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Expo. Math. 24 (2006) 273–280

EXPOSITIONES MATHEMATICAE

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Contact Dehn surgery, symplectic fillings, and Property P for knots

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Received 1 July 2005; received in revised form 4 October 2005

Abstract

These are notes of a talk given at the Mathematische Arbeitstagung 2005 in Bonn. Following ideas of Özbağcı–Stipsicz, a proof based on contact Dehn surgery is given of Eliashberg's concave filling theorem for contact 3-manifolds. The role of the theorem in the Kronheimer–Mrowka proof of Property P for nontrivial knots is sketched.

MSC 2000: 57R17; 57R65; 53D35

Keywords: Contact surgery; Symplectic filling; Knots; Property P

1. Property P for knots

According to a fundamental theorem of Lickorish and Wallace from the 1960s, every closed, connected, orientable 3-manifold can be obtained by performing Dehn surgery on a link in the 3-sphere. Previous to the recent work of Perelman, which is expected to close the coffin on the Poincaré conjecture, it was a natural question for geometric topologists whether one might be able to produce a counterexample to that conjecture by a single

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0723-0869/$ - see front matter
doi:10.1016/j.exmath.2005.11.002

Dehn surgery. This led to the definition of the following property, whose name is generally regarded as a little unfortunate.

Definition. A knot K in S^3 has *Property P* if every nontrivial surgery along K yields a nonsimply connected 3-manifold.

Our knots are always understood to be smooth, or at least tame, i.e. equivalent to a smooth one.

Let me briefly recall the notion of Dehn surgery along a knot K in the 3-sphere S^3. Write $\nu K \cong S^1 \times D^2$ for a (closed) tubular neighbourhood of K. On the boundary $\partial(\nu K) \cong T^2$ of this tubular neighbourhood there are two distinguished curves (which we implicitly identify with the classes they represent in the homology group $H_1(T^2)$):

1. The meridian μ, defined as a simple closed curve that generates the kernel of the homomorphism on H_1 induced by the inclusion $T^2 \to \nu K$.
2. The preferred longitude λ, defined as a simple closed curve that generates the kernel of the homomorphism on H_1 induced by the inclusion $T^2 \to C := \overline{S^3 \backslash \nu K}$.

This preferred longitude can also be characterised by the property that it has linking number zero with K. The knot K bounds an embedded surface in S^3 (called a *Seifert surface* for K), and λ can be obtained by pushing K along that surface. For that reason, the trivialisation of the normal bundle of K defined by λ is called the *surface framing* of K.

Given an orientation of S^3, orientations of μ and λ are chosen such that (μ, λ) is a positive basis for $H_1(T^2)$, with T^2 oriented as the boundary of νK. In the contact geometric setting below, the orientation of S^3 will be the one induced by the contact structure.

Let p, q be coprime integers. The manifold $K_{p/q}$ obtained from S^3 by *Dehn surgery* along K with *surgery coefficient* $p/q \in \mathbb{Q} \cup \{\infty\}$ is defined as

$$K_{p/q} := \overline{S^3 \backslash \nu K} \cup_g S^1 \times D^2,$$

where the gluing map g sends the meridian $* \times \partial D^2$ to $p\mu + q\lambda$. The resulting manifold is completely determined by the knot and the surgery coefficient.

A simple Mayer–Vietoris argument shows that $H_1(K_{p/q}) \cong \mathbb{Z}_{|p|}$. Therefore, saying that a knot K has Property P is equivalent to

$$\pi_1(K_{1/q}) = 1 \text{ only for } q = 0.$$

(Observe that $p/q = \infty$ corresponds to a trivial surgery.)

Example. The unknot does *not* have Property P. Indeed, every $(1/q)$-surgery on the unknot yields S^3, which is seen as follows. If K is the unknot, then the closure C of $S^3 \backslash \nu K$ is also a solid torus. Write μ_C and λ_C for meridian and preferred longitude on ∂C. We may assume $\mu = \lambda_C$ and $\lambda = \mu_C$. When performing $(1/q)$-surgery on K, a solid torus is glued to C by sending its meridian μ_0 to $\mu + q\lambda = \lambda_C + q\mu_C$. Now, there clearly is a diffeomorphism of C that sends μ_C to itself and λ_C to $\lambda_C + q\mu_C$. It follows that the described surgery is equivalent to the one where we send μ_0 to $\lambda_C = \mu$, which is a trivial ∞-surgery.

In the early 1970s, Bing and Martin, as well as González-Acuña, conjectured that every nontrivial knot has Property P. By work of Kronheimer and Mrowka [1], this is now a theorem.

Theorem 1 (*Kronheimer–Mrowka*). *Every nontrivial knot in S^3 has Property P.*

Before describing the role that contact geometry has played in the proof of this theorem, I want to indicate the importance of this theorem beyond the negative statement that counterexamples to the Poincaré conjecture cannot result from a single surgery.

Proposition 2. *If two knots K, K' in S^3 have homeomorphic complements and one of the knots has Property P, then the knots are equivalent, i.e. there is a homeomorphism of S^3 mapping K to K'.*

Proof. According to a result of Edwards [2], two compact 3-manifolds with boundary are homeomorphic if and only if their interiors are homeomorphic. Thus, if $S^3 \setminus K$ is homeomorphic to $S^3 \setminus K'$, then there is a homeomorphism $\varphi\colon C \to C'$, where $C := \overline{S^3 \setminus \nu K}$ and $C' := \overline{S^3 \setminus \nu K'}$.

Suppose K has Property P. This implies that there is a unique way of attaching a solid torus $S^1 \times D^2$ to C such that the resulting manifold is the 3-sphere. Hence, φ extends to a homeomorphism $S^3 \to S^3$ mapping K to K'. $\square$

Observe that in this proof we did not actually use the full strength of Property P, but only the weaker property that nontrivial surgery along K does not yield the standard 3-sphere. This had been proved earlier (for K different from the unknot) by Gordon and Luecke [3].

Theorem 3 (*Gordon–Luecke*). *Nontrivial Dehn surgery along a nontrivial knot in S^3 never yields S^3.*

It is clear that Theorem 1 implies Theorem 3. A positive answer to the Poincaré conjecture would give the opposite implication. Either theorem, together with (the proof of) Proposition 2, implies that nontrivial knots are classified by their complement. Since the unknot is characterised by its complement being a solid torus, this statement actually holds for all knots:

Corollary 4 (*Gordon–Luecke*). *If two knots in S^3 have homeomorphic complements, then the knots are equivalent.*

2. Contact Dehn surgery

This section gives a brief report on joint work with Fan Ding [4]. Recall that a (coorientable) *contact structure* ξ on a differential 3-manifold is a tangent 2-plane field defined as the kernel of a global differential 1-form α that satisfies the nonintegrability condition $\alpha \wedge d\alpha \neq 0$ (meaning that $\alpha \wedge d\alpha$ vanishes nowhere). An example is the standard contact structure

$$\xi_{\mathrm{st}} = \ker(x\,\mathrm{d}y - y\,\mathrm{d}x + z\,\mathrm{d}t - t\,\mathrm{d}z)$$

on $S^3 \subset \mathbb{R}^4$. This can also be characterised as the complex line in the tangent bundle TS^3 of S^3 with respect to complex multiplication induced from the inclusion $TS^3 \subset T\mathbb{C}^2|_{S^3}$.

A (smooth) knot K in a contact 3-manifold (M, ξ) is called *Legendrian* if it is everywhere tangent to ξ. The normal bundle of such a knot has a canonical trivialisation, determined by a vector field along K that is everywhere transverse to ξ. This will be referred to as the *contact framing*. We now consider Dehn surgery along K with coefficient p/q as before, but we define the surgery coefficient with respect to the contact framing.

There is a dichotomy between so-called *tight* and *overtwisted* contact structures on 3-manifolds. I need the notion of tightness to give a precise statement concerning the definition of a contact structure on the surgered manifold. The actual definition of tightness, however, is irrelevant for an understanding of what follows. For more details on contact geometry see the introductory lectures by Etnyre [5], the historical survey [6], or a forthcoming *Handbook* chapter by the present author [7].

It turns out that for $p \neq 0$ one can always extend the contact structure $\xi|_{M\setminus \nu K}$ to one on the surgered manifold in such a way that the extended contact structure is tight on the glued-in solid torus $S^1 \times D^2$. Moreover, subject to this tightness condition there are but finitely many choices for such an extension, and for $p/q = 1/k$ with $k \in \mathbb{Z}$ the extension is in fact unique. These observations hinge on the fact that $\partial(\nu K)$ is a convex surface in the sense of Giroux, i.e. a surface admitting a transverse flow preserving the contact structure. On solid tori with convex boundary condition, tight contact structures have been classified by Giroux and Honda. Furthermore, one knows how to glue contact manifolds along convex surfaces, since the germ of a contact structure along a convex surface is determined by some simple data on that surface. Again I refer the reader to [5] for more details on convex surface theory.

We can therefore speak sensibly of *contact* $(1/k)$-*surgery*. The following theorem is proved in [4].

Theorem 5. *Let* (M, ξ) *be a closed, connected contact* 3-*manifold. Then* (M, ξ) *can be obtained from* (S^3, ξ_{st}) *by contact* (± 1)-*surgery along a Legendrian link.*

Remarks. (1) There is a related theorem, due to Lutz–Martinet in the early 1970s, cf. [7], saying that every (closed, orientable) 3-manifold admits a contact structure in each homotopy class of tangent 2-plane fields. The original proof is based on surgery along a link in S^3 *transverse* to ξ_{st}. For an alternative proof using Legendrian surgery see [8].

(2) From the topological point of view, surgeries with integer surgery coefficient are best, since they correspond to attaching 2-handles to the boundary of a 4-manifold. Thus, contact (± 1)-surgeries are best from both the topological and contact geometric viewpoint.

(3) If (M', ξ') is obtained from (M, ξ) by a contact $(1/k)$-surgery, one can recover (M, ξ) by a suitable contact $(-1/k)$-surgery on (M', ξ'), see [4].

(4) Contact (-1)-surgery is symplectic handlebody surgery in the sense of Eliashberg and Weinstein, cf. [8], and preserves the property of being strongly symplectically fillable (see below).

3. Symplectic fillings

Contact geometry enters the proof of Theorem 1 via the notion of symplectic fillings. Observe that a contact 3-manifold (M, ξ) is naturally oriented – the sign of the volume form $\alpha \wedge \mathrm{d}\alpha$ does not depend on the choice of 1-form α defining a given ξ; similarly, a symplectic 4-manifold (W, ω), i.e. with ω a closed 2-form satisfying $\omega^2 \neq 0$, is naturally oriented by the volume form ω^2.

Definition. (a) A compact symplectic 4-manifold (W, ω) is called a *weak* (*symplectic*) *filling* of the contact manifold (M, ξ) if $\partial W = M$ as oriented manifolds (outward normal followed by orientation of M gives orientation of W) and $\omega|_\xi \neq 0$.

(b) A compact symplectic 4-manifold (W, ω) is called a *strong* (*symplectic*) *filling* of the contact manifold (M, ξ) if $\partial W = M$ and there is a Liouville vector field X defined near ∂W, pointing outwards along ∂W, and satisfying $\xi = \ker(i_X\omega|_{TM})$. Here *Liouville vector field* means that the Lie derivative $\mathscr{L}_X\omega$, which is the same as $\mathrm{d}(i_X\omega)$ because of $\mathrm{d}\omega = 0$ and Cartan's formula, is required to equal ω.

For instance, (S^3, ξ_{st}) is strongly filled by the standard symplectic 4-disc D^4 with $\omega_{\mathrm{st}} = \mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}z \wedge \mathrm{d}t$. The Liouville vector field here is the radial vector field $X = r\partial_r/2$.

It is clear that every strong filling is also a weak filling. The converse is false: There are contact structures that are weakly but not strongly fillable; such examples are due to Eliashberg and Ding-Geiges.

The contact geometric result that allowed Kronheimer and Mrowka to conclude their proof of Property P was established by Eliashberg [9].

Theorem 6 (*Eliashberg*). *Any weak symplectic filling of a contact* 3-*manifold embeds symplectically into a closed symplectic* 4-*manifold.*

An alternative proof was given by Etnyre [10]. Both proofs rely on open book decompositions adapted (in the sense of Giroux) to contact structures. Theorem 6 being a cobordism theoretic result, it is arguably more natural to give a surgical proof. Özbağcı and Stipsicz [11] were the first to observe that such a proof, based on Theorem 5, can indeed be devised. In the remainder of this section, I shall sketch this surgical argument.

Theorem 6 is proved by showing that any contact 3-manifold has what is called a *concave* filling that can be glued to the given (convex) filling. (For instance, a strong concave filling corresponds to a Liouville vector field pointing inwards along the boundary.) Such a "cap", attached to the (convex) symplectic filling of the contact manifold, gives the desired closed symplectic manifold.

(i) *Strong fillings can be capped off*: Let (W, ω) be a strong filling of (M, ξ). By Theorem 5, there is a Legendrian link $\mathbb{L} = \mathbb{L}^- \sqcup \mathbb{L}^+$ in (S^3, ξ_{st}) such that contact (-1)-surgery along the components of $\mathbb{L}^-$ and contact $(+1)$-surgery along those of $\mathbb{L}^+$ produces (M, ξ). By Remarks (3) and (4) we can attach symplectic 2-handles to the boundary (M, ξ) of (W, ω) corresponding to contact (-1)-surgeries that undo the contact $(+1)$-surgeries along $\mathbb{L}^+$.

The result will be a symplectic manifold (W', ω') strongly filling a contact manifold (M', ξ'), and the latter can be obtained from $(S^3, \xi_{\mathrm{st}}) = \partial(D^4, \omega_{\mathrm{st}})$ by performing contact (-1)-surgeries (along $\mathbb{L}^-$) only.

A handlebody obtained from $(D^4, \omega_{\mathrm{st}})$ by attaching symplectic handles in this way is in fact a Stein filling of its boundary contact manifold, and for those a symplectic cap had been found earlier by Akbulut–Özbağcı and Lisca–Matić. The cap that fits on the Stein filling also fits on the strong filling (W', ω'), since strongly convex and strongly concave fillings of a given contact manifold can always be glued together, using the Liouville flow to define collar neighbourhoods of the boundary.

(ii) *Reduce the problem to the consideration of homology spheres only*: Let (W, ω) be a weak filling of (M, ξ). We want to attach a (weak) symplectic cobordism from (M, ξ) to some integral homology sphere Σ^3 with contact structure ξ', so as to get a weak filling of (Σ^3, ξ') containing (M, ξ) as a separating hypersurface.

We start from a contact surgery presentation of (M, ξ) as in (i). For each component L_i of $\mathbb{L}$ we choose a Legendrian knot K_i in (S^3, ξ_{st}) only linked with that component, with linking number 1. These K_i can be chosen in such a way that surgery with coefficient -1 relative to the contact framing is the same as surgery with coefficient 0 relative to the surface framing. (In case you know the term: the Thurston–Bennequin invariant of K_i can be chosen to be equal to 1). Performing these surgeries has the effect of killing the first integral homology.

I shall indicate presently that these surgeries can be performed by attaching symplectic 2-handles as in the case of a strong filling; the collection of these handles then gives the desired (weak) symplectic cobordism. The claim about the surgeries follows from Lemma 2.4 of [12], where it is shown that by a C^∞-small perturbation of ξ in a neighbourhood of the K_i one can achieve that (W, ω) is a strong filling near these Legendrian knots. By Gray stability, cf. [7], the perturbed contact structure is isotopic to ξ. (The argument in [12] is rather sketchy; full details can be found in a forthcoming book on contact topology by the present author.)

(iii) *Pass from a weak filling of a homology sphere to a strong filling*: We begin with a weak filling (W, ω) of an integral homology sphere (Σ^3, ξ), for instance the one obtained in (ii); beware that we retain the original notation for the filling. We want to modify ω in a collar neighbourhood $\Sigma^3 \times [0, 1]$ of the boundary $\Sigma^3 \equiv \Sigma^3 \times \{1\}$ such that the resulting symplectic manifold is a strong filling of the new induced contact structure on the boundary. By (i) this can then be capped off.

Since $H^2(\Sigma^3) = 0$, we can write $\omega = \mathrm{d}\eta$ with some 1-form η in a collar neighbourhood as described. (We see that it would be enough to have Σ^3 a rational homology sphere.) Choose a 1-form α on Σ^3 with $\xi = \ker \alpha$ and $\alpha \wedge \omega|_{T\Sigma^3} > 0$, which is possible for a weak filling. Then set

$$\tilde{\omega} = \mathrm{d}(f\eta) + \mathrm{d}(g\alpha)$$

on $\Sigma^3 \times [0, 1]$, where the smooth functions $f(t)$ and $g(t)$, $t \in [0, 1]$, are chosen as follows: Fix a small $\varepsilon > 0$. Choose $f: [0, 1] \to [0, 1]$ identically 1 on $[0, \varepsilon]$ and identically 0 near $t = 1$. Choose $g: [0, 1] \to \mathbb{R}_0^+$ identically 0 near $t = 0$ and with $g'(t) > 0$ for $t > \varepsilon/2$.

We compute

$$\tilde{\omega} = f'\,\mathrm{d}t \wedge \eta + f\omega + g'\,\mathrm{d}t \wedge \alpha + g\,\mathrm{d}\alpha,$$

whence

$$\begin{aligned}\tilde{\omega}^2 = {} & 2ff'\,\mathrm{d}t\wedge\eta\wedge\omega + 2f'g\,\mathrm{d}t\wedge\eta\wedge\mathrm{d}\alpha + f^2\omega^2\\ & + 2fg'\omega\wedge\mathrm{d}t\wedge\alpha + 2fg\omega\wedge\mathrm{d}\alpha + 2gg'\,\mathrm{d}t\wedge\alpha\wedge\mathrm{d}\alpha.\end{aligned}$$

The terms appearing with the factors f^2, fg' and gg' are positive volume forms. By choosing g small on $[0,\varepsilon]$ and g' large compared with $|f'|$ and g on $[\varepsilon,1]$, one can ensure that these positive terms dominate the three terms we cannot control. Then $\tilde{\omega}$ is a symplectic form on the collar and, in terms of the coordinate $s := \log g(t)$, the symplectic form looks like $\mathrm{d}(\mathrm{e}^s\alpha)$ near the boundary, with Liouville vector field ∂_s.

4. Proof of Property P for nontrivial knots

Here is a very rough sketch of the proof by Kronheimer and Mrowka. It relies heavily on pretty much everything known under the sun about gauge theory. All relevant references can be found in [1].

Let K be a nontrivial knot. It had been proved earlier by Culler–Gordon–Luecke–Shalen that $\pi_1(K_{1/q})$ is nontrivial for $q \notin \{0, \pm 1\}$. It therefore suffices to find a nontrivial homomorphism $\pi_1(K_1) \to \mathrm{SO}(3)$.

Arguing by contradiction, we assume that no such homomorphism exists. This implies the vanishing of the instanton Floer homology group $\mathrm{HF}(K_1)$. By the Floer exact triangle one finds that the group $\mathrm{HF}(K_0)$ vanishes likewise, and so does the Fukaya–Floer homology group.

For K nontrivial, results of Gabai say that K_0 is different from $S^1 \times S^2$ and admits a taut 2-dimensional foliation. Eliashberg and Thurston, in their theory of confoliations, deduce from this the existence of a symplectic structure on $K_0 \times [-1, 1]$ weakly filling contact structures on the boundary components. According to Theorem 6, by capping off these boundaries we find a closed symplectic 4-manifold V containing K_0 as a separating hypersurface (and satisfying some mild cohomological conditions).

Now, on the one hand, the Donaldson invariants of V can be expressed as a pairing on the Fukaya–Floer homology group of K_0 and therefore have to vanish.

On the other hand, results of Taubes say that the Seiberg–Witten invariants of V are nontrivial. By a conjecture of Witten, proved in the relevant case by Feehan–Leness, the Donaldson invariants are likewise nontrivial. This contradiction proves Theorem 1.

Acknowledgments

I thank Fan Ding for his eagle-eyed reading of the original manuscript.

Previous to the Arbeitstagung, I presented similar talks in seminars at the ETH Zürich, the Universidad Autónoma de Madrid, and Aarhus Universitet. I thank Dietmar Salamon, Jesús Gonzalo and Jørgen Ellegaard Andersen for the respective invitations. The stay in Madrid was partially supported by Grant No. GE 1245/1-1 of the Deutsche Forschungsgemeinschaft within the framework of the Schwerpunktprogramm 1154 "Globale Differentialgeometrie".

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