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Mean Estimation of Brownian Sheet

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Abstract—We observe in a domain $G \subset \mathbb{R}_0^{+^2}$ a standard Brownian sheet disturbed by an unknown constant (mean). We determine explicitly, the maximum-likelihood estimator of the mean by the help of the method of stochastic partial differential equations.

Keywords—Brownian sheet, Maximum-likelihood, Gaussian random fields, Mean estimation, Stochastic partial differential equations.

1. INTRODUCTION

The Brownian sheet is one of the most important examples of the Gaussian random fields. In many papers, some fundamental results were proved on the properties of these fields (see [1–3] and others). For example, Rozanov proposed a method to study Brownian sheet by the help of linear stochastic partial differential equations (in the sequel LSPDE).

We say $W(t_1, t_2)(t_1, t_2 \geq 0)$ is a standard Brownian sheet if it is a Gaussian random field with mean-zero

$$\mathbf{E}W(t_1, t_2) = 0, \qquad \forall (t_1, t_2) \in \mathbb{R}_0^{+^2},$$

and with covariance function

$$\mathbf{cov}\,(W(t_1, t_2), W(s_1, s_2)) = \min(t_1, s_1) \cdot \min(t_2, s_2).$$

In this paper, we discuss the problem of the estimation of an unknown mean. We assume, that instead of the standard Brownian sheet $W(t_1, t_2)$, we observe in a domain $G \subset \mathbb{R}_0^{+^2} = \{(t_1, t_2) : t_1, t_2 \geq 0\}$ the Brownian sheet with mean m:

$$\widetilde{W} = W(t_1, t_2) + m, (t_1, t_2) \in G,$$

where m is an unknown constant. Generally, a Brownian sheet is given by $\sigma W + m$, $\sigma > 0$, $-\infty < m < \infty$. It is well-known that σ may be estimated with probability 1 [4].

The main result of our paper is the following (assuming a few smooth properties of domain G).

The maximum-likelihood estimator of m depends on the values and the generalized normal derivatives of the field only on a part of the boundary of G.

In Section 2, we give shortly the results corresponding to the Radon-Nikodym derivative of Gaussian measures with different means. In Section 3, we describe Rozanov's scheme of LSPDE. In Section 4, we determine the maximum-likelihood (in the sequel ML) estimator of the mean of fields given by LSPDE. Applying this result, we prove in the last section the main theorem of our work with respect to the ML estimator of the mean of a Brownian sheet.

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2. EQUIVALENCE OF GAUSSIAN MEASURES

The systematic investigation of general conditions of equivalence of Gaussian measures was started with the works of Feldman [5] and Hajek [6]. Earlier such problems were discussed by Cameron and Martin. In the sequel, we describe some important facts about orthogonality and equivalence of Gaussian measures, which we have to use. We follow the second chapter of [7].

Let $(\Omega, \mathcal{A})$ be a measurable space with two probability measures $\mathbf{P}$ and $\mathbf{P}_1$, and $\xi(\omega, t)$, $\omega \in \Omega, t \in T$, be a (random) function on the parameter space T. Assume on the space $(\Omega, \mathcal{A}, \mathbf{P})$ ξ is a Gaussian random function with zero mean

$$\mathbf{E}_{\mathbf{P}}\xi(t) = 0, \qquad t \in T,$$

and covariance function B

$$\mathbf{cov}(\xi(t), \xi(s)) = B(t, s), \qquad t, s \in T.$$

Further, on the space $(\Omega, \mathcal{A}, \mathbf{P}_1)$ ξ is also a Gaussian random function, but with a different expectation function

$$\mathbf{E}_{\mathbf{P}_1}\xi(t) = a(t), \qquad t \in T,$$

and with the same covariance function B.

To give sufficient and necessary conditions for the equivalence of measures $\mathbf{P}$ and $\mathbf{P}_1$, we introduce the linear space $\mathbf{U}$ and the Hilbert space $\overline{\mathbf{U}}$:

$$\mathbf{U} := \left\{ u : \Omega \to \mathrm{R},\ u(\omega) = \sum_{k=1}^{n} c_k \xi(\omega, t_k),\ n \in \mathrm{N},\ c_k \in \mathbb{R},\ t_k \in T \right\},$$

and $\overline{\mathbf{U}}$ is the closure of $\mathbf{U}$ by the help of scalar product

$$\langle u, v \rangle = \int_\Omega u(\omega) v(\omega)\, d\mathbf{P}.$$

The following statement is true.

STATEMENT. (See [7, p. 33].) The measures $\mathbf{P}$ and $\mathbf{P}_1$ are equivalent, if and only if there exists $\psi \in \overline{\mathbf{U}}$ such that

$$a(t) = \int_\Omega \xi(\omega, t) \psi(\omega)\, d\mathbf{P}, \qquad t \in T. \tag{2.1}$$

In this case, the Radon-Nikodym derivative is given by

$$\frac{d\mathbf{P}_1}{d\mathbf{P}}(\omega) = \frac{e^{\psi(\omega)}}{e^{\langle \psi, \psi \rangle / 2}}.$$

Particularly if $\mathbf{P}_m$, $m \in \mathbb{R}$, are probability measures, and

$$\mathbf{E}_{\mathbf{P}_m}\xi(t) = m a(t), \qquad t \in T,$$

then the measures $\mathbf{P}$ and $\mathbf{P}_m$ are equivalent, if and only if there exists an element $\psi \in \overline{\mathbf{U}}$ which satisfies the equation (2.1), and the Radon-Nikodym derivative and ML estimator $\widehat{m}$ of m, respectively, are the following:

$$\frac{d\mathbf{P}_m}{d\mathbf{P}}(\omega) = \frac{e^{m\psi(\omega)}}{e^{m^2 \langle \psi, \psi \rangle / 2}}, \tag{2.2}$$

$$\widehat{m} = \frac{\psi}{\langle \psi, \psi \rangle}. \tag{2.3}$$

We can apply this Statement, also in the case when the parameter space T is a Hilbert space $\mathbf{X}$ with scalar product $\langle .,. \rangle_{\mathbf{X}}$ and $\xi(\omega, x)$, $\omega \in \Omega$, $x \in \mathbf{X}$ is a Gaussian linear continuous random functional on $\mathbf{X}$, with covariance operator B:

$$\mathbf{E}_{\mathbf{P}_m}(\xi, x) = m\langle x, a\rangle_{\mathbf{X}}, \qquad m \in \mathbb{R}, \qquad x \in \mathbf{X}, \quad a \in \mathbf{X},$$
$$\mathbf{cov}_{\mathbf{P}_m}((\xi, x), (\xi, y)) = \langle x, By\rangle_{\mathbf{X}}, \qquad m \in \mathbb{R}, \quad x,\, y \in \mathbf{X}.$$

(ξ, x) means the value of the random functional ξ at x. If there exists an element $z \in \mathbf{X}$, such that

$$Bz = a,$$

then the measures $\mathbf{P}_m$ and $\mathbf{P}$ ($\mathbf{P} = \mathbf{P}_0$) are equivalent, and the formulae (2.2) and (2.3) remain valid

$$\frac{d\mathbf{P}_m}{d\mathbf{P}}(\omega) = \frac{e^{m(\xi,z)}}{e^{m^2\langle z,Bz\rangle/2}}, \tag{2.4}$$
$$\widehat{m} = \frac{(\xi, z)}{\langle z,\, Bz\rangle}. \tag{2.5}$$

3. LINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

In the eighties, Rozanov proposed a new method to study generalized Gaussian fields given by LSPDE. In his paper [1], one can find a detailed description of these equations. In the sequel, we describe his scheme.

Let $G_0 \subseteq \mathbb{R}^q$ be a domain, and

$$L = \sum_{|\alpha| \le p} a_\alpha(.)D^\alpha, \qquad (a_\alpha \in C^\infty(G_0)),$$

be a nondegenerate linear differential operator. The term "nondegenerate" means that if $\varphi_n \in C_0^\infty(G_0)$ and $\lim_{n\to\infty} \|L\varphi_n\|_{L_2} = 0\, (L_2 = L_2(G_0))$, then for every $\psi \in C_0^\infty(G_0)$,

$$\lim_{n\to\infty} \langle \varphi_n, \psi\rangle_{L_2} = 0.$$

The operator

$$L^* = \sum_{|\alpha|\le p} (-1)^{|\alpha|} D^\alpha(a_\alpha .)$$

is a formal conjugate differential operator. The Hilbert space of test functions

$$\mathbf{X} = \left\{x \in D'(G_0) : \varphi_n \in C_0^\infty(G_0),\ \lim_{n\to\infty} \|L\varphi_n\|_{L_2} = 0 \Rightarrow (\varphi_n, x) \to 0\right\}$$

is the closure of $L^*LC_0^\infty(G_0)$ by the help of scalar product

$$\langle L^*L\varphi, L^*L\psi\rangle_{\mathbf{X}} = \langle L^*L\varphi, \psi\rangle_{L_2}, \qquad \varphi, \psi \in C_0^\infty(G_0).$$

If $G \subseteq G_0$, we can define the following subspace:

$$\mathbf{X}(G) = \left\{x \in \mathbf{X} : \operatorname{supp} x \subseteq \overline{G}\right\}.$$

Then, the following identity is true:

$$\mathbf{X}(G) = \overline{C_0^\infty(G)} = L^*L_2(G) + \mathbf{X}^+(\Gamma), \tag{3.1}$$

where $\Gamma = \partial G$ and $\mathbf{X}^+(\Gamma) \subseteq \mathbf{X}(\Gamma)$.

Now, it is possible to speak about LSPDE. Let

$$\eta : L_2(G_0) \to \mathbf{H},$$

be a linear continuous Gaussian functional and $\mathbf{E}(\eta, f) = 0$, $\mathbf{E}(\eta, f)(\eta, g) = \langle f, g\rangle_{L_2}, \forall f, g \in L_2(G_0)$ (η is a standard white noise), where $\mathbf{H} = \mathcal{L}_2(\Omega, \mathcal{A}, \mathbf{P})$ is a Hilbert space of random variables with finite second moments.

Let us say that ξ is a solution of the following equation

$$L\xi = \eta, \qquad \text{in } G_0, \tag{3.2}$$

if $\xi : \mathbf{X} \to \mathbf{H}$ is a linear, continuous Gaussian functional and

$$(\xi, L^*\varphi) = (\eta, \varphi), \qquad \forall \varphi \in C_0^\infty(G_0).$$

As $L^*C_0^\infty(G_0)$ is dense in $\mathbf{X}$, the equation (3.2) can be solved and the solution is unique. From the identity (3.1), we get that if $G \subset G_0$, $\Gamma = \partial G$, and $x \in \mathbf{X}(\Gamma)$, then there exists $\varphi_n \in C_0^\infty(G)$ such that

$$\mathbf{E}\left((\xi, x) - (\xi, \varphi_n)\right)^2 = \|x - \varphi_n\|_{\mathbf{X}}^2 \to 0. \tag{3.3}$$

If $G \subset G_0$, setting $\eta : L_2(G) \to \mathbf{H}$ is a standard Gaussian white noise, further $\xi_\Gamma : \mathbf{X}^+(\Gamma) \to \mathbf{H}$ is a linear, continuous Gaussian functional with zero mean ($\mathbf{E}(\xi_\Gamma, x) = 0$).

Let us say that ξ is a solution of the following random boundary problem:

$$L\xi = \eta, \qquad \text{in } G, \tag{3.4}$$

$$(\xi, x) = (\xi_\Gamma, x), \qquad x \in \mathbf{X}^+(\Gamma), \tag{3.5}$$

if $\xi : \mathbf{X}(G) \to \mathbf{H}$ is a linear continuous Gaussian functional

$$(\xi, L^*\varphi) = (\eta, \varphi), \qquad \forall \varphi \in C_0^\infty(G),$$

and $(\xi, x) = (\xi_\Gamma, x)$, for all $x \in \mathbf{X}^+(\Gamma)$.

In [1], it is proven that the problem given by (3.4),(3.5) can be solved and the solution is unique

$$(\xi, x) = (\eta, f) + (\xi_\Gamma, x_\Gamma),$$

where $x \in \mathbf{X}(G)$, $f \in L_2(G)$, $x_\Gamma \in \mathbf{X}^+(\Gamma)$, and

$$x = L^* f + x_\Gamma.$$

4. ML ESTIMATION OF THE MEAN

In this section, we determine the ML estimate of the mean if the observable Gaussian field satisfies LSPDE.

Let G_0, G, and L be as in Section 3, $(\Omega, \mathcal{A})$ is a measurable space and

$$(\xi, x) = \xi(\omega, x), \qquad \omega \in \Omega, \quad x \in \mathbf{X}(G)$$

is a (random) functional on the Hilbert space $\mathbf{X}(G)$. On the probability space $(\Omega, \mathcal{A}, \mathbf{P}_m)$, $m \in \mathbf{R}$,

$$\xi : \mathbf{X}(G) \to \mathcal{L}_2(\Omega, \mathcal{A}, \mathbf{P}_m)$$

is a linear continuous Gaussian functional and

$$\xi = \zeta_m + m \cdot u, \tag{4.1}$$

where ζ_m is the solution of the random boundary problem (3.4),(3.5)

$$L\zeta_m = \eta_m, \qquad \text{in } G,$$
$$(\zeta_m, x) = (\zeta_{m,\Gamma}, x), \qquad x \in \mathbf{X}^+(\Gamma),$$

η and $\zeta_{m,\Gamma}$ are independent functionals, and the covariance structure of $\zeta_{m,\Gamma}$ does not depend on m. u is the solution of the following deterministic boundary problem:

$$Lu = h, \qquad \text{in } G \ (h \in L_2(G)), \tag{4.2}$$
$$(u, x) = (u_\Gamma, x), \qquad x \in \mathbf{X}^+(\Gamma). \tag{4.3}$$

In this case, we get for $x = L^*f + x_\Gamma$, $y = L^*g + y_\Gamma \in \mathbf{X}(G)$ $(f, g \in L_2(G),\ x_\Gamma, y_\Gamma \in \mathbf{X}^+(\Gamma))$

$$\begin{aligned} m\langle x, a\rangle_{\mathbf{X}} &= \mathbf{E}_{\mathbf{P}_m}(\xi, x) = \mathbf{E}_{\mathbf{P}_m}\left[(\zeta, x) + m(u, x)\right] = m(u, x) = m\left[\langle f, h\rangle_{L_2(G)} + (u_\Gamma, x_\Gamma)\right], \\ \langle x, By\rangle_{\mathbf{X}} &= \mathbf{cov}_{\mathbf{P}_m}\left[(\xi, x), (\xi, y)\right] = \mathbf{cov}_{\mathbf{P}_m}\left[(\zeta, x), (\zeta, y)\right] \\ &= \langle f, g\rangle_{L_2(G)} + \mathbf{cov}_{\mathbf{P}_m}\left[(\zeta_\Gamma, x), (\zeta_\Gamma, y)\right] = \langle f, g\rangle_{L_2(G)} + \langle x_\Gamma, By_\Gamma\rangle_{\mathbf{X}}. \end{aligned}$$

This means that there exists $z \in \mathbf{X}(G)$ such that

$$Bz = a,$$

if and only if there exists $z_\Gamma \in \mathbf{X}^+(\Gamma)$ for which

$$\langle x_\Gamma, B_\Gamma z_\Gamma\rangle_{\mathbf{X}} = (u_\Gamma, x_\Gamma), \qquad \forall x_\Gamma \in \mathbf{X}^+(G), \tag{4.4}$$

and in this case

$$z = L^*h + z_\Gamma.$$

From here and (2.4),(2.5), we get

$$\widehat{m} = \frac{(\xi, z)}{\langle z, Bz\rangle_{\mathbf{X}}} = \frac{(\xi, L^*h) + (\xi, z_\Gamma)}{\|h\|^2_{L_2} + \langle z_\Gamma, B_\Gamma z_\Gamma\rangle_{\mathbf{X}}} = \frac{(\xi, L^*h) + (\xi, z_\Gamma)}{\|h\|^2_{L_2} + (u_\Gamma, z_\Gamma)}.$$

So, we proved the following theorem.

THEOREM 1. *The Radon-Nikodym derivative of Gaussian measures $\mathbf{P}_m$ and $\mathbf{P}$ corresponding to the field ξ which is given by (4.1), one can get if there exists a boundary functional $z_\Gamma \in \mathbf{X}^+(G)$ which fulfills (4.4). In this case, the maximum likelihood estimator of the unknown m is the following*

$$\widehat{m} = \frac{(\xi, L^*h) + (\xi, z_\Gamma)}{\|h\|^2_{L_2} + (u_\Gamma, z_\Gamma)}.$$

The distribution of $\widehat{m}$ is Gaussian. Let us consider the moments of the ML estimator,

$$\begin{aligned} \mathbf{E}_{\mathbf{P}_m}\widehat{m} &= \frac{\mathbf{E}_{\mathbf{P}_m}(\xi, L^*h) + \mathbf{E}_{\mathbf{P}_m}(\xi, z_\Gamma)}{\|h\|^2_{L_2} + (u_\Gamma, z_\Gamma)} \\ &= \frac{\mathbf{E}_{\mathbf{P}_m}(\zeta, L^*h) + \mathbf{E}_{\mathbf{P}_m}(\zeta, z_\Gamma) + m(u, L^*h) + m(u, z_\Gamma)}{\|h\|^2_{L_2} + (u_\Gamma, z_\Gamma)} = m, \end{aligned}$$

which means $\widehat{m}$ is unbiased;

$$D^2_{\mathbf{P}_m}\widehat{m} = \mathbf{E}_{\mathbf{P}_m}\left[\frac{(\zeta, L^*h) + (\zeta, z_\Gamma)}{\|h\|^2_{L_2} + (u_\Gamma, z_\Gamma)}\right]^2 = \frac{\|h\|^2_{L_2} + (u_\Gamma, z_\Gamma)}{\left[\|h\|^2_{L_2} + (u_\Gamma, z_\Gamma)\right]^2} = \frac{1}{\|h\|^2_{L_2} + (u_\Gamma, z_\Gamma)}.$$

EXAMPLE 1. Let us consider the case of the well-known stationary Ornstein-Uhlenbeck process. We can observe the process on the interval $[0,T]$:

$$\xi(t) = \zeta(t) + m, \qquad 0 \le t \le T,$$

where

$$d\zeta(t) = -\rho\zeta(t)\,dt + dw(t), \qquad t > 0,$$

where w is a standard Wiener-process. In this case $G_0 = (-\infty,\infty)$, $G = (0,T)$, $\mathbf{X}^+(G) = \langle\delta_0\rangle$ (δ_0 is the Dirac function), $u = 1$ and the equations (3.4),(3.5) has the form

$$\left(\frac{\partial}{\partial t} + \rho\right)\zeta = \eta, \qquad \text{in } (0,T),$$
$$(\zeta,\delta_0) = \zeta(0) = \zeta_0,$$

and the equations (4.2),(4.3) give

$$\left(\frac{\partial}{\partial t} + \rho\right)u = \rho, \qquad \text{in } (0,T),$$
$$(u,\delta_0) = u(0) = 1.$$

In the stationary case

$$D^2\zeta(0) = D^2\zeta_0 = \frac{1}{2\rho},$$

from equation (4.4), we have

$$\langle\delta_0, B_\Gamma(\alpha\delta_0)\rangle_{\mathbf{X}} = \alpha\langle\delta_0,\delta_0\rangle_{\mathbf{X}} = \alpha D^2\zeta(0) = (u,\delta_0) = 1.$$

From here, we get

$$z_\Gamma = 2\rho\delta_0$$

and

$$z = L^*\rho + 2\rho\delta_0.$$

Finally, we can calculate $L^*\rho$:

$$(L^*\rho,\varphi) = \langle L\varphi,\rho\rangle_{L_2} = \int_0^T (\varphi'(t) + \rho\varphi(t))\,\rho\,dt = \rho^2\int_0^T \varphi(t)\,dt + \rho\,(\varphi(T) - \varphi(0)).$$

This means

$$L^*\rho = \rho^2 + \rho(\delta_T - \delta_0)$$

and

$$z = \rho^2 + \rho(\delta_T + \delta_0).$$

From here and from Theorem 1, we get that the ML estimator of m is the following

$$\widehat{m} = \frac{\rho(\xi(0) + \xi(T)) + \rho^2\int_0^T \xi(t)\,dt}{\rho^2 T + 2\rho} = \frac{\xi(0) + \xi(T) + \rho\int_0^T \xi(t)\,dt}{\rho T + 2}.$$

This formula is the same as the well-known Grenander formula (see [8]).

5. THE BROWNIAN SHEET

The standard Brownian sheet—as a generalized field—satisfies (3.2) type equation with operator $L = L^* = \frac{\partial^2}{\partial t_1 \partial t_2}$:

$$\frac{\partial^2 W}{\partial t_1 \partial t_2} = \eta, \qquad \text{in } \mathbb{R}_0^{+^2}.$$

If $G \subset \mathbb{R}_0^{+^2}$ is an open domain, $\Gamma_0 \subset \partial G$ is a part of the boundary, where

$$\begin{aligned} \Gamma_0 &= \{(t_1,t_2) : a_1 \le t_1 \le b_1,\ t_2 = \gamma(t_1)\} \\ &= \{(t_1,t_2) : a_2 \le t_2 \le b_2,\ t_1 = \gamma^{-1}(t_2)\}, \end{aligned} \tag{5.1}$$

where γ is twice as differentiable strictly monotone decreasing function, then (see [1]) the following functionals are elements of the "boundary space" $\mathbf{X}(\Gamma)$.

(i) The Dirac functions (functionals)

$$\delta_{(s_1,s_2)} = \left(\delta_{(s_1,s_2)}, \varphi\right) = \varphi(s_1,s_2), \qquad (s_1,s_2) \in \Gamma_0,$$

$\delta_{(s_1,s_2)} = L^* g = \frac{\partial^2 g}{\partial t_1 \partial t_2}$, where

$$g(t_1,t_2) = \begin{cases} 1, & (t_1,t_2) \in [0,s_1] \times [0,s_2], \\ 0, & (t_1,t_2) \notin [0,s_1] \times [0,s_2]. \end{cases}$$

(ii) The weighted Dirac functionals

$$\begin{aligned} x = (x,\varphi) &= \int_{\Gamma_0} y(s)\varphi(s)\, ds, \\ y(s_1,\gamma(s_1)) &= \frac{h(s_1)}{\sqrt{1+\gamma'(s_1)^2}}, \qquad (h \in L_2(a_1,b_1)), \end{aligned} \tag{5.2}$$

$x = L^* g = \frac{\partial^2 g}{\partial t_1 \partial t_2}$, where

$$g(t_1,t_2) = \begin{cases} \displaystyle\int_{t_1}^{\gamma^{-1}(t_2)} h(s_1)\, ds_1, & a_1 < t_1 < b_1, \quad a_2 < t_2 < \gamma(t_1), \\ \displaystyle\int_{t_1}^{b_1} h(s_1)\, ds_1, & a_1 < t_1 < b_1, \quad 0 < t_2 < a_2, \\ \displaystyle\int_{a_1}^{\gamma^{-1}(t_2)} h(s_1)\, ds_1, & 0 < t_1 < a_1, \quad a_2 < t_2 < b_2, \\ \displaystyle\int_{a_1}^{b_1} h(s_1)\, ds_1, & 0 < t_1 < a_1, \quad 0 < t_2 < a_2, \\ 0, & \text{otherwise}. \end{cases} \tag{5.3}$$

(iii) The "derivative" functionals

$$x = (x,\varphi) = \int_{\Gamma_0} y(s) \frac{\partial \varphi}{\partial t_1}(s)\, ds, \qquad y(s_1,\gamma(s_1)) = \frac{f_1(s_1)}{\sqrt{1+\gamma'(s_1)^2}}, \qquad (f_1 \in L_2(a_1,b_1)),$$

$x = L^* g = \frac{\partial^2 g}{\partial t_1 \partial t_2}$, where

$$g(t_1,t_2) = \begin{cases} f_1(t_1), & a_1 < t_1 < b_1, \quad 0 < t_2 < \gamma(t_1), \\ 0, & \text{otherwise}. \end{cases}$$

Similarly, we get

$$x = (x,\varphi) = \int_{\Gamma_0} y(s)\frac{\partial \varphi}{\partial t_2}(s)\,ds, \qquad y(s_1,\gamma(s_1)) = \frac{f_2(\gamma(s_1))}{\sqrt{1+1/\gamma'(s_1)^2}}, \qquad (f_2 \in L_2(a_2,b_2)),$$

$x = L^* g = \frac{\partial^2 g}{\partial t_1 \partial t_2}$, where

$$g(t_1,t_2) = \begin{cases} f_2(t_2), & 0 < t_1 < \gamma^{-1}(t_2), \quad a_2 < t_2 < b_2, \\ 0, & \text{otherwise.} \end{cases}$$

(iv) The "normal derivative" functionals

$$x = (x,\varphi) = \int_{\Gamma_0} y(s)\frac{\partial \varphi}{\partial n}(s)\,ds,$$

$x = L^*(g+f) = \frac{\partial^2 (g+f)}{\partial t_1 \partial t_2}$, where

$$g(t_1,t_2) = \begin{cases} y\left(t_1,\gamma(t_1)\right)\gamma'(t_1), & a_1 < t_1 < b_1, \quad 0 < t_2 < \gamma(t_1), \\ 0, & \text{otherwise,} \end{cases} \tag{5.4}$$

$$f(t_1,t_2) = \begin{cases} \dfrac{y\left(\gamma^{-1}(t_2),t_2\right)}{\gamma'\left(\gamma^{-1}(t_2)\right)}, & 0 < t_1 < \gamma^{-1}(t_2), \quad a_2 < t_2 < b_2, \\ 0, & \text{otherwise.} \end{cases} \tag{5.5}$$

We say that $(W,x) = \int_{\Gamma_0} y(s)\frac{\partial W}{\partial n}(s)\,ds$ is the weighted generalized normal derivative of the Brownian sheet W. A similar definition can be found in [9].

In the following, we assume that we observe not a standard Brownian sheet in a domain G, but a Brownian sheet $\widetilde{W}$ with mean m. In the sequel, we determine the ML estimator of this mean.

Thus, we assume that on the probability space $(\Omega, \mathcal{A}, \mathbf{P}_m)$, $m \in \mathbb{R}$,

$$\widetilde{W}(t_1,t_2) = W_m(t_1,t_2) + m, \qquad (t_1,t_2) \in G,$$

where W_m means a standard Brownian sheet. We assume that G has the following form:

$$G \subseteq \{(t_1,t_2) : a_1 \le t_1 \le b_1,\ \gamma(t_1) \le t_2\} \cup \{(t_1,t_2) : b_1 \le t_1, a_2 \le t_2\}, \tag{5.6}$$

where the function γ determines the part of boundary of G (see (5.1)). The main result of our work is the following.

THEOREM 2. *Let us observe a Brownian sheet with an unknown mean m in the domain G, which satisfies condition (5.6). Then, the Gaussian measures corresponding to different means are equivalent and the maximum-likelihood estimator of the mean depends on the values and the generalized normal derivatives of the field $\widetilde{\mathbf{W}}$ only on the part Γ_0 of the boundary of G, where Γ_0 is given by (5.1)*

$$\widehat{m} = d_1\widetilde{\mathbf{W}}(a_1,b_2) + d_2\widetilde{\mathbf{W}}(b_1,a_2) + \int_{\Gamma_0}\left(y_1(s)\widetilde{\mathbf{W}}(s) + y_2(s)\frac{\partial \widetilde{\mathbf{W}}}{\partial n}(s)\right)ds. \tag{5.7}$$

PROOF OF THEOREM 2. If $x \in \mathbf{X}(G)$, then from (3.3), we get that there exists $\varphi_n \in C_0^\infty(G)$ such that

$$\mathbf{E}\left(\left(\widetilde{\mathbf{W}},x\right) - \left(\widetilde{\mathbf{W}},\varphi_n\right)\right)^2 \underset{n\to\infty}{\longrightarrow} 0.$$

As $(\widetilde{\mathbf{W}}, \varphi_n) = \int_G \widetilde{\mathbf{W}}(t)\varphi_n(t)\,dt$, thus there exists $t(k) \in G$, $a_k \in \mathbf{R}$ such that

$$\mathbf{E}\left(\left(\widetilde{\mathbf{W}}, x\right) - \sum_{k=1}^{n} a_k \widetilde{\mathbf{W}}(t(k))\right)^2 \underset{n\to\infty}{\longrightarrow} 0. \tag{5.8}$$

Thus, if there exists $x^* \in \mathbf{X}(G)$ one gets

$$\mathbf{E}\left(\left(\widetilde{\mathbf{W}}, x^*\right)\widetilde{\mathbf{W}}(t)\right) = 1, \qquad \forall t \in G, \tag{5.9}$$

then from the formula (2.1) of the Statement the Gaussian measures $\mathbf{P}_m$ are equivalent, and from (2.3) the ML estimator of m is the following:

$$\widehat{m} = \frac{\left(\widetilde{\mathbf{W}}, x^*\right)}{D^2_{\mathbf{P}_0}\left(\left(\widetilde{\mathbf{W}}, x^*\right)\right)}. \tag{5.10}$$

On the basis of Theorem 1, we may assume that $x^* \in \mathbf{X}(G)$ which satisfies the identity (5.9) is an element of the boundary subspace $\mathbf{X}(\Gamma)$. For this reason, we try to find this x^* in the following form:

$$x^* = L^*g = \alpha\delta_{(a_1,b_2)} + \beta\delta_{(b_1,a_2)} + L^*(g_1 + g_2 + g_3), \tag{5.11}$$

where g_1 is a function of form (5.4), g_2 is a function of form (5.5), and g_3 is a function of form (5.3):

$$g_1(t_1,t_2) = \begin{cases} g_1(t_1), & a_1 < t_1 < b_1, \quad 0 < t_2 < \gamma(t_1), \\ 0, & \text{otherwise}, \end{cases} \tag{5.12}$$

$$g_2(t_1,t_2) = \begin{cases} g_2(t_2), & 0 < t_1 < \gamma^{-1}(t_2), \quad a_2 < t_2 < b_2, \\ 0, & \text{otherwise}, \end{cases} \tag{5.13}$$

where

$$g_2(\gamma(t_1)) = \frac{g_1(t_1)}{(\gamma'(t_1))^2}, \qquad a_1 < t_1 < b_1, \tag{5.14}$$

$$g_3(t_1,t_2) = \begin{cases} \displaystyle\int_{t_1}^{\gamma^{-1}(t_2)} h(s_1)\,ds_1, & a_1 < t_1 < b_1, \quad a_2 < t_2 < \gamma(t_1), \\ \displaystyle\int_{t_1}^{b_1} h(s_1)\,ds_1, & a_1 < t_1 < b_1, \quad 0 < t_2 < a_2, \\ \displaystyle\int_{a_1}^{\gamma^{-1}(t_2)} h(s_1)\,ds_1, & 0 < t_1 < a_1, \quad a_2 < t_2 < b_2, \\ \displaystyle\int_{a_1}^{b_1} h(s_1)\,ds_1, & 0 < t_1 < a_1, \quad 0 < t_2 < a_2, \\ 0, & \text{otherwise}. \end{cases} \tag{5.15}$$

We can write the identity (5.9) in the following form:

$$\begin{aligned} \mathbf{E}\left(\left(\widetilde{\mathbf{W}}, x^*\right)\widetilde{\mathbf{W}}(t)\right) &= \langle g, \chi((0,t_1)\times(0,t_2))\rangle_{L_2} \\ &= \int_0^{t_1}\int_0^{t_2} g(s_1,s_2)\,ds_2\,ds_1 = 1, \qquad \forall t = (t_1,t_2) \in G. \end{aligned}$$

As

$$\operatorname{supp} g \subseteq \{(t_1,t_2) : a_1 \le t_1 \le b_1,\ 0 \le t_2 \le \gamma(t_1)\} \cup \{(t_1,t_2) : 0 \le t_1 \le a_1, 0 \le t_2 \le b_2\},$$

we have that g satisfies the previous identity, if

$$\int_0^{a_1}\int_0^{b_2} g(s_1,s_2)\,ds_2\,ds_1 = 1, \tag{5.16}$$

$$\int_0^{\gamma(t_1)} g(t_1,s_2)\,ds_2 = 0, \qquad \forall a_1 \le t_1 \le b_1, \tag{5.17}$$

$$\int_0^{\gamma^{-1}(t_2)} g(s_1,t_2)\,ds_1 = 0, \qquad \forall a_2 \le t_2 \le b_2. \tag{5.18}$$

For $(t_1,t_2) \in \{(t_1,t_2) : a_1 \le t_1 \le b_1,\ \gamma(t_1) \le t_2 \le b_2\}$, we obtain

$$\begin{aligned}
&\int_0^{t_1}\int_0^{t_2} g(s_1,s_2)\,ds_2\,ds_1 \\
&= \int_0^{t_1}\int_0^{t_2} g(s_1,s_2)\,ds_2\,ds_1 + \int_{t_2}^{b_2}\int_0^{\gamma^{-1}(s_2)} g(s_1,s_2)\,ds_1\,ds_2 \\
&= \int_0^{t_1}\left(\int_0^{t_2} g(s_1,s_2)\,ds_2 + \int_{t_2}^{b_2} g(s_1,s_2)\,ds_2\right) ds_1 = \int_0^{t_1}\int_0^{b_2} g(s_1,s_2)\,ds_2\,ds_1 \\
&= \int_0^{a_1}\int_0^{b_2} g(s_1,s_2)\,ds_2\,ds_1 - \int_{a_1}^{t_1}\int_0^{\gamma(t_1)} g(s_1,s_2)\,ds_2\,ds_1 = \int_0^{a_1}\int_0^{b_2} g(s_1,s_2)\,ds_2\,ds_1 = 1,
\end{aligned}$$

and for $(t_1,t_2) \in \{(t_1,t_2) : b_1 \le t_1, a_2 \le t_2\}$ it follows

$$\int_0^{t_1}\int_0^{t_2} g(s_1,s_2)\,ds_2\,ds_1 = \int_0^{b_1}\int_0^{a_2} g(s_1,s_2)\,ds_2\,ds_1 = \int_0^{a_1}\int_0^{b_2} g(s_1,s_2)\,ds_2\,ds_1 = 1,$$

and finally for $(t_1,t_2) \in \{(t_1,t_2) : a_1 \le t_1, b_2 \le t_2\}$, one has

$$\int_0^{t_1}\int_0^{t_2} g(s_1,s_2)\,ds_2\,ds_1 = \int_0^{a_1}\int_0^{b_2} g(s_1,s_2)\,ds_2\,ds_1 = 1.$$

Assuming that g is differentiable and by derivation of (5.17) with respect to t_1, we get

$$g\left(t_1,\gamma(t_1)\right)\gamma'(t_1) + \int_0^{\gamma(t_1)} \frac{\partial g(t_1,s_2)}{\partial t_1}\,ds_2 = 0, \qquad \forall a_1 < t_1 < b_1. \tag{5.19}$$

Similarly, by derivation of (5.18) with respect to t_2, we get

$$g\left(\gamma^{-1}(t_2),t_2\right)\left(\gamma^{-1}(t_2)\right)' + \int_0^{\gamma^{-1}(t_2)} \frac{\partial g(s_1,t_2)}{\partial t_2}\,ds_1 = 0, \qquad \forall a_2 < t_2 < b_2,$$

and thus,

$$g\left(t_1,\gamma(t_1)\right)\frac{1}{\gamma'(t_1)} + \int_0^{t_1} \frac{\partial g(s_1,\gamma(t_1))}{\partial t_2}\,ds_1 = 0, \qquad \forall a_1 < t_1 < b_1. \tag{5.20}$$

From (5.11)–(5.13), and (5.15), we get

$$\begin{aligned}
&g(t_1,\gamma(t_1)) = g_1(t_1) + g_2(\gamma(t_1)), \\
&\frac{\partial g(t_1,t_2)}{\partial t_1} = g_1'(t_1) - h(t_1), \\
&\frac{\partial g(t_1,t_2)}{\partial t_2} = g_2'(t_2) + h\left(\gamma^{-1}(t_2)\right)\frac{1}{\gamma'(\gamma^{-1}(t_2))}, \qquad a_1 < t_1 < b_1.
\end{aligned}$$

From here and from (5.19),(5.20)

$$(g_1(t_1)+g_2(\gamma(t_1)))\,\gamma'(t_1)+\gamma(t_1)\,(g_1'(t_1)-h(t_1))=0,$$
$$(g_1(t_1)+g_2(\gamma(t_1)))\,\frac{1}{\gamma'(t_1)}+t_1\left(g_2'(\gamma(t_1))+h(t_1)\frac{1}{\gamma'(t_1)}\right)=0,$$

which gives together with (5.14) a first-order ordinary differential equation for g_1

$$g_1'(t_1)+g_1(t_1)\left(\frac{1}{t_1}-2\frac{\gamma''(t_1)}{\gamma'(t_1)}\frac{1}{1+\gamma'(t_1)^2}+\frac{\gamma'(t_1)}{\gamma(t_1)}\right)=0,\qquad a_1<t_1<b_1.$$

From this relation, we get

$$\begin{aligned} g_1(t_1)&=c\frac{1}{t_1\gamma(t_1)}\frac{\gamma'(t_1)^2}{1+\gamma'(t_1)^2}, && a_1<t_1<b_1,\\ g_2(\gamma(t_1))&=c\frac{1}{t_1\gamma(t_1)}\frac{1}{1+\gamma'(t_1)^2}, && a_1<t_1<b_1, \quad (5.21)\\ h(t_1)&=\frac{c}{1+\gamma'(t_1)^2}\left(\frac{\gamma'(t_1)}{t_1\gamma(t_1)^2}+\frac{2}{t_1\gamma(t_1)}\frac{\gamma''(t_1)\gamma'(t_1)}{1+\gamma'(t_1)^2}-\frac{\gamma'(t_1)^2}{t_1^2\gamma(t_1)}\right), && a_1<t_1<b_1,\end{aligned}$$

where c is a constant. (5.17) gives for $t_1=b_1$

$$0=\int_0^{\gamma(b_1)} g(b_1,s_2)\,ds_2=g_1(b_1)a_2+\beta a_2.$$

This means

$$\beta=-g_1(b_1), \tag{5.22}$$

and similarly

$$\alpha=-g_2(b_2). \tag{5.23}$$

Thus, if we choose α and β in this way, and g_1, g_2, and h as in (5.21), then g satisfies equations (5.17) and (5.18). Further, for suitable chosen c, g satisfies (5.16)

$$1=\int_0^{a_1}\int_0^{b_2} g(s_1,s_2)\,ds_2\,ds_1.$$

This gives that $(\widetilde{\mathbf{W}},x^*)=(\widetilde{\mathbf{W}},L^*g)$ fulfills equation (5.9), and so $\mathbf{P}_m$ and $\mathbf{P}_0$ are equivalent, and the ML estimator of m has the form (5.10).

From (5.22) and (5.23), we get

$$\begin{aligned}\left(\widetilde{\mathbf{W}},x^*\right)&=-g_2(b_2)\widetilde{\mathbf{W}}(a_1,b_2)-g_1(b_1)\widetilde{\mathbf{W}}(b_1,a_2)+\int_{\Gamma_0}\frac{h(s_1)}{\sqrt{1+\gamma'(s_1)^2}}\widetilde{\mathbf{W}}(s_1,\gamma(s_1))d(s_1,\gamma(s_1))\\ &\quad+\int_{\Gamma_0}\frac{g_1(s_1)}{\gamma'(s_1)}\frac{\partial\widetilde{\mathbf{W}}}{\partial n}(s_1,\gamma(s_1))d(s_1,\gamma(s_1)).\end{aligned}$$

(5.9) and (5.8) give

$$D^2_{\mathbf{P}_0}\left(\left(\widetilde{\mathbf{W}},x^*\right)\right)=(1,x^*)=\alpha+\beta+\int_{a_1}^{b_1}h(s_1)\,ds_1=-c\left(\frac{1}{a_1b_2}+\int_{a_1}^{b_1}\frac{1}{t_1^2\gamma(t_1)}\,ds_1\right).$$

So, (5.21) and (5.10) give (5.7), where

$$
\begin{aligned}
d_1 &= \frac{1}{a_1 b_2 (1+\gamma'(a_1)^2) d},\\
d_2 &= \frac{\gamma'(b_1)^2}{b_1 a_2 (1+\gamma'(b_1)^2) d},\\
y_1(t_1,\gamma(t_1)) &= \frac{-1}{(1+\gamma'(t_1)^2)^{3/2}} \left(\frac{\gamma'(t_1)}{t_1\gamma(t_1)^2} + \frac{2}{t_1\gamma(t_1)} \frac{\gamma''(t_1)\gamma'(t_1)}{1+\gamma'(t_1)^2} - \frac{\gamma'(t_1)^2}{t_1^2\gamma(t_1)} \right) \frac{1}{d}, \qquad (5.24)\\
y_2(t_1,\gamma(t_1)) &= \frac{-1}{t_1\gamma(t_1)} \frac{\gamma'(t_1)}{1+\gamma'(t_1)^2} \frac{1}{d},\\
d &= \left(\frac{1}{a_1 b_2} + \int_{a_1}^{b_1} \frac{1}{t_1^2 \gamma(t_1)}\, ds_1 \right).
\end{aligned}
$$

Thus, the theorem is proved.

EXAMPLE 2. First, let us consider the case when $G = (q_1, r_1) \times (q_2, r_2)$. We cannot apply Theorem 2 because G does not satisfy the condition (5.1). But Theorem 1 (or Statement) immediately show that

$$\widehat{m} = \widetilde{\mathbf{W}}(q_1, q_2).$$

EXAMPLE 3. Let us consider the case, when a part of the boundary is a hyperbola and G satisfies condition (5.6):

$$\gamma(t_1) = \frac{1}{t_1}, \qquad a_1 = a_2 = \frac{1}{2}, \qquad b_1 = b_2 = 2.$$

From this

$$\gamma'(t_1) = \frac{-1}{t_1^2}, \qquad \gamma''(t_1) = \frac{2}{t_1^3}.$$

(5.24) gives

$$
\begin{aligned}
\Gamma_0 &= \left\{ \left(t_1, \frac{1}{t_1}\right) : \frac{1}{2} \le t_1 \le 2 \right\},\\
d &= \ln(4e),\\
d_1 &= d_2 = \frac{1}{17\ln(4e)},\\
y_1\left(t_1, \frac{1}{t_1}\right) &= \left(\frac{1}{t_1^2} + t_1^2 \right)^{-3/2} \left(\frac{1}{t_1^2} + t_1^2 + \frac{4}{1/(t_1^2) + t_1^2} \right) \frac{1}{\ln(4e)},\\
y_2\left(t_1, \frac{1}{t_1}\right) &= \left(\frac{1}{1/(t_1^2) + t_1^2} \right) \frac{1}{\ln(4e)}.
\end{aligned}
$$

This means

$$\widehat{m} = \frac{1}{17\ln(4e)} \left(\widetilde{\mathbf{W}}\left(\frac{1}{2}, 2\right) + \widetilde{\mathbf{W}}\left(2, \frac{1}{2}\right) \right) + \int_{\Gamma_0} \left\{ y_1(t)\widetilde{\mathbf{W}}(t) + y_2(t) \frac{\partial \widetilde{\mathbf{W}}}{\partial n}(t) \right\} dt.$$

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