see Theorem 5.2), \( K[S] \) is still a noetherian PI algebra for a class of semigroups of skew type essentially wider than those satisfying the cyclic condition.

3. Noetherian implies PI

It is well known, that the Gelfand–Kirillov dimension of a finitely generated PI-algebra is finite (see [14]). One of our aims is to show that the converse holds for every algebra \( K[S] \) of a semigroup \( S \) of skew type, provided that \( K[S] \) is right noetherian. Surprisingly, the following theorem shows that this can be proved in the more general context of finitely generated monoids defined by homogeneous relations. Clearly, in such a semigroup we have a natural degree function given by \( s \mapsto |s| \), where \( |s| \) is the length of \( s \in S \) as a word in the generators of \( S \).

In the proof of the theorem we rely on the rich structure of linear semigroups [17].

**Theorem 3.1.** Let \( S \) be a monoid such that the algebra \( K[S] \) is right noetherian and \( \text{GK}(K[S]) < \infty \). Then \( S \) is finitely generated. If, moreover, \( S \) has a monoid presentation of the form

\[
S = \langle x_1, \ldots, x_n | R \rangle
\]

with \( R \) a set of homogeneous relations, then \( K[S] \) satisfies a polynomial identity.

**Proof.** The first assertion follows from [12, Theorem 2.2]. So assume \( S \) has a monoid presentation \( S = \langle x_1, \ldots, x_n | R \rangle \). Note that the unit group \( U(S) \) is trivial. Let \( T = S^0 \), the semigroup with zero \( \theta \) adjoined. We define a congruence \( \rho \) on \( T \) to be homogeneous if \( spt \) and \( (s, \theta) \notin \rho \) imply that \( |s| = |t| \).

The contracted semigroup algebra \( K_0[T] \) may be identified with \( K[S] \). Suppose that \( K_0[T] \) is not a PI algebra. Then, by the noetherian condition, there exists a maximal homogeneous congruence \( \eta \) on \( T \) such that \( K_0[T/\eta] \) is not PI. So, replacing \( T \) by \( T/\eta \), we may assume that every proper homogeneous homomorphic image of \( T \) yields a PI algebra.
Since there are only finitely many minimal prime ideals of \( K_0[T] \) and the prime radical \( B(K_0[T]) \) is nilpotent, there exists a minimal prime \( P \) such that \( K_0[T]/P \) is not a PI algebra. As \( K_0[T] \) can be considered in a natural way as a \( Z \)-graded algebra (with respect to the length function on \( S \)), it is well known [19], that \( P \) is a homogeneous ideal of \( K_0[T] \). Therefore, the congruence \( \rho_P \) determined by \( P \) is homogeneous. (Recall that \( \mu P \) if \( s - t \in P \), for \( s, t \in T \)). Since \( K_0[T]/P \) is a homomorphic image of \( K_0[T]/\rho_P \), and because of the preceding paragraph of the proof, we get that \( T = T/\rho_P \). As \( K_0[T] \) is right noetherian, we thus get

\[
T \subseteq K_0[T]/P \subseteq M_t(D)
\]

for some division algebra \( D \), where \( M_t(D) = Q_d(K_0[T]/P) \), the classical ring of quotients of \( K_0[T]/P \). Let \( I \) be the set of all elements of \( T \) (with \( \theta \)) that are of minimal nonzero rank as matrices in \( M_t(D) \). Consider \( K(I) \), the subalgebra of \( K_0[T]/P \) generated by \( I \). Clearly, \( K(I) \) is an ideal of \( K_0[T]/P \). Then \( M_t(D) = Q_d(K(I)) \). So \( K(I) \) is not a PI algebra, as otherwise its ring of quotients would also satisfy a polynomial identity.

Since not all elements of \( I \) can be nilpotent, it follows from the theory of linear semigroups that \( I \) has a nonempty intersection \( C \) with a maximal subgroup \( G \) of the multiplicative monoid \( M_t(D) \). So \( G \) is the group of units of the monoid \( eM_t(D)e \) for some \( e = e^2 \) in \( M_t(D) \). Let \( F \subseteq G \) be the group generated by \( C \). Define

\[
Z = \{ ex \mid x \in T, \ Cx \subseteq C \}.
\]

If \( g = ex \in G \cap eT \) then \( Cx = Cex \subseteq G \cap T = C \). Hence \( g \in Z \) and \( G \cap eT \subseteq Z \). It is easy to see that \( Z \subseteq G \), so that \( Z = G \cap eT \). We claim that the monoid \( Z \) satisfies the ascending chain condition on right ideals. Fix some \( c \in C \). Let \( J \) be a right ideal of \( Z \). Notice that \( cJ \) is a right ideal of \( T \). Then \( cJ \cap Z = cJeT \cap Z = cJZ = cJ \) because \( cJ \) is a monoid with the ascending chain condition on right ideals, the claim follows.

One verifies that \( F \) is a finitely generated group. This follows from [17, Proposition 3.16] (the result is proved for a field \( D \) only, but the proof works also for division rings \( D \)). Since \( GK(K_0[T]) < \infty \), we also have \( GK(C) < \infty \). It is then known that \( F \) also has finite Gelfand–Kirillov dimension [8]. Moreover, as \( F \) is finitely generated, it follows from [9] that \( F \) is nilpotent-by-finite.

Next we claim that the group of units \( U(Z) \) of \( Z \) is a periodic group. For this, suppose \( g, g^{-1} \in Z \). Then \( Cg \subseteq C \) and \( Cg^{-1} \subseteq C \). So \( Cg = C \). Write \( g = ab^{-1} \) with \( a, b \in C \). Then \( Ca = Cb \) and so \( Ma = Mb \), where \( M \) is the subset consisting of the elements of minimal length in \( C \). Clearly, \( Mg = M \). As \( M \) is finite, we get \( g^k = e \) for some \( k \geq 1 \), which proves the claim.

So \( U(Z) \) is a periodic subgroup of the finitely generated nilpotent-by-finite group \( F \). Hence \( U(Z) \) is finite. Since also \( Z \) satisfies the ascending chain condition on right ideals, it follows from the remark in [12, p. 550] that \( F \) is finite-by-abelian-by-finite. Hence, \( F \) is abelian-by-finite and thus \( K[F] \) is a PI algebra.

Finally, as \( T \) satisfies the ascending chain condition on right ideals, \( I \) intersects finitely many \( R \)-classes of the monoid \( M_t(D) \). It is then known that \( I \) embeds into a completely
0-simple semigroup with finitely many \( R \)-classes and with a maximal subgroup \( F \).

It follows that \( K \langle I \rangle \) is a PI algebra, see [16, Proposition 20.6], a contradiction. This completes the proof of the theorem. \( \square \)

The proof shows that the theorem remains valid in the more general situation of an algebra generated by a finite set of elements \( x_1, \ldots, x_n \) subject to a system \( R \) of relations of the form \( u = v \), where \( u \) and \( v \) are words of the same length in the generators \( x_1, \ldots, x_n \), or \( u = 0 \) where \( u \) is a word in \( x_1, \ldots, x_n \).

**Corollary 3.2.** Let \( S \) be a semigroup of skew type such that \( \text{GK}(K[S]) < \infty \). If \( K[S] \) is right noetherian, then it satisfies a polynomial identity. In particular, \( K[S] \) embeds into a matrix ring over a field and \( \text{GK}(K[S]) = \text{GK}(K[S]/B(K[S])) \) is an integer, where \( B(K[S]) \) is the prime radical of \( K[S] \). Moreover, \( S \) satisfies a semigroup identity.

**Proof.** \( K[S] \) is a PI algebra by Theorem 3.1. Hence, [1] implies that \( K[S] \) is a sub-algebra of \( M_t(L) \) for a field \( L \) and \( t \geq 1 \). Then, by a result of Markov \( \text{GK}(K[S]) = \text{GK}(K[S]/B(K[S])) \) is an integer, see [14, Section 12.10]. The last assertion now follows from [17, Proposition 7.10]. \( \square \)

In Sections 4 and 5 we will show that for a wide class semigroups \( S \) of skew type \( \text{GK}(K[S]) < \infty \) and \( K[S] \) is right and left noetherian. So the corollary is applicable in this situation.

**4. Non-degeneracy and the ascending chain condition**

Assume \( S = \langle x_1, \ldots, x_n \rangle \) is a semigroup of skew type that satisfies the cyclic condition. If \( x \in \{ x_1, \ldots, x_n \} \), then for every \( y_1 \in \{ x_1, \ldots, x_n \} \) we get a cycle

\[
xy_1 = y_2t, \quad xy_2 = y_3t, \quad \ldots, \quad xy_s = y_1t
\]

with \( t, y_i \in \{ x_1, \ldots, x_n \} \). As noticed before, for every \( x_k \neq x \) there exists a relation of the form \( xx_i = x_kx_l \) for some \( i, l \).

A semigroup of skew type satisfying the latter condition will be said to be right non-degenerate. Left non-degenerate semigroups are defined dually. However, these two conditions are not equivalent, see Example 7.1.

A symmetric argument shows that the cyclic condition implies that \( S \) is left non-degenerate as well. Notice that if \( S \) is right non-degenerate then every \( x \in \{ x_1, \ldots, x_n \} \) defines a bijection \( f_x \) of \( \{ x_1, \ldots, x_n \} \) as follows: if \( xx_i = x_kx_l \) then \( f_x(x_i) = x_k \).

There are many examples of right and left non-degenerate \( S \) which do not satisfy the cyclic condition. For example, \( S = \langle x_1, x_2, x_3, x_4 \rangle \) defined by the relations: \( x_2x_1 = x_1x_3, x_3x_1 = x_2x_4, x_4x_1 = x_1x_2, x_3x_2 = x_1x_4, x_4x_2 = x_2x_3, x_4x_3 = x_3x_4 \).

First we prove some technical and combinatorial properties of non-degenerate semigroups.
Let $S$ be a semigroup of skew type. Let $Y = \langle X \rangle$, $X = \{x_1, \ldots, x_n\}$, be a free monoid of rank $n$. (So, we use the same notation for the generators of $Y$ and of $S$, if unambiguous.) For any $m \geq 2$ and any $y_1, \ldots, y_m \in X$ define

$$g_i(y_1 \cdots y_m) = y_1 \cdots y_i y_{i+1} y_{i+2} \cdots y_m$$

for $i = 1, \ldots, m - 1$, where

$$y_i y_{i+1} = y_{i+1} y_i$$

is one of the defining relations of $S$ (if $y_i \neq y_{i+1}$ or $y_i = y_{i+1} = y_i = y_{i+1}$). Let

$$g(y_1 \cdots y_m) = g_{m-1} \cdots g_2 g_1(y_1 \cdots y_m).$$

(Notice that $g$ is used for all $m = 2, 3, \ldots$.) If $g(y_1 \cdots y_m) = s_1 \cdots s_m$, $s_i \in X$, then we set

$$f_{s_1}(y_2 \cdots y_m) = s_1 \cdots s_{m-1}.$$

So $f_{s_1} : X^{m-1} \to X^{m-1}$ can be considered as a function on the subset $X^{m-1}$ of $Y$ consisting of all words of length $m - 1$.

**Lemma 4.1.** Assume that $S$ is a right non-degenerate semigroup of skew type. If $y_1 \in X$, then $f_{y_1} : X^{m-1} \to X^{m-1}$ is a one-to-one mapping, for any $m \geq 2$.

**Proof.** We proceed by induction on $m$. The case $m = 2$ is clear because $S$ is right non-degenerate.

Assume now that $m > 2$. Let

$$g(y_1 \cdots y_m) = s_1 \cdots s_m.$$

We will show that $s_1 \cdots s_{m-1}$ and $y_1$ determine $y_2 \cdots y_m$. Notice that $g_1(y_1 \cdots y_m) = s_1 h(y_1 y_2) y_3 \cdots y_m$ where

$$y_1 y_2 = s_1 h(y_1 y_2)$$

is a relation in $S$ or $y_1 = y_2 = s_1 = h(y_1 y_2)$. Moreover,

$$s_1 g(h(y_1 y_2) y_3 \cdots y_m) = g(y_1 \cdots y_m) = s_1 \cdots s_m.$$

Then $g(h(y_1 y_2) y_3 \cdots y_m) = s_2 \cdots s_m$ and hence by the induction hypothesis it follows that $s_2 \cdots s_{m-1}$ and $h(y_1 y_2)$ determine $y_3 \cdots y_m$. Since $S$ is right non-degenerate, $y_1$ and $s_1$ determine $h(y_1 y_2)$ and $y_2$. Hence $y_1$ and $f_{y_1}(y_2 \cdots y_m) = s_1 \cdots s_{m-1}$ determine $y_2 \cdots y_m$, as desired. \qed

Our aim is to investigate when $K[S]$ is noetherian. Hence, we first study the weaker condition that $S$ satisfies the ascending chain condition on right ideals.
Let $S = \langle x_1, \ldots, x_n \rangle$. We shall consider the following (right) over-jumping property:

for every $a \in S$ and every $i$ there exist $k \geq 1$ and $w \in S$ such that $aw = x_i^k a$.

This property is formally stronger than the following immediate consequence of the ascending chain condition on right ideals in $S$:

for every $a \in S$ and every $i$ there exist positive integers $q, p$ and $w \in S$ such that $x_i^p a w = x_i^{p+q} a$.

(Indeed, this condition immediately follows from the ascending chain condition applied to $I_j = \bigcup_{k=1}^j x_i^k a S$.) We show that the over-jumping property holds for the class of right non-degenerate semigroups of skew type.

**Proposition 4.2.** Assume that $S$ is a right non-degenerate semigroup of skew type. Then $S$ has the over-jumping property.

**Proof.** Fix some $y_1 \in X = \{x_1, \ldots, x_n\}$. We have shown that, if $m \geq 2$, then $f = f_{y_1} : X^{m-1} \to X^{m-1}$ is a permutation. Therefore, $f^r$ is the identity map for some $r \leq (|X|^{m-1})! = (n^{m-1})!$. So, for any $y_2, \ldots, y_m \in X$, we have

$$f^r(y_2 \cdots y_m) = y_2 \cdots y_m.$$

Now, interpreting $X$ as the generating set of $S$, we get the following equality in $S$

$$y_1 y_2 \cdots y_m = f_{y_1}(y_2 \cdots y_m)s_m.$$

Next

$$y_1^2 y_2 \cdots y_m = y_1 f_{y_1}(y_2 \cdots y_m)s_m = f_{y_1}(f_{y_1}(y_2 \cdots y_m))s_m + 1 s_m$$

for some $s_{m+1} \in X$. Proceeding this way, we come to

$$y_1^r y_2 \cdots y_m = f_{y_1}^r(y_2 \cdots y_m)s_m + \cdots + 1 s_m + 1 s_m$$

for some $s_i \in X$, $i = m, \ldots, m + r - 1$. This means that in $S$ we have

$$y_1^r y_2 \cdots y_m = y_2 \cdots y_m w$$

for some $w \in S$.

This can be also repeated for $f$ considered as a map $f_{y_1} : X \cup \cdots \cup X^{m-1} \to X \cup \cdots \cup X^{m-1}$. We have thus shown that $S$ has the following property:

for every $m \geq 1$ there exists $r \geq 1$ ($r \leq (n^{m-1})!$) such that if $a \in S$ has length less than $m$ in the generators $x_1, \ldots, x_n$ and $i \in \{1, \ldots, n\}$ then we have $aw = x_i^r a$ for some $w \in S$.

The result follows. □
Lemma 4.3. Assume that $S$ is a right non-degenerate semigroup of skew type. Then for every $x, y \in S$ there exist $t, w \in S$ such that $|w| = |y|$ and $xw = yt$.

Proof. Suppose first $|x| = 1$, so that $x = x_j$ for some $j$. Then the assertion follows from Lemma 4.1. So, suppose $|x| > 1$. We now proceed by induction on the length of $x$ as a word in $x_1, \ldots, x_n$. So, suppose that the assertion holds for all $x \in S$ of length $< m$. Let $x \in S$ be such that $|x| = m$, say $x = z_1 \cdots z_m$ for some $z_i \in \{x_1, \ldots, x_n\}$. By the induction hypothesis, $z_1 \cdots z_{m-1}u = yw$ for some $u, w \in S$ with $|u| = |y|$. We know also that $z_m v = us$ for some $v, s \in S$ such that $|v| = |u|$. Then

$$xv = z_1 \cdots z_{m-1}z_m v = z_1 \cdots z_{m-1}us = yws.$$ 

Since $|v| = |y|$, this proves the assertion. □

The following result, together with its proof, provide the first insight into the structure of non-degenerate semigroups and their algebras. This will be heavily exploited and strengthened in Section 5.

In the proof the following sets will play a crucial role.

Definition 4.4. Let $S = \langle x_1, x_2, \ldots, x_n \rangle$ be a semigroup of skew type. For a subset $Y$ of $X = \{x_1, \ldots, x_n\}$ define

$$SY = \bigcap_{y \in Y} yS \quad \text{and} \quad DY = \{s \in SY \mid \text{if } s = xt \text{ for some } x \in X \text{ and } t \in S \text{ then } x \in Y\}.$$ 

The left–right symmetric duals of these sets will be denoted by $S'_Y$ and $D'_Y$, respectively.

Notice that because of Lemma 4.3 each such set $SY$ is nonempty. However, it may happen that $SY = SZ$ for different subsets $Y$ and $Z$ of $X$; possibly it can occur that $DY = \emptyset$.

Theorem 4.5. Let $S = \langle x_1, \ldots, x_n \rangle$ be a semigroup of skew type. If $S$ is right non-degenerate then

(1) for each integer $i$, with $1 \leq i \leq n$, $S_i = \bigcup_{Y: |Y| = i} SY$ is an ideal of $S$, and

$$SX = S_n \subseteq S_{n-1} \subseteq \cdots \subseteq S_1 \subseteq S,$$

(2) $S$ is the union of sets of the form $\{y_1^{a_1} \cdots y_k^{a_k} \mid a_i \geq 0\}$, where $y_1, \ldots, y_k \in X$ and $k \leq n$.

In particular, $\text{GK}(K[S]) \leq n$.

Proof. Let $Y$ be a subset of $X = \{x_1, \ldots, x_n\}$. If $x \in X$ then let $Z \subseteq X$ be the largest subset such that $x SY \subseteq SZ$. Since $X$ is right non-degenerate, it follows that $|Z| \geq |Y|$. Moreover, if $x \notin Y$, then $|Z| > |Y|$. Consequently, $S_j = \bigcup_{Y: |Y| = j} SY$ are ideals of $S$ such that

$$SX = S_n \subseteq S_{n-1} \subseteq \cdots \subseteq S_1 \subseteq S.$$
Note that if $j = |Y|$ then $D_{Y} = S_{Y} \setminus S_{j+1}$ (we let $S_{n+1} = \emptyset$). So

$$S_{j} \setminus S_{j+1} = \bigcup_{Z: |Z| = j} D_{Z}$$

is a disjoint union.

Suppose first that $|Y| = 1$. Let $w \in D_{Y} \setminus \langle y \rangle$, where $Y = \{y\}$. Then $w = y^{q}xt$ for some $x \in X$, $t \in S$, and $y^{q}x \in D_{Y} \setminus \langle y \rangle$. Since $S$ is right non-degenerate, there exist $r \geq 1$ and distinct elements $u_{1}, \ldots, u_{r} = x \in X$ such that $yx = u_{1}w_{1}$, $yu_{1} = u_{2}w_{2}$, $\ldots$, $yu_{r-1} = u_{r}w_{r}$. Therefore, $y^{q}x \in \bigcup u_{i}S$ for every $q \geq 1$. But $u_{i} \neq y$ for all $i \geq 1$, so $y^{q}x \notin D_{Y}$, a contradiction. It follows that $D_{Y} = \langle y \rangle \setminus \{1\}$.

Fix some $y \in Y$. Suppose $s \in D_{Y}$ and $j = |Y|$. Let $r \geq 1$ be the maximal integer such that $s = y^{r}t$ for some $t \in S$. Suppose $t \in S_{Z}$ for some $Z \subseteq X$ with $|Z| = |Y|$. If $y \notin Z$ then $yt \in S_{j+1}$ and, therefore, $s \in S_{j+1}$, a contradiction. So, we have $y \in Z$. Then $t \in yS$, which contradicts the maximality of $r$. Hence, we have shown that $t \notin S_{j}$. It follows that

$$D_{Y} \subseteq \langle y \rangle(S \setminus S_{j}).$$

By induction on $|Y|$ this easily implies $D_{Y}$ is contained in a union of sets of the form $\{y^{a_{1}}_{i_{1}} \cdots y^{a_{k}}_{i_{k}} \mid a_{j} \geq 0\}$, where $|Y| = j$ and $y_{i} \in X$. So $S$ is the (finite) union of sets of the form $\{y^{a_{1}}_{i_{1}} \cdots y^{a_{k}}_{i_{k}} \mid a_{j} \geq 0\}$, where $y_{1}, \ldots, y_{k} \in X$ and $k \leq n$.

The assertion on the Gelfand–Kirillov dimension of $K[S]$ is now an easy consequence.

It is clear that $S \setminus S_{2} = \bigcup_{i=1}^{n}(x_{i}).$ Hence, there are $nm + 1$ elements of $S$ that are words of length at most $m$ in the generators $x_{1}, \ldots, x_{n}$ and that lie in $S \setminus S_{2}$. Proceeding by induction on $j$, assume that the number of elements of $S \setminus S_{j}$, that are words of length at most $m$, is bounded by a polynomial of degree $j - 1$ in $m$. Let $|Y| = j$, $Y \subseteq X$. Since $D_{Y} \subseteq \langle y \rangle(S \setminus S_{j})$ for $y \in Y$, it is easy to see that the number of elements of $D_{Y}$, that are words of length at most $m$, is bounded by a polynomial of degree $j$. As $S_{j} \setminus S_{j+1}$ is a finite union of such $D_{Y}$, the same is true of the elements of the set $S_{j} \setminus S_{j+1}$. This proves the inductive claim. It follows that the growth of $S$ is polynomial of degree not exceeding $n$, so that $\text{GK}(K[S]) \leq n$. \hfill \Box

The left–right symmetric dual of $S_{i}$ will be denoted by $S'_{i}$. Of course, if $S$ is a semigroup of skew type which is left non-degenerate then we obtain that each $S'_{i}$ also is an ideal of $S$.

The following technical result turns out to be very useful.

**Lemma 4.6.** Let $S$ be a right non-degenerate semigroup of skew type. Let $Y$ be a subset of $X$ and assume $|Y| = i - 1$. Let $b \in D_{Z}$, for some subset $Z$ of $Y$. Assume that $k$ is the length of $b$ in the generators of $X$. Then $(S_{i-1})^{k} \cap D_{Y} \subseteq bS$. Furthermore,

$$(S_{i-1})^{k+1} \cap D_{Y} \subseteq bS_{i-1} \quad \text{and} \quad (S_{i-1} \cap S'_{i-1})^{k+1} \cap D_{Y} \subseteq b(S_{i-1} \cap S'_{i-1}).$$

**Proof.** If $k = 1$, the assertion is clear. So assume $k \geq 2$. Write $b = y_{k} \cdots y_{1}$ with each $y_{j} \in X$. Let $q \geq k$ and $a = a_{q} \cdots a_{1} \in D_{Y}$ with each $a_{j} \in S_{i-1} \setminus S_{i}$. Since $b \in D_{Z}$ with
Suppose we have already shown that \( \text{Proposition 4.7.} \) Let \( S \) be a right non-degenerate semigroup of skew type. Then \( S \) has the ascending chain condition on right ideals.

**Proof.** Suppose we know already that \( S/S_i \) has the ascending chain condition on right ideals for some \( i \). We will show that \( S/S_{i+1} \) also has this property. Recall that by definition \( S\_{i+1} = \emptyset \) and \( S/S_n = S \). Then with \( i = n + 1 \) the assertion follows.

From Theorem 4.5 we know that \( S = \{ z_1^1 z_2^2 \cdots z_m^m | a_j \geq 0 \} \) for some \( m \geq 1 \) and \( z_1, \ldots, z_m \in X \) (not all \( z_j \) are necessarily different). We claim that \( S_j/S_{j+1} \) is finitely generated as a right ideal of \( S/S_{j+1} \). To prove this, it is sufficient to show by induction on \( m - k \) that the right ideal of \( S/S_{j+1} \) generated by \( C_k \cap (S_i \setminus S_{i+1}) \) is finitely generated, where \( C_k = \{ z_j^{a_1} \cdots z_j^{a_m} | a_j \geq 0 \} \). The case \( m - k = 0 \) is clear. The case \( m - k = m - 1 \) gives the assertion.

So assume \( 1 \leq k < m \). Let \( B = \{ b \in C_{k+1} | z_j^b \in S \) for some \( a \} \). If \( y \in C_k \cap (S_i \setminus S_{i+1}) \) then \( y \in z_j^y (B \cap (S \setminus S_i)) \) for some \( a \) (see the proof of Theorem 4.1) and \( (B \cap (S \setminus S_i))S \subseteq b_1 S \cup \cdots \cup b_r S \) for some \( b_i \in B \cap (S \setminus S_i) \) because \( S/S_i \) has the ascending chain condition on right ideals. Since \( S_i \) is an ideal of \( S \), it follows that

\[
C_k \cap (S_i \setminus S_{i+1}) \subseteq \bigcup_{r \geq N, j=1}^r z_j^b S \cup \bigcup_{j=0}^{N-1} z_j^b B_j,
\]
where \( N \) is chosen so that \( z^N_k b_j \in S_i \) for \( j = 1, \ldots, r \) and \( B_j = \{ y \in C_{k+1} \mid z^i_k y \in S_i \} \). By the inductive hypothesis, every \( B_j \cap S_i \) generates a finitely generated right ideal modulo \( S_{i+1} \). On the other hand, \((B_j \setminus S_i)S\) is a finitely generated right ideal because \( S/S_i \) has the ascending chain condition on right ideals. Hence, \( B_j \) and thus also \( z^i_k B_j \) generates a finitely generated right ideal modulo \( S_{i+1} \).

Next we show that the double union above is a finitely generated right ideal of \( S \). Because of Proposition 4.2 we know that \( S \) has the over-jumping property. Consequently, for every \( j \) there exist \( w_j \in S \) and a positive integer \( q_j \) such that

\[
 b_j w_j = z^q_j b_j.
\]

Hence,

\[
 z^N_k b_j \subseteq b_j S \quad \text{and so} \quad z^{p+q_j}_k b_j \subseteq z^q_k b_j S
\]

for every \( p \geq 0 \). It follows that the right ideal

\[
 \bigcup_{t \geq N} z^t_k b_j S = \bigcup_{t = N} z^{N+q_j}_k b_j S
\]

is finitely generated, as claimed.

As the left and the right side in (4) generate modulo \( S_{i+1} \) the same right ideal, it follows that \( C_k \cap (S_i \setminus S_{i+1}) \) generates a finitely generated right ideal modulo \( S_{i+1} \). So we proved our claim that \( S_i/\Si \) is a finitely generated right ideal of \( S/S_{i+1} \).

Suppose there is an infinite sequence \( a_1, a_2, \ldots \in S \setminus S_{i+1} \) such that we have proper inclusions

\[
 a_1 S \subset a_1 S \cup a_2 S \subset \cdots \subset a_1 S \cup \cdots \cup a_2 S \subset \cdots.
\]

Since \( S \) is the union of finitely many sets \( D_Y, Y \subseteq X \), we may assume that all \( a_j \in D_Y \) for some \( Y \). As \( S/S_i \) has the ascending chain condition on right ideals, it follows that \( \Si \subseteq S_i \setminus S_{i+1} \). Lemma 4.6 implies that \( a_j \notin S^t_i \) where \( t \) denotes the length of \( a_1 \). This leads to a contradiction with the fact that \( S_i/S_{i+1} \) is a finitely generated right ideal of \( S/S_{i+1} \) and \( S/S_i \) has the ascending chain condition on right ideals. (Namely, if \( S_i = s_1 S \cup \cdots \cup s_p S \cup S_{i+1} \) then \( \{a_j\} \) has a subsequence contained in \( s_{k_1} \cdots s_{k_p} (S \setminus S_i) \) for some \( p < t \) and some \( k_j \), leading to a contradiction.)

This proves that \( S/S_{i+1} \) has the ascending chain condition on right ideals, completing the inductive argument, and proving the result. \( \square \)

5. Non-degenerate implies noetherian

Our main aim in this section is to show that left and right non-degenerate semigroups \( S \) yield left and right noetherian algebras \( K[S] \). To prove this, we will rely on a general result
[18] that makes use of ideal chains in $S$ of a special type. Before stating the latter, we recall some terminology. Let $E = \mathcal{M}(G, t, t; \text{Id})$ be an inverse semigroup over a group $G$ with $t \geq 1$ (see [10]). In other words, $E = \{(g)_{ij} | g \in G, i, j = 1, \ldots, t \} \cup \{0\}$, where $(g)_{ij}$ denotes the $t \times t$-matrix with $g$ in the $(i, j)$-component and zeros elsewhere. The multiplication on $E$ is the ordinary matrix multiplication. A semigroup $S$ is said to be a generalised matrix semigroup if it is a subsemigroup of a semigroup $E$ of the above type and for every $i, j$ there exists $g \in G$ so that $(g)_{ij} \in S$. So, in the terminology of [16], $S$ is a uniform subsemigroup of $E$.

**Theorem 5.1** [18, Theorem 3.3]. Assume that $M$ is a finitely generated monoid with an ideal chain $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_r = M$ such that $M_1$ and every factor $M_j / M_{j-1}$ is either nilpotent or a generalised matrix semigroup. If $M$ has the ascending chain condition on right ideals, and $\text{GK}(K[M])$ is finite, then $K[M]$ is right noetherian.

Now we assume that $S$ is a semigroup of skew type that is right and left non-degenerate. Recall that for a $Y \subseteq X$ we write $S_Y = \bigcap_{y \in Y} S_y$. Further, $S_k = \bigcup_{Y \subseteq X, |Y|=k} S_Y$ is an ideal of $S$. We claim that $S'_k \subseteq S_k$ for all $k$ and symmetrically $(S'_k)^k \subseteq S_k$. Since $S_1 = S'_1$ (see the description of $S_1$ in the proof of Theorem 4.5), we may assume that $k > 1$. Let $1 \neq a \in S$ and $b \in S_k$. Let $Y \subseteq X$ be maximal such that $a \in S'_Y$. If $|Y| < k$ then there exists $x \in X$ such that $b \in xS$ and $x \notin Y$. So we may write $b = xc$, $c \in S$. Now $ax \subseteq Sx$ but, as $x \notin Y$ and $a \in S'_Y$, the element $ax$ is also contained in $|Y|$ different left ideals of the form $Sw$, $x \neq w \in X$ (use the left non-degeneracy of $S$). Therefore, $ax \in S'_Z$ for some $Z \subseteq X$ with $|Z| > |Y|$. Hence $ab = axc \in S'_Z$. By induction it follows easily that $S_k^k \subseteq S'_k$, as desired.

We have shown that $I_k = S_k \cap S'_k$ is an ideal of $S$ such that $S_k / I_k$ is nilpotent.

**Theorem 5.2.** Let $S = \langle x_1, \ldots, x_n \rangle$ be a semigroup of skew type. If $S$ is right and left non-degenerate, then $K[S]$ is a right and left noetherian PI-algebra.

**Proof.** From Theorem 4.5 we know that $\text{GK}(K[S])$ is finite. Because of Proposition 4.7 we also know that $S$ satisfies the ascending chain condition on one sided ideals. In view of Theorem 5.1 and its dual, to prove that $K[S]$ is right and left noetherian it is sufficient to show that $S$ has an ideal chain with each factor either nilpotent or a generalised matrix semigroup.

Write $S_{n+1} = S_{i+1} = \emptyset$ and adopt the convention $S/\emptyset = S$. By induction on $i$, we will prove that $S/S_i$ has an ideal chain of the desired type. The case $i = n + 1$ then yields the result. As noticed in the proof of Theorem 4.5, $S \setminus S_2$ is the disjoint union of all $D_{\{x_i\}} = \{x_i\} \setminus \{1\}$ and $\{1\}$. So $S/S_2$ has an ideal chain with commutative 0-cancellative factors, hence it has a chain of the type described in Theorem 5.1. So now assume that we have shown this for the semigroup $S/S_{i-1}$ for some $i \geq 3$.

Let $J'$ be the ideal of $S$ such that $S_i \subseteq J'$ and $J'/S_i$ is the maximal nil ideal of $S/S_i$. We consider the following ideals of $S$:

$$S_i \cap I \subseteq (S_i \cup S'_i) \cap I \subseteq I \subseteq I = S_{i-1} \cap S'_{i-1} \subseteq S_{i-1},$$

where $g_{ij}$ denotes the $t \times t$-matrix with $g$ in the $(i, j)$-component and zeros elsewhere. The multiplication on $E$ is the ordinary matrix multiplication. A semigroup $S$ is said to be a generalised matrix semigroup if it is a subsemigroup of a semigroup $E$ of the above type and for every $i, j$ there exists $g \in G$ so that $(g)_{ij} \in S$. So, in the terminology of [16], $S$ is a uniform subsemigroup of $E$. 


where $J = J' \cap I$. (Notice that the first and the last Rees factor are nilpotent by the comment after Theorem 5.1.) Then $J/S_i$ is nilpotent because of the ascending chain condition on one-sided ideals in $S$, see [3, Theorem 17.22].

For $Y \subseteq X$, let $D_Y$ and $D'_Y$ be the subsets of $S$ introduced in Definition 4.4. Let $I_Y W = D_Y \cap D'_W$ for $Y, W \subseteq X$ and $I_Y = I_{YY}$. If $|Y| = i - 1$, $D'_Y \neq \emptyset$, and $x \in X \setminus Y$ then $D'_yx \subseteq S'_i$. (Use the left non-degeneracy of $S$; $D'_yx \subseteq Sx$ but also it is contained in $|Y|$ different left ideals of the form $Syx$, $y \in Y$, and thus $yx = y'x'$ with $y' \neq x$.) So $D'_Y D_Z \subseteq S'_i$ for every $Y, Z \subseteq X$ of cardinality $i - 1$ with $Z \neq Y$, provided that $D_Y \neq \emptyset$ and $D'_Z \neq \emptyset$. Hence, we get a generalised matrix structure $(S_{i-1} \cap S'_{i-1}) \setminus (S_i \cup S'_i) = \bigcup_{Y, Z} I_{YZ}$.

Now, for $Y \subseteq X$ with $|Y| = i - 1$ there are two mutually exclusive cases:

**Case 1:** either $I_Y = \emptyset$ or there exist $b \in D'_Y$ and $x \in D_Y$ such that $bx \in J$.

**Case 2:** $I_Y \neq \emptyset$ and $D'_Y D_Y \subseteq (S_{i-1} \cap S'_{i-1}) \setminus J$, so in particular $I_Y$ is a subsemigroup of $(S_{i-1} \cap S'_{i-1}) \setminus J$.

In Case 1 we claim that $D_Y \cap I$ and $D'_Y \cap I$ are contained in $J$. If $I_Y = \emptyset$, then the generalised matrix structure easily yields that $(D_Y \cap I)^2$ and $(D'_Y \cap I)^2$ are contained in $J$. As both $D_Y \cap I$ and $D'_Y \cap I$ are one-sided ideals modulo $J$, we get $D_Y \cap I$, $D'_Y \cap I \subseteq J$. So assume $I_Y \neq \emptyset$ and that there exist $b \in D'_Y$ and $x \in D_Y$ so that $bx \in J$. Let $q$ be the maximum of the lengths of $b$ and $x$. Then, by Lemma 4.6 and its right–left dual, we get

$$I_Y^{2q} \subseteq (D_Y)^q D_Y' \subseteq (Sb)(xS) \cup S_i \cup S'_i \subseteq J'.$$

So $I_Y$ is nilpotent modulo $J$. Then again $D'_Y \cap I$ (with zero) is a left ideal of $S/S'_i$ and it is nil modulo $J$ (use the generalised matrix pattern), so we must have $D'_Y \cap I \subseteq J$. Similarly, $D_Y \cap I \subseteq J$.

In Case 2 we will show that $I' \cap I_Y$ is a cancellative semigroup, for some $r \geq 1$.

Before proving this, we introduce some notation and develop some machinery. For $a, b \in S$ we write $ab$ if there exists $z \in I = S'_{i-1} \cap S_{i-1}$ so that $az = bz \neq J$. Notice $a$ and $b$ have the same length. Hence, for a given $a \in S$, there are only finitely many $b$ so that $ab$.

Let $A$ be the set of all elements $d \in I$ such that every proper initial segment of $d$ is not in $I$. In other words, $A$ is the (unique) minimal set of generators of $I$ as a right ideal of $S$. By the ascending chain condition on right ideals in $S$ this is a finite set.

Let $ab$ for some $a, b \in S$; so $az = bz \neq J$ for some $z \in I$. Let $Y, Z \subseteq X$ be such that $a \in D'_Y$ and $b \in D'_Z$. Because $S$ is left non-degenerate, $az \notin J$ implies that $z \in D_Y$. Since $S$ is also right non-degenerate and because $bz \notin J$, we thus obtain $Z \subseteq Y$. In particular, $a, b \in Sx$ for every $x \in Z$.

Choose $s \in S$ such that $a = a's$, $b = b's$ for some $a' \in I$, $b' \in S$ such that if $a'd = ax$, $b = b'x$ for $x \in X$, $a'' = b''$ then $a'' \notin I$. Notice that $a' b'$. The previous paragraph implies that $a' \in A$.

Let $a_j, b_j$, $j = 1, \ldots, q$, be all pairs such that $a_j b_j$, $a_j \neq b_j$ and $a_j \in A$. As remarked earlier, there are only finitely many such pairs of elements. Let $z_j \in I$ so that $a_j z_j = b_j z_j \notin J$. By the above, for every $a \in I$, $b \in S$ such that $a \tau b$ and $a \neq b$, we have
If $b \in A_J$ because $c$ satisfies $az$ then $a, b \in D'_Y$. We claim that
\[ at = bt \notin J \quad \text{for every } t \in I^N \cap I_Y, \]
where $N$ is the maximum of all $|z_j|$, $j = 1, \ldots, q$.

So $a = a_J s$ and $b = b_J s$ for some $j$ and some $s \in S$. Since $a_J b \in I \setminus J$, there exist $W, z \subseteq X$, each of cardinality $i - 1$, so that $a_J z \in I_W z$. As $a \in D'_Y \setminus J$, we thus get that a $\in I_W Y$. Moreover, $ajz_j = bjz_j \notin J$ yields $z_j \in I_{ZJ}$ for some $V \subseteq X$. Now $a_J z = a \in D'_Y$ implies $a_I Y = a_J z$ and, because $Y$ satisfies Case 2, the former does not intersect $J$. In particular, $s I_Y \subseteq I_{ZJ}$.

Let $t \in I^N \cap I_Y$. Then $st \in s(I^N \cap I_Y) \subseteq D_Z \cap I^{|z_j|} \subseteq z_j S$ by Lemma 4.6. So $st = z_j u$ for some $u \in S$. Now $at = a JS = a_J z ju$ and $at \in I_{WY} I_Y$, and thus $at \notin J$. Similarly, $bt = b JS = b_J z ju$ whence $at = bt$. This proves the claim.

For every $Y$ that satisfies Case 2 choose $c_Y \in I^N \cap I_Y$. Write $r = \max(|c_Y| + 1)$ and let $T' = I'$. Then $T / (J \cap T) = I' / (J \cap I')$ has a matrix pattern $T / (J \cap T) = \bigcup T / (J \cap T)$ where $T / (J \cap T) = (I_{WY} \cap T) \setminus J$ and $Y, W$ run through a subset of the set of $i - 1$-element subsets of $X$. The 'diagonal components' are $T_Y = T \cap I_Y$. We know that if there exist $a \in T_{YZ}$ and $b \in T_{ZW}$ (so $Z$ satisfies Case 2) then $ab \in T_{YW}$. In particular, if $T_{YZ}$ and $T_{ZW}$ are nonempty, then also $T_{YW}$ is nonempty.

Let $A$ be a maximal subsemigroup of $T / (J \cap T)$ of the form $A = \bigcup_{T / (J \cap T) \in P} T / (J \cap T)$ where $P$ is a set of $i - 1$-element subsets of $X$ such that every $T / (J \cap T)$ is empty. Let $Y \in P$. Suppose that $T_{YW} \neq \emptyset$ for some $W \in P$ of cardinality $i - 1$. Then $\emptyset \neq T_{ZJ} T_{YW} \subseteq T_{ZW}$ for every $Z \in P$. Clearly, $T_{YW} \neq \emptyset$ because $W$ satisfies Case 2. Using the maximality of $P$ it is now easy to see that $B = \bigcup_{T / (J \cap T) \in P \notin P} T / (J \cap T)$ is a right ideal of $T / (J \cap T)$. However, if $b \in B$ and $0 \neq bx \in A$ for some $s \in S / J$, then $0 \neq bx \in A$ for some $x \in A$. Since $ss x \in T / (J \cap T)$, it follows that $bx \in B$, a contradiction. This shows that $B$ is a right ideal of $S / J$. From the matrix pattern it follows that it is nilpotent and this contradicts with the definition of $J$. Consequently, $T_{TV} = \emptyset$, and similarly $T_{TV} = \emptyset$ for every $V \notin P$ of cardinality $i - 1$. Therefore, we get a decomposition $T / (J \cap T) = A_1 \cup \cdots \cup A_k$ for some $k$, where each $A_j$ is of the 'square type,' as $A$ above. This union is 0-disjoint. $A_k$ are ideals of $S / J$, and $A_j A_k = 0$ for $j \neq k$. Fix some $A = A_j$, say $j = 1$.

Let $T'$ be such that $T' \subseteq A$. Let $c = c_{Y'} \in I^N \cap I_Y$. We now prove that if $a, b \in T \cap D'_Y$ satisfy $a z = b z \notin J$ for some $z \in T$ then $a = b$. As $z \in D_Y \setminus J$, we must have $z \in A$. Then $z \cap D'_Y \cap T' \neq \emptyset$, so we may assume that $z \in T_Y$. Since $r \geq |c| + 1$, by the dual of Lemma 4.6, we may write $a = a' c$, $b = b' c$ for some $a', b' \in I$. Moreover, $a', b' \in D'_Y$ because $c \in I_Y$ and $a', b' \in I$ (use the generalised matrix pattern). Then $a' c z = b' c z$ and $a' t b'$. As $a' c = b' c$ by $(5)$, we obtain $a = b$. Repeating this for every $Y$ with $T_Y \subseteq A$, we show that $A$ has the property that $a z = b z \not\in 0$ implies $a = b$. By a symmetric argument, we may also obtain that $A$ has the property
\[ a, b, z \in A \quad \text{and} \quad a z = b z \neq 0 \quad \text{or} \quad z a = z b \neq 0 \quad \text{then} \quad a = b. \]

In particular, if $Y$ satisfies Case 2, then the diagonal components $T_Y = I_Y \cap T$ are cancellative semigroups.
Let \( Q = \{a_1 + \cdots + a_m \mid a_i \in T_Y\} \) where \( T = \{Y_1, \ldots, Y_m\} \). Let \( Z = (T/(J \cap T))/(A_2 \cup \cdots \cup A_k) \). Then \( Z \) may be identified with \( A \). Because of (6), \( Q \) consists of regular elements in the algebra \( K_0[A] \). Furthermore, the diagonal components of \( T_Y \) form cancellative right and left Ore semigroups. Indeed, from Lemma 4.6 it follows that every two right ideals of each \( T_Y \) intersect nontrivially. This implies easily that the same holds for the semigroup \( T_Y \), and a symmetric argument works for left ideals. It is then readily verified that \( Q \) is an Ore subset of the algebra \( K_0[A] \). The localization of \( A \) with respect to \( Q \) is an inverse semigroup (it has a matrix pattern and each diagonal component is a group, namely the group of quotients of the corresponding \( T_Y \)). Therefore, \( A \), and thus each \( A_i \) is a semigroup of generalised matrix type. Hence, \( T/(J \cap T) \) has an ideal chain whose factors are of generalised matrix type and which is determined by certain ideals of \( S \). Consider the ideal chain

\[
S_i \subseteq S_i \cup (J \cap T) \subseteq S_i \cup T \subseteq S_{i-1} \subseteq S.
\]

We know that \( S_i \cup (J \cap T) \subseteq S_i \cup T \). The factor \( (S_i \cup J)/(S_i \cup (J \cap T)) \) is naturally identified with \( T/(J \cap T) \) because \( S_i \cap T \subseteq J \). It follows that \( S/S_i \) has an ideal chain of the type described in Theorem 5.1. This completes the inductive step, and thus we have shown that \( K[S] \) is right and left noetherian.

Finally, from Theorem 3.1 it now follows that \( K[S] \) satisfies a polynomial identity. \( \square \)

In the last paragraph of the proof we have shown that each \( A_i \) is an order in a completely 0-simple inverse semigroup, in the sense of Fountain and Petrich. While this is an easy consequence of the properties of \( T \) proved before and of the main results of [4], we used a simple localization technique at the semigroup algebra level, rather than referring to these nontrivial semigroup theoretical results.

The following is a direct consequence of the proof of Theorem 5.2.

**Corollary 5.3.** Assume that \( S \) is a right and left non-degenerate semigroup of skew type. Then \( S \) has a cancellative ideal \( I \). Namely, \( S_X^N \) is such an ideal for some \( N \geq 1 \).

6. Cancellative congruence and the prime radical

Let \( \rho \) be the least cancellative congruence on a semigroup of skew type \( S \). So it is the intersection of all congruences \( \sim \) on \( S \) such that \( S/\sim \) is cancellative. Let \( \rho_1 \) be the smallest congruence on \( S \) containing all \((s,t)\) such that \( su = tu \) or \( us = ut \) for some \( u \in S \). Suppose we have already constructed \( \rho_n \). Let \( \rho_{n+1} \) be the smallest congruence on \( S \) that contains all \((s,t)\) with \((su, tu) \in \rho_n \) or \((us, ut) \in \rho_n \) for some \( u \in S \). We claim that \( \rho = \bigcup_{n \geq 1} \rho_n \). Indeed, if \((su, tu) \in \bigcup_{n \geq 1} \rho_n \), then \((su, tu) \in \rho_n \) for some \( n \geq 1 \). Hence \((s, t) \in \rho_{n+1} \). It follows that \( \bigcup_{n \geq 1} \rho_n \) is right cancellative. Similarly, it is left cancellative, so that \( \rho \subseteq \bigcup_{n \geq 1} \rho_n \). For the converse first note that \( \rho_1 \subseteq \rho \). Then, by induction one shows easily that \( \rho_n \subseteq \rho \) for every \( n \geq 1 \). Hence \( \rho = \bigcup_{n \geq 1} \rho_n \), as claimed.

It is easy to see (by induction) that every \( \rho_n \) is homogeneous, because the defining relations of \( S \) are homogeneous. It follows that \( \rho \) is homogeneous.
From now on we assume that $S$ is left and right non-degenerate. So, by Lemma 4.3, $S$ satisfies $xS \cap yS \neq \emptyset$ for every $x, y \in S$. Define a relation $\sim$ on $S$ by $a \sim b$ if $ax = bx$ for some $x \in S$. We claim that $\sim$ is a congruence on $S$. Suppose $a \sim b$ and $b \sim c$. Then $ax = bx$, $by = cy$ for some $x, y \in S$. There exist $u, w \in S$ such that $xu = yw$. Thus,

$$axu = bxu = byw = cyw = cxu$$

and so $a \sim c$. Next, if $z \in S$, then $zs = xt$ for some $s, t \in S$. Then

$$azs = axt = bxt = bzs$$

and $az \sim bz$. It follows that $\sim$ is a congruence on $S$. It is clear that it is the least congruence on $S$ such that $S/\sim$ is right cancellative.

**Lemma 6.1.** Let $T$ be a semigroup with a cancellative ideal $J$. Assume that $J$ has a group of quotients $G$. Define $\hat{T} = (T \setminus J) \cup G$. Then $\hat{T}$ has a semigroup structure extending that of $T$.

**Proof.** The multiplication on $\hat{T}$ is defined by

$$t(ab^{-1}) = (ta)b^{-1}$$

for $a, b \in J$ and $t \in T$. Similarly, one defines the left multiplication by elements of $G$. Associativity can be easily checked. $\square$

By Lemma 5.3 there exists $N \geq 1$ such that $I = S_N^S$ is a cancellative ideal of $S$. We know that $I$ has a group of quotients and thus by Lemma 6.1 we have the semigroup $\hat{S} = (S \setminus I) \cup I^{-1}$. Let $e = e^2 \in I^{-1}$. For any $a, b, x \in S$ we get that $(a - b)x = 0$ implies $(a - b)xI = 0$, and thus $(a - b)e = 0$. Since $e$ is a central idempotent this yields $e(a - b) = 0$ and therefore $x(a - b) = 0$. So, by symmetry, we obtain that the following conditions are equivalent: (1) $(a - b)x = 0$, (2) $x(a - b) = 0$, (3) $Ix = 0$, and (4) $xI = 0$. It follows that the least right cancellative congruence coincides with the least left cancellative congruence on $S$. Note that $\rho$ is finitely generated, as a right congruence, since $K[S]$ is right noetherian. So, we have proved the following result.

**Proposition 6.2.** Let $S$ be a left and right non-degenerate semigroup of skew type. Then the least right cancellative congruence on $S$ coincides with the least cancellative congruence on $S$ and it is defined by $a \rho b$ if $ax = bx$ for some $x \in S$. Moreover, the ideal of $K[S]$ determined by $\rho$ is of the form $I(\rho) = \sum_{i=1}^{k} (a_i - b_i)K[S]$ for some $k \geq 1$ and $a_i, b_i \in S$.

Recall that by definition $I(\rho)$ is the kernel of the natural homomorphism $K[S] \to K[S/\rho]$. The congruence $\rho$ is actually important for the description of the prime radical $B(K[S])$ of $K[S]$. As in the proof of Theorem 3.1, we get that $S/\rho$ has an abelian-by-finite group of quotients. Moreover, if $\text{char}(K) = 0$, then $K[S/\rho]$ is semiprime (see, for example, [16, Theorem 7.19]). In particular, $B(K[S]) \subseteq I(\rho)$, the ideal of $K[S]$ determined by $\rho$. 
We have seen that \( apb \) if and only if \((a - b)I = 0 = I(a - b)\). So \( I(\rho)I = I(I(\rho)) = 0 \). Hence, if \( P \) is a prime ideal of \( K[S] \) with \( P \cap S = \emptyset \) then \( I(\rho) \subseteq P \). If, on the other hand, \( P \) is a prime ideal with \( P \cap S \neq \emptyset \), then there exists \( b \in P \cap S \). So, by Lemma 4.6, \( S_k^X = bS \subseteq P \) for some positive integer \( k \). It follows that \( S_X \subseteq P \).

Suppose that \( \alpha \in K[S] \) belongs to the left annihilator \( \text{ann}_L(I) \) of \( I \) in \( K[S] \). Then \( \alpha s = 0 \) for all \( s \in I \). Write \( \alpha = \alpha_1 + \cdots + \alpha_m \) with \( |\text{supp}(\alpha_i)s| = 1 \) and \( \text{supp}(\alpha_i)s \neq \text{supp}(\alpha_j)s \) for \( i \neq j \). It follows that \( \alpha_i s = 0 \) for all \( i \). So the augmentation of \( \alpha_i \) is zero and it is clear that \( \alpha_i \in I(\rho) \). From all the above it follows that \( I(\rho) \subseteq \text{ann}_L(I) \subseteq I(\rho) \).

Hence, \( I(\rho) = \text{ann}_L(I) \). By symmetry, \( I(\rho) = \text{ann}_R(I) \), the two-sided annihilator of \( I \). If \( \text{char}(K) = 0 \), then \( I(\rho) \) is a semiprime ideal and thus \( I(\rho) = \bigcap_{P \in \text{I}(\rho) \subseteq P} P \). So we have proved the following result. By \( X^0(K[S]) \) we denote the set of all the minimal primes of \( K[S] \).

**Proposition 6.3.** If \( S \) is a left and right non-degenerate semigroup of skew type, then

1. \( I(\rho) = \text{ann}(S_X^N) \) for some \( N \geq 1 \).
2. \( I(\rho) \subseteq P \) for any \( P \in X^0(K[S]) \) with \( P \cap S = \emptyset \).
3. \( S_X \subseteq P \) for any \( P \in X^0(K[S]) \) with \( P \cap S \neq \emptyset \).

If, furthermore, \( \text{char}(K) = 0 \) then

\[
B(K[S]) = I(\rho) \cap \bigcap_{P \in X^0(K[S]), P \cap S \neq \emptyset} P = I(\rho) \cap \left( \bigcap_{P \in X^0(K[S]), S_X \subseteq P} P \right).
\]

Note that if \( S \) is a left and right non-degenerate semigroup of skew type then there is at least one minimal prime \( P \) so that \( P \cap S = \emptyset \). Indeed, for otherwise the proposition implies that \( S_X \subseteq B(K[S]) \). This yields a contradiction as \( S_X \) is not nil.

7. Examples

Our first example shows that \( K[S] \) can be a noetherian PI algebra even if \( S \) is neither left nor right non-degenerate.

**Example 7.1.** Let \( S = \langle x_1, x_2, x_3, x_4 \rangle \) be the semigroup of skew type defined by

\[
x_3x_2 = x_1x_4 \quad \text{and} \quad x_4x_1 = x_2x_3
\]

with all the remaining relations in \( S \) of the form \( xy = yx \). Then, for every field \( K \), \( K[S] \) is a noetherian PI-algebra. Moreover, \( B(K[S]) = I(\rho) \), and \( I(\rho) \) is the only minimal prime of \( K[S] \).

**Proof.** By the defining relations, we get

\[
x_3x_1x_2 = x_3x_1x_4 = x_1x_3x_4 = x_1x_4x_3 = x_3x_2x_3 = x_3x_4x_1 = x_4x_3x_1 = x_4x_1x_3 = x_2x_3x_3.
\]
So $x_3^2 \in Z(S)$. Similarly, $x_i^2 \in Z(S)$ for $i = 1, 2, 3, 4$. It is easy to see that

$$S = \{ x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4} \mid a_i \geq 0 \}.$$  

It follows that $K[S]$ is a finite module over $K[A]$, where $A = \langle x_1^2, x_2^2, x_3^2, x_4^2 \rangle$. Therefore, $K[S]$ is right and left noetherian. Notice that $S$ is right and left degenerate.

We claim that $B(K[S]) = I(\rho)$ and the least cancellative congruence $\rho$ on $S$ coincides with the congruence determined by the natural homomorphism $\phi : K[S] \to K[C]$, where $C$ is the commutative monoid obtained from $S$ by adding all the commutator relations to the defining relations of $S$. So $C = \langle a_1, a_2, a_3, a_4 \mid a_i a_j = a_j a_i, a_1 a_4 = a_3 a_2 \rangle$. In fact, we have seen above that

$$x_3(x_2 x_3 - x_3 x_2) = 0 = (x_2 x_3 - x_3 x_2)x_3.$$  

Similarly one shows that

$$y(x_2 x_3 - x_3 x_2) = 0 = (x_2 x_3 - x_3 x_2)y$$  

for every $y \in \{ x_1, x_2, x_3, x_4 \}$. Also,

$$y(x_1 x_4 - x_4 x_1) = 0 = (x_1 x_4 - x_4 x_1)y.$$  

So $\ker(\phi) \subseteq I(\rho)$. Since $C$ embeds in a torsion-free group, $K[C]$ is a domain and we get $I(\rho) = \ker(\phi)$. Also $B(K[S]) \subseteq \ker(\phi)$, while $(\ker(\phi))^2 = 0$ by the displayed formulas. It follows that $B(K[S]) = I(\rho)$. In particular, $I(\rho)$ is the only minimal prime of $K[S]$, so $K[S]$ has no minimal primes intersecting $S$.  

The following example shows that a right non-degenerate semigroup $S$ of skew type does not always yield a right noetherian algebra $K[S]$ and neither it always has a cancellative ideal (note that the latter property is an essential feature used in the proofs of the main results in the previous sections).

**Example 7.2.** Let $S = \langle x_1, x_2, x_3 \rangle$ be the monoid defined by the relations

$$x_2 x_1 = x_3 x_1, \quad x_1 x_2 = x_3 x_2, \quad x_1 x_3 = x_2 x_3.$$  

Then $S$ is right non-degenerate but not left non-degenerate and $K[S]$ is neither right nor left noetherian. Moreover, $B(K[S]) = I(\eta)$, where $\eta$ is the least commutative congruence on $S$, $B(K[S])^2 = \{ 0 \}$, and thus $K[S]$ is a PI algebra. Furthermore, $\text{GK}(K[S]) = 2$ and $S$ is left cancellative but does not have a cancellative ideal.

**Proof.** We note that

$$x_i x_i x_k = x_i x_j x_k = x_k x_j x_i = x_k x_i x_k = x_j x_i x_k = x_j x_j x_k$$

for $i, j, k = 1, 2, 3$.
for all \(i, j, k\) with \([i, j, k] = \{1, 2, 3\}\). Next we show that \(x^n_j x_j = x^n_k x_k\) for all \(n \geq 1\) and \([i, j, k] = \{1, 2, 3\}\). For \(n = 1\) or 2 this is clear and for \(n \geq 3\), using induction, we get

\[
x^n_j x_k = x_j x_j^{n-1} x_k = x_j x_j^{n-2} x_j x_k = x_j x_j^{n-3} x_j x_k = x_j x_j^{n-2} x_j x_k = x_j x_j x_k.
\]

Now we show that \(xx_j x_k = x^{i+1}_j x_k = x^{i+1}_j x_k\), for every \(i, j, k\) so that \([i, j, k] = \{1, 2, 3\}\). We prove this by induction on the length of \(x\). By the above the claim holds if \(|x| = 1\). Now assume \(n \geq 2\) and the claim holds for all \(x \in S\) with \(|x| < n\). So now let \(x \in S\) and \(|x| = n\). Write \(x = x_m y\) with \(|y| = n-1\). So, because of the induction hypothesis, we get

\[
xx_j x_k = x_m y x_j x_k = x_m x^n_j x_k.
\]

If \(m = i\) then using the first claim several times, we get

\[
xx_j x_k = x_i x_j^n x_k = x_i x_j x^n_k = x_i x_j x^{i+1}_j x_k.
\]

If \(m = k\) then

\[
xx_j x_k = x_k x_j^n x_k = x_j x_i^n x_k = x_j x_i x^{n+1}_j x_k = x_j x_i x^{n+1}_j x_k.
\]

This shows the claim.

It is easy to see that \(S/\eta\) is a monoid with presentation \(\langle x, y, z \mid xy = yz = xz = yx = zx = zy \rangle\) and \(S/\eta = \{1\} \cup x \cup y \cup z \cup \{x^n y \mid n \geq 1\}\), a semilattice of torsion-free cancellative semigroups. So \(K[S/\eta]\) is semiprime. Because of the second claim, we also know that \((x_i x_j - x_j x_i)Sx_i x_j = 0\) for every \(i, j, k, l\) with \(k \neq l\). Hence it follows that \(B(K[S]) = I(\eta)\) and \(B(K[S])^2 = \{0\}\). In particular, \(K[S]\) is a PI algebra.

Consider the elements \(a_n = x_1 x_2^{n+1} - x_1^n x_2^n \in K[S]\), \(n = 2, 3, \ldots\). Because of the second claim, we have that \(a_n x_3^k = a_n x_3^k = 0\) for every \(k \geq 0\). Hence, it follows that \(a_n K[S] = \text{lin}_{K} \{a_n x_3^k \mid j \geq 0\}\). Now each element \(x_1 x_2^q\) can only be rewritten as \(s y x_3^q\) for some \(s \in S\) and \(y \in \{x_1, x_3\}\). It then easily follows that for \(n \geq 3\) there do not exist \(\lambda_j \in K\) so that

\[
a_n = \sum_{j=2}^{n-1} \lambda_j (x_1 x_2^{j+1} - x_1^n x_2^n) x_3^{n-j}.
\]

Therefore, \(a_n \notin \sum_{j=2}^{n-1} a_j K[S]\) for every \(n\). So, indeed \(K[S]\) is not right noetherian (however, \(S\) satisfies the ascending chain condition by Proposition 4.7).

If \(k < n\) then \(x_2 x_1^n \notin S x_2 x_1^n\). This is clear from the defining relations. Namely, for every \(s \in S\) the element \(s x_2 x_1^n\) can only be rewritten in the form \(t x_2 x_1^n\) or \(t x_3 x_1^n\) for some \(t \in S\). So \(S\) does not satisfy the ascending chain condition on left ideals and \(K[S]\) is not left noetherian.
Again from the second claim it easily follows that
\[ S = \langle x_1 \rangle \cup \langle x_2 \rangle \cup \langle x_3 \rangle \cup \{ x_2^k x_1^m \mid k, m \geq 1 \} \cup \{ x_1^k x_2^m \mid k, m \geq 1 \} \cup \{ x_1^k x_3^m \mid k, m \geq 1 \}. \]
So, clearly every ideal \( I \) of \( S \) intersects each of the last three sets. It is readily verified that therefore \( I \) is not cancellative.

Obviously, \( \text{GK}(K[S]) = 2 \). Furthermore, from the relations it also follows easily that \( S \) is left cancellative. \( \square \)

Our third example satisfies the cyclic condition, but the defining relations do not yield a Gröbner basis, so it is not of binomial type studied in [5,11]. The aim is to show that one can get important structural information on \( K[S] \). In particular, we determine all minimal primes and the prime radical of \( K[S] \). Recall that \( K[S] \) is an affine PI algebra which is left and right noetherian by Theorem 3.1 and Proposition 2.4.

**Example 7.3.** Let \( S = \langle x_1, x_2, x_3, x_4 \rangle \) be given by the presentation
\[
\begin{align*}
x_4 x_3 &= x_1 x_4, & x_4 x_2 &= x_2 x_4, & x_4 x_1 &= x_3 x_4, \\
x_3 x_2 &= x_1 x_3, & x_3 x_1 &= x_2 x_3, & x_2 x_1 &= x_1 x_2.
\end{align*}
\]
The minimal primes of \( K[S] \) are the ideals \( P_1 = \langle x_1 - x_2, x_2 - x_3 \rangle = I(\rho) \), \( P_2 = \langle x_4 \rangle \), \( P_3 = \langle x_1, x_3 \rangle \), and \( P_4 = \langle x_2 \rangle \). Moreover, \( K[S] \) is semiprime, has dimension three and
\[ S = \langle x_1, x_2, x_3 \rangle \cup \langle x_1 \rangle \langle x_4 \rangle \cup \langle x_2 \rangle \langle x_4 \rangle \cup \langle x_3 \rangle \langle x_4 \rangle \cup \langle x_1 \rangle \langle x_2 \rangle \langle x_4 \rangle. \]

**Proof.** First, note that the following equalities hold in \( S \):
\[ x_1 x_3 x_4 = x_1 x_4 x_1 = x_4 x_3 x_1 = x_4 x_2 x_3 = x_2 x_4 x_3 = x_2 x_1 x_4 = x_1 x_2 x_4 \]  
(7)
and
\[ x_1 x_2 x_4 = x_1 x_4 x_2 = x_4 x_3 x_2 = x_4 x_1 x_3 = x_3 x_4 x_3 = x_3 x_1 x_4 = x_2 x_3 x_4. \]
(8)
So \( x_1 x_3 x_4 = x_1 x_2 x_4 \) and \( x_2 x_1 x_4 = x_2 x_3 x_4 \). Therefore, \( P_1 = \langle x_2 - x_3, x_1 - x_3 \rangle \subseteq I(\rho) \). As \( K[S]/P_1 \cong K[Y_1, Y_4] \), a polynomial ring in two commuting variables, we get that \( P_1 \) is a prime ideal of \( K[S] \). So, by Proposition 6.3 and its following remark, \( P_1 \) is a minimal prime ideal of \( K[S] \) (it has depth 2), \( I(\rho) = P_1 \), and \( P_1 \) is the only minimal prime of \( K[S] \) intersecting \( S \) trivially.

Second, note that \( x_4 \) is a normalizing element of \( S \) and thus also a normalizing element of \( K[S] \). Also \( K[S]/(x_4) \cong K[\{ x_1, x_2, x_3 \}] \) and because \( \langle x_1, x_2, x_3 \rangle \) is a binomial semigroup, we get that \( \langle x_4 \rangle \) is a prime ideal of depth 3.

Now, suppose \( P \) is a prime ideal of \( K[S] \) that does not contain \( x_4 \). Equations (7) and (8) yield that
\[ I = (x_1(x_2 - x_3), (x_1 - x_2)x_3) \subseteq P. \]
In the classical ring of quotients $Q_{el}(K[S]/P)$ the element $\bar{x_4}$ is invertible (as it is regular in $K[S]/P$) and this element acts via conjugation on the set $\{\bar{x_1}, \bar{x_2}, \bar{x_3}\}$. Applying this conjugation action on the equations $\bar{x_1}\bar{x_2} = \bar{x_1}\bar{x_3} = \bar{x_2}\bar{x_3}$ yields $\bar{x_1}\bar{x_2} = \bar{x_2}\bar{x_3} = \bar{x_3}\bar{x_1}$. As $x_1x_2 = x_2x_1$ and thus $\bar{x_1}\bar{x_2} = \bar{x_2}\bar{x_1}$, we get that the monoid $\langle \bar{x_1}, \bar{x_2}, \bar{x_3} \rangle$ is abelian. It is easily verified that $\langle \bar{x_1}, \bar{x_2}, \bar{x_3} \rangle = \langle \bar{x_1} \rangle \cup \langle \bar{x_2} \rangle \cup \langle \bar{x_3} \rangle$. It follows that $K[S]/P$ is an epimorphic image of $K[S/\tau]$, where $\tau$ is the smallest congruence generated by the relations in $S$ and the extra relations $x_1x_2 = x_1x_3 = x_2x_1 = x_2x_3 = x_3x_2$. Denote the image of $x_j$ in $S/\tau$ by $y_j$. Then we get $S/\tau = ((y_1) \cup (y_2) \cup (y_3) \cup (y_1)y_2)(y_4)$. Moreover, $y_4$ acts on $T = (y_1, y_2, y_3) \subseteq S/\tau$ via an automorphism $\sigma$ of finite order. It follows that $K[S/\tau] = (K[T])[y_4, \sigma]$, a skew polynomial ring. Now the commutative semigroup $T$ is a semilattice of cancellative semigroups, each yielding an algebra which is a domain. Hence, $K[T]$ is semiprime and thus so is the skew polynomial ring $K[S/\tau]$. Moreover,

$$y_1(y_2 - y_3) = (y_1 - y_2)y_3 = 0.$$  

Thus, if $Q$ is a minimal prime ideal of the abelian algebra $K[T]$ then $Q$ contains one of the following ideals:

$$(y_1, y_3), \quad (y_1, y_1 - y_2) = (y_1, y_2), \quad (y_3, y_2 - y_3) = (y_3, y_2), \quad (y_2 - y_3, y_1 - y_3).$$

It is easily seen that each of these ideals is a prime ideal of $K[T]$ of depth 1. Hence, these are all the minimal prime ideals of $K[T]$. Under the action of $\sigma$ there are thus precisely three orbits of minimal primes in $K[T]$. Hence, the minimal $\sigma$-primes of $K[T]$ are $(y_1, y_3)$, $(y_1, y_2) \cap (y_2, y_3) = (y_2)$, and $(y_1 - y_3, y_2 - y_3)$. Note also that

$$T \cap (y_4) \cap (y_2) \cap (y_1, y_3) = \{y_1^\alpha y_2 y_3^\gamma \mid \alpha, \gamma > 0\}. \quad (9)$$

It is easily seen (and well known from standard results on $\mathbb{Z}$-graded rings) that the minimal primes of the skew polynomial algebra $K[S/\tau] = (K[T])[y_4, \sigma]$ are all ideals of the type $M[y_4, \sigma]$ with $M$ a minimal $\sigma$-prime ideal of $K[T]$. Therefore, the minimal primes of $K[S/\tau]$ are $(y_2)$, $(y_1, y_3)$, and $(y_1 - y_3, y_2 - y_3)$.

All the above implies that if $P$ is a prime ideal of $K[S]$ that does not contain $x_4$, then $P$ contains one of the following incomparable prime ideals of depth 2:

$$J + (x_2) = (x_2) = P_4, \quad J + (x_1, x_3) = (x_1, x_3) = P_3 \text{ or } P_1,$$

where $J$ is the kernel of the natural epimorphism $K[S] \to K[S/\tau]$. As all these primes are incomparable with the prime $P_2 = (x_4)$, we get that indeed $P_1, P_2, P_3, P_4$ are all the minimal prime ideals of $K[S]$. Because $P_2$ has maximal depth, $K[S]$ has dimension 3.

From (9) we get that

$$(x_4) \cap (x_2) \cap (x_1, x_3) \subseteq J + K\{x_1^\alpha x_2 x_4^\gamma \mid \alpha, \gamma > 0\} \subseteq (x_4) \cap (x_2) \cap (x_1, x_3).$$
Hence, it follows easily that

\[ S \cap (x_4) \cap (x_2) \cap (x_1, x_3) = \{ x_1^\alpha x_2 x_4^\gamma \mid \alpha, \gamma > 0 \}. \]

Since \( K[S]/(x_1 - x_2, x_2 - x_3) \cong K[Y_1, Y_4] \), a polynomial algebra in commuting variables, it also easily follows that \( x_1^\alpha x_2 x_4^\gamma = x_1^\alpha' x_2 x_4^\gamma' \) if and only if \( \alpha = \alpha' \) and \( \gamma = \gamma' \). Hence,

\[ (x_1 - x_2, x_2 - x_3) \cap K \left[ \{ x_1^\alpha x_2 x_4^\gamma \mid \alpha, \gamma > 0 \} \right] = \{0\}. \]

As

\[ P_2 \cap P_3 \cap P_4 = K \left[ \{ x_1^\alpha x_2 x_4^\gamma \mid \alpha, \gamma > 0 \} \right], \]

we indeed obtain that \( K[S] \) is semiprime.

Now earlier we have shown that \( x_1 x_2 x_4 = x_1 x_3 x_4 = x_2 x_3 x_4 \). Using these and the defining relations for \( S \), it is easy to check that \( x_1^\alpha x_2^\beta x_4^\gamma = x_1^{\alpha+\beta+\gamma-1} x_2 x_4 \) in case that at least two of the exponents \( \alpha, \beta, \gamma \) are nonzero. Hence,

\[ S = (x_1, x_2, x_3) \cup \{ x_1^\alpha x_2 x_4^\gamma, x_1 x_2^\beta x_4^\gamma, x_1^\alpha x_2^\beta x_4^\gamma, x_1^\alpha x_2 x_4^\gamma \mid \alpha, \gamma \geq 0 \}. \]

\[ \blacksquare \]

References