On the effect of fiber creep-compliance in the high-temperature deformation of continuous fiber-reinforced ceramic matrix composites

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1. Introduction

Advanced composites with ceramic matrices have been developed with a view to introduce ceramics in structural parts used in severe environments, such as rocket and jet engines, gas turbines for power plants, heat shields for space vehicles, fusion reactor first wall, aircraft brakes, heat treatment furnaces, etc. Although CMCs are promising thermostructural materials, their applications are still limited by the lack of suitable reinforcements, processing difficulties, sound material databases, lifetime and cost. Comprehension of the effect of the basic properties of the constituent phases and “bundle-like effects” to the creep behavior and the expected rupture lifetime is critical for the proper design and use of CMCs.

Creep models for unidirectional ceramic matrix composites reinforced by long creeping fibers with weak interfaces are presented. These models extend the work of Du and McMeeking (1995) to include the effect of fiber primary creep present in the required operational temperatures for ceramic matrix composites (CMCs). The effects of fiber breaks and the consequential stress relaxation around the breaks are incorporated in the models under the assumption of global load sharing and time-independent stochastics for fiber failure. From the set of problems analyzed, it is found that the high-temperature deformation of CMCs is sensitive to the creep-compliance of the fibers. High fiber creep-compliance drives the composite to creep faster, leading however to greater lifetimes and greater overall strains at rupture. This behavior is attributed to the fact that the greater the creep-compliance of the fibers, the higher the creep rate but the slower the matrix stress relaxation – since the matrix must deform with a rate compatible with the more creep-resistant fibers – and therefore the less the load carried by the main load-bearing phase, the fibers. As a result, fewer fibers fail and less damage is accumulated in the system. Moreover, the greater the creep-compliance of the fibers, the slower the matrix shear stress relaxation – and thus the lower the levels of applied stress for which this effect becomes important. The slower the shear stress relaxes, the slower the “slip” length increases. Due to the Weibull nature of the fibers, the fiber strengths at the smaller gauge length of the slip length are stronger; therefore fewer fibers undergo damage. Hence, high fiber creep-compliance is desirable in terms of composite lifetime but not in terms of overall strain. These results are considered of importance for composite design and optimization.
1993; Okabe et al., 1999; Henager et al., 2001; Wilshire, 2002). In Holmes et al. (1993), it was observed that the creep behavior is strongly influenced by the initial loading rate. At low rates of initial loading the relaxation of the matrix stress reduces the likelihood of matrix fracture; however, it is possible for the fiber stress to increase, as load is shed from the matrix, to a level that causes fiber fracture within the composite. At higher rates of initial loading, matrix fracture is pronounced, influencing stress redistribution and resulting in shorter composites lifetimes and higher creep rates due to accelerated fiber damage. Fiber breakage occurs along with progressive debonding (Wu and Holmes, 1993; Ohno et al., 1994; Weber et al., 1993; Weber et al., 1996) due to the low fiber-matrix cohesion – a weak fiber-matrix interface has been found to be essential to ensure good longitudinal strength for both MMCs and CMCs (Jansson et al., 1994; He et al., 1993). The decohesion of the interface allows the fibers to slip relative to the matrix. Due to the sliding resistance, load is transferred back onto the fibers that are broken and thus the broken fibers still have substantial load-carrying capability. For short-term loading, the sliding resistance can be well represented as being constant (Marshall et al., 1992). For long term loading, however, the relaxation of the matrix and interface shear stresses and the subsequent additional induced damage become an important aspect of the problem. As time progresses the matrix shear stress at the matrix-fiber interface relaxes and the recovery length – the distance required for an isolated broken fiber to recover its elastic load – increases. This reduces the load-carrying capacity of broken fibers. The more the stress of the broken fibers relaxes the higher the load carried by the unbroken fibers and the matrix. This causes additional increases in damage, affecting the creep strain to rupture and the rupture lifetime.

There have been analytical attempts to incorporate all these effects for predicting the mechanical behavior of fiber-reinforced composites subjected to a uniaxial constant tensile loading in high-temperature environments. These treatises, generally are one-dimensional and consider the composite reinforced by continuous (stochastic) fibers aligned parallel to the axis of tensile loading. The matrix is considered as viscous reinforced by elastic/viscous fibers (McLean, 1985, 1989). Global load sharing rule – according to which only time-dependent spatial averages of the uniaxial stress and strain, rather than the full time-dependent spatially-varying fields – is assumed which simplifies considerably the study of the stochastics of fiber failure. Such an approach is justified by the extensive fiber damage prior to failure of composites with weak interfaces, suggesting that the global (average) effects of fiber damage on stress and strain are most important. The low shear strength of composites with weak interfaces actually diminishes stress concentrations in fibers adjacent to breaks and thus damage in the fibers occurs in an uncoordinated manner (He et al., 1993; Du and McMeeking, 1994). Analytical solutions for the stress concentrations around fiber breaks were given in Landsis and McMeeking (1999) and Ohno et al. (2004b). Curtin (1991) established, under global load sharing rule, fiber failure stochastics for time-independent fibers dependent upon the gauge length and a constant shear stress across the slipped portion of the interface. The time-dependent breakdown of the fibers was included in a number of studies based mainly on the Coleman’s lifetime distribution (Coleman, 1958; Ibnabdeljalil and Phoenix, 1995; Newman and Phoenix, 2001; Mahesh and Phoenix, 2004) and an empirical crack growth power law (Iyengar and Curtin, 1997a; Halverson and Curtin, 2002). Moreover, the effects of fiber pullout and that of the matrix damage state were also included in Halverson and Curtin (2002). Du and McMeeking (1995) have solved a single broken fiber model (cell model) to understand the evolution of interface shear and then developed an averaging approach for the entire composite (the effect of interface relaxation was accounted for by an average time-dependent sliding interfacial shear stress). This issue was also addressed in later works (Iyengar and Curtin, 1997b; Ohno and Miyake, 1999, Ohno et al., 2000, 2004a). Cell models have been used extensively for the time-dependent deformation of composites reinforced by either discontinuous or continuous fibers and particles, especially for finite elements simulations (Aboudi, 1991; Park and Holmes, 1992; Kondo et al., 1994; Aboudi, 1995; Aravas et al., 1995; Nimmagadda and Sofronis, 1996; Cheng and Aravas, 1997b,a; Bednarczyk and Arnold, 2002).

At the required operational temperatures (up to 1500 °C) for CMCs, the fibers usually exhibit primary creep while the matrix both primary and steady-state creep. In this paper, the creep models of Du and McMeeking (1995) are extended with the aim of gaining insight into the effect of creeping fibers in the overall composite creep behavior, in an effort to facilitate effective composite design. The present models describe the creep behavior of CMCs on the basis of the load transfer from broken fibers to other intact fibers through a constant or relaxing friction at the fiber/matrix interface under global load sharing and time-independent stochastics for fiber failure. Fiber degradation with time and matrix damage are not included in the models since they do not alter the trends observed in the overall creep response with variation of the fiber creep-compliance although they do alter the relevant time scale for failure (Baxevanis and Charalambakis, 2010).

The remainder of the paper is organized as follows. In Section 2 the assumptions utilized for the development of the models are itemized and mathematically formulated. In Section 3 two models are developed for composites with creeping fibers, one based on a constant and another one on a relaxing matrix shear stress. Finally, in Section 4 the results on the composite creep response are further discussed.

2. Assumptions utilized for the development of the models

We consider a unidirectional composite consisting of an elastic-power law creeping matrix

\[ \dot{\varepsilon}_m = \frac{\sigma_m}{E_m} + B \sigma_m^n, \]

where \( E_m \) is the Young’s modulus of the matrix, and \( B \) and \( n \) are constants that characterize the inelastic behavior, reinforced by a volume fraction \( f \) of continuous ceramic fibers of diameter \( D \), aligned parallel to the axis of loading. Here, \( \dot{\varepsilon}_m \) is the axial strain in the matrix and \( \sigma_m \) is the axial stress in the matrix. Superscribed dot denotes differentiation with respect to time \( t \). The mechanical analogue of equation (1) resembles the Maxwell unit and consists of a linear spring connected in series with a nonlinear dashpot. Such power law response functions have been used to describe the slow, steady-state, uniaxial creep of metals and ceramics subjected to constant uniaxial tensile stresses. It is assumed that since the primary creep is small compared with the steady-state creep it can be neglected. Moreover, matrix cracking, prevalent in CMCs (or matrix yielding in MMCs), is not included in the constitutive relation. The matrix is actually considered in the model as a deforming material which is assumed to rupture when the strain on average goes to infinity.

The fibers are assumed to exhibit primary creep characterized by the nonlinear deformation relationship

\[ \dot{\varepsilon}_f = \frac{\sigma_f}{E_f} + C (\dot{\varepsilon}_f - \frac{\sigma_f}{E_f})^\mu. \]

where \( \dot{\varepsilon}_f \) is the axial strain, \( E_f \) the Young’s modulus, \( C, v \) and \( \mu \) are constants for given temperature characterizing the inelastic response (note that only \( C \) is dependent on the temperature). This
constitutive relation, valid for time-dependent stress $\sigma_\ell$, results from the well known relation
\begin{equation}
\dot{\varepsilon}_\ell(t) = A\sigma_\ell^p,
\end{equation}
which connects the creep strain $\dot{\varepsilon}_\ell$ to a constant load $\sigma_\ell$ (see Appendix A). This empirical relation has been validated by experiments on several high strength ceramics (Dicarlo, 1986, 1994. A, l and $p$ are constants characterizing the inelastic response at times $t$ from the application of the constant stress. The relations between the exponents in these two equations read as
\begin{equation}
C = (p + 1)^{1/(p+1)}A^{1/(p+1)}, \quad \gamma = 1/(p + 1), \quad \mu = -p/(p + 1).
\end{equation}

Moreover, we assume that the gross structural integrity of the composite is maintained during loading and we neglect inertial and wave propagation effects. Hence
\begin{equation}
\dot{\varepsilon} = \dot{\varepsilon}_m = \dot{\varepsilon}_f, \quad \sigma = (1 - f)\sigma_m + f\sigma_f, \quad (5a)
\end{equation}
\begin{equation}
(5b)
\end{equation}
for all times $t$, where $\varepsilon$ and $\sigma$ are the composite strain and stress respectively. However, the fiber bundle consists of fibers that are broken in various locations. Thus, in order to study the effect of stochastic fiber fracture with global load sharing on the response of the composite, we replace equation (5b) by
\begin{equation}
\sigma = (1 - f)\sigma_m + f\sigma_f, \quad (6)
\end{equation}
which gives the stress of the composite in an arbitrary cross-sectional plane, where $\sigma_f$ is the average stress supported by the fiber bundle at the cross-section under consideration.

Hence, the system of governing equations for the composite reads as
\begin{equation}
\dot{\varepsilon} = \dot{\varepsilon}_m + B\sigma_m^n, \quad (7a)
\end{equation}
\begin{equation}
\dot{\varepsilon}_f = C\sigma_f \left(\dot{\varepsilon} - \frac{\sigma_f}{\dot{\varepsilon}_f}\right)^{-\mu}, \quad (7b)
\end{equation}
\begin{equation}
\sigma = (1 - f)\sigma_m + f\sigma_f, \quad (7c)
\end{equation}
The composite is subjected to a step uniaxial tensile loading defined by
\begin{equation}
\sigma = \begin{cases} 0, & t < t_0, \\ \sigma_0, & t > t_0. \end{cases} \quad (8)
\end{equation}

When the load is suddenly applied at $t = 0$, all of the state variables suffer jump discontinuities. We consider the limits, as time approaches zero from the right, to be the initial values of the state variables in order to create a smooth initial boundary value problem
\begin{equation}
\varepsilon_0 = \lim_{t \to t_0^-} \varepsilon(t), \quad \sigma_{f0} = \lim_{t \to t_0^-} \sigma_f(t), \quad \sigma_{m0} = \lim_{t \to t_0^-} \sigma_m(t). \quad (9)
\end{equation}

2.1. Stochastic fiber failure based upon the gauge length and the interfacial shear stress

As noted above, an important determinant of composite behavior is the statistical fiber strength. For time-independent fiber break down (no strength degradation), Curtin (1991) showed that the average stress in the fibers at a cross-section of the composite can be approximated as
\begin{equation}
\sigma_f(t) \approx \left[1 - \frac{L_f(t)}{L_0} \left(\frac{\max \sigma_f(t)}{S_0}\right)^m\right] \sigma_f(t), \quad (10)
\end{equation}
for all times $t$, if (a) the fraction of fibers that have been broken more than once within the distance $L_f$ from the cross sectional plane is negligible and the fiber length $L$ is much larger than $L_f$, (b) there are no stress concentrations in fibers adjacent to the broken ones (global load sharing assumption), (c) the finite population of fiber strengths is adequately described by a Weibull distribution. $L_0$ and $S_0$ are reference values of length and strength, respectively, and $m$ describes the variability in fiber strengths. It should be noted that more elaborate cross-sectional fiber average stress solutions have been worked out, e.g. in Hui et al. (1995) and Phoenix et al. (1997).

3. Models based on specific cases of matrix shear stress

3.1. Rupture model with constant matrix shear stress

By equilibrium, if the shear stress $\tau$ is assumed constant ($\tau = \tau_0$), the stress recovery distance $L_f$ is
\begin{equation}
L_f(t) = \frac{D\sigma_f(t)}{4\tau_0}. \quad (11)
\end{equation}

Substituting $L_f$ of (11) in (10) gives
\begin{equation}
\sigma(t) \approx \left[1 - \frac{1}{2} \left(\frac{\max \sigma_f(t)}{S_0}\right)^{m+1}\right] \sigma_f(t), \quad (12)
\end{equation}
where
\begin{equation}
S_0 = \left(\frac{2\tau_0 S_0^\delta}{D}\right)^{1/\delta}, \quad (13)
\end{equation}
is the fiber strength at the critical gauge length $\delta = S_0 D L_0^\delta / (2\tau_0)^{\delta/(m+1)}$.

Using (12), system (7) becomes after simple calculations and rearrangement
\begin{equation}
\dot{\varepsilon} = \frac{B}{\sigma_0 - f\sigma_f} \left[1 - \left(\frac{\sigma_f}{\sigma_0}\right)^m\right] \left(1 + \left(\frac{\sigma_f}{\sigma_0}\right)^m\right)^{-1} \left[1 - \left(1 + \frac{\sigma_f}{\sigma_0}\right)^{m+1}\right] \sigma_f \left(\frac{\varepsilon - \sigma_f}{\sigma_0}\right)^{-\mu}, \quad (14a)
\end{equation}
\begin{equation}
\dot{\sigma}_f = \frac{E_f \dot{\varepsilon}}{S_0} \left(\frac{\varepsilon - \sigma_f}{\sigma_0}\right)^{-\mu}. \quad (14b)
\end{equation}

We find it advantageous to introduce the following dimensionless variables
\begin{equation}
\hat{\varepsilon} = \frac{E_f \varepsilon}{S_0}, \quad \hat{\sigma} = \frac{\sigma}{S_0}, \quad \hat{\tau} = \frac{\tau}{S_0}, \quad \hat{\dot{\varepsilon}} = \frac{\dot{\varepsilon}}{E_f}, \quad \hat{\dot{\sigma}} = \frac{\dot{\sigma}}{S_0}, \quad \hat{\dot{\sigma}_f} = \frac{\dot{\sigma}_f}{S_0}, \quad \hat{\sigma}_f = \frac{\sigma_f}{S_0}, \quad \hat{\tau}_f = \frac{\tau_f}{S_0}, \quad (15)
\end{equation}
whereupon, system (14) reads as
\begin{equation}
\hat{\dot{\varepsilon}} = \frac{B}{\hat{\sigma}_0 - f\hat{\sigma}_f} \left[1 - \left(\frac{\hat{\sigma}_f}{\hat{\sigma}_0}\right)^m\right] \left(1 + \left(\frac{\hat{\sigma}_f}{\hat{\sigma}_0}\right)^m\right)^{-1} \left[1 - \left(1 + \frac{\hat{\sigma}_f}{\hat{\sigma}_0}\right)^{m+1}\right] \hat{\sigma}_f \left(\hat{\varepsilon} - \hat{\sigma}_f\right)^{-\mu}, \quad (16a)
\end{equation}
\begin{equation}
\hat{\dot{\sigma}}_f = \hat{\dot{\varepsilon}} - \hat{\dot{\sigma}}_f \left(\hat{\varepsilon} - \hat{\sigma}_f\right)^{-\mu}. \quad (16b)
\end{equation}

If the fibers are assumed elastic then the above system reduces to the Curtin–McLean rupture model (Du and McMeeking, 1995)
\begin{equation}
\hat{\dot{\varepsilon}} = \frac{\hat{\sigma}_0 - f\hat{\varepsilon} \left(1 - \frac{1}{2} \hat{\varepsilon}^{m+1}\right)}{1 + \frac{\hat{\varepsilon}^{m+1}}{1 - \left(1 + \frac{\hat{\varepsilon}}{2}\right)^{m+1}}} \left(\hat{\varepsilon} - \hat{\sigma}_f\right). \quad (17)
\end{equation}

Note that for both system (16) and Eq. (17) no solutions of physical significance can be extended through the singular point for $\sigma_f$. 


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at which the denominators of both (16) and (17) become zero. It is not surprising that this value depends solely upon the elastic properties of the constituent phases because the response of all materials exhibiting memory effects is approximately elastic when they are subjected to rapid or impulsive deformations such as those in the vicinity of the singular point.

Moreover, it should be noted that the applied stress \( \sigma_0 \) cannot exceed the fast-fracture effective tensile strength of the composite

\[
\sigma_{\text{m} + 1} = \frac{1}{m + 2} \left[ \frac{E_{\text{m}}(1 - f)}{E_f} + f \right] \frac{2}{(m + 2)^{\frac{1}{m^*}}}. \tag{19}
\]

If the applied stress is above this value then the composite fails immediately upon loading. However, in ceramic matrix composites under fast loading conditions, the matrix cracks well prior to composite failure and thus the composite strength is actually decreased to the bundle strength of the fibers alone (Curtin, 1993)

\[
\sigma_{\text{m} + 1} = \frac{1}{m + 2} \left[ \frac{2}{(m + 2)^{\frac{1}{m^*}}} \right]. \tag{20}
\]

Eq. (20) is the maximum value of \( \sigma_0 \) for which the numerator of (17), \( \sigma_0 - f\sigma_{\text{f}}(1 - \frac{1}{m^{*+1}}) = 0 \) can be solved for \( \dot{\varepsilon} \). Hence, it is actually a critical value for the applied stress, below which the composite suffers only partial failure and the stress on the unbroken fibers tends towards a finite value giving rise to an infinite lifetime. Fabeny and Curtin (1996) have constructed a system similar to (16) describing the overall creep behavior of a composite with fibers exhibiting steady-state creep. They proved that there exists again a critical value (greater than (20)) for the applied stress, below which the composite has an infinite lifetime with some steady-state creep rate. The stress in the matrix in that case does not decay to zero but rather to a value at which the matrix and the fibers creep at the same rate (the composite never ceases to creep).

Before proceeding further, it is important to recognize the following point regarding application of the above mentioned analysis to ceramic matrix composites. The ceramic composite, according to the Curtin–McLean and the Fabeny–Curtin models, will either fail instantly on loading, if the applied stress is above the bundle strength, or will have an infinite lifetime, if the applied stress is below the bundle strength. This is so because the bundle strength (fast-fracture strength (20)) is equal to (for the Curtin–McLean model) and less than (for the Fabeny–Curtin model) the corresponding critical values of the two models for infinite lifetime. Thus, the intermediate regime of finite lifetimes disappears. In the Curtin–McLean model an elastic response is assumed for the fibers while in the Fabeny–Curtin model a steady-state creep response is assumed, respectively, hence it is conjectured that the situation is analogous for the model (16) described herein, that assumes a decreasing creep rate response for the fibers; a response that lies in between the two aforementioned ones. Unfortunately, no analytical results can confirm this conjecture which is however supported by numerical experiments. In conclusion, the main result of interest herein is not the application of the model to experimental data from CMCs, but rather the basic dependencies of the overall creep deformation time-scale, the creep rate and the stress transfer between the fibers and the matrix on the parameters of the constitutive phases, as given by integration of (16).

Numerical integration of (16) indicates that the creep response of CMCs is sensitive to the creep-compliance of the fibers. High fiber creep-compliance drives composite to creep faster, leading however to greater lifetimes and greater overall strains at rupture (Fig. 1). The fibers get more creep-compliant as parameters \( A \), \( l \), and \( p \) increase, as it can be easily deduced from (A.1). The dependence of fiber creep-compliance on parameters \( C \), \( v \) and \( \mu \) is given through relations (4). Since the matrix must deform compatibly with the fibers, the more creep-compliant the fibers, the slower the matrix stress relaxation. Therefore the load carried by the fibers increases more slowly (Fig. 2) and fewer fibers fail. The values of the parameters used in the numerical calculations are chosen so as to conform with those of a composite system comprised SCS-6 SiC fibers of diameter \( D = 142 \mu m \) in a hot-pressed Si3N4 matrix at 1200 °C. Therefore,

\[
m = 5, \quad S_0 = 1.47 \text{ GPa}, \quad E_f = 367.16 \text{ MPa}, \quad E_m = 274 \text{ MPa}, \quad B = 2.833 \times 10^{-25} \text{ Pa}^2 \text{s}^{-1}, \quad n = 2.
\]

according to Dicarlo (1986) and Weber et al. (1996), and

\[
E_m = 274 \text{ MPa}, \quad B = 2.833 \times 10^{-25} \text{ Pa}^2 \text{s}^{-1}, \quad n = 2. \quad v = 0.27.
\]

according to Kossowsky et al. (1975) and Park and Holmes (1992). The interfacial shear strength, used in the next section, at this temperature is measured by Morscher et al. (1990) and found to be \( \tau_0 = 12 - 32 \text{ MPa} \).
3.2. Model with matrix shear stress relaxation

This model is based upon a cell model which concerns the shear stress relaxation in a single broken fiber surrounded by intact neighbors. It gives the governing system of a partial and an ordinary differential equation for the evolution of the stress in the broken fiber. Then, an approximate model is developed that averages over the effect of initially broken, progressively broken and intact fibers on the time evolution of the state variables during creep.

3.2.1. Cell model

Consider the cell illustrated in Fig. 3. The displacement on the lateral surface of the unit cell is \( u(x, t) = \varepsilon(t)x \), where \( \varepsilon(t) \) is the strain of the composite. The displacement on the lower segment of the unit cell is \( u_b(x, t) \) and the shear strain \( \gamma_m \) in the matrix for high fiber volume fractions is

\[
\gamma_m = \frac{u_c(x, t) - u_b(x, t)}{w},
\]

where

\[
w = D \left( \frac{\pi}{2\sqrt{3}} - 1 \right),
\]

is the spacing between fiber surfaces for hexagonally packed fibers (neighboring fibers experience nearly the composite strain rate).

The shear strain rate in the matrix (of Von-Mises type) can be expressed as

\[
\dot{\gamma}_m = \frac{\dot{e}_c}{C_m} + 3B\sigma_e^{-1}\tau,
\]

where \( C_m \) is the shear modulus of the matrix, \( \tau \) is the shear stress in the matrix and

\[
\sigma_e = \sqrt{\frac{3}{2} S_0 S_{ij}},
\]

is the effective stress, where \( S_0 \) is the deviatoric stress. The effective stress for the present problem reads as

\[
\sigma_e = \sqrt{\sigma_m^2 + 3\tau^2}.
\]

It is assumed that \( \sigma_m \) depends only on \( t \), while \( \tau \) depends on \( t \) and \( x \).

Equilibrium gives

\[
\frac{\partial \sigma_m^2}{\partial x} = -\frac{4\tau}{D},
\]

where \( \sigma_m^2 = E_f \varepsilon_f^2(x, t) \),

\[
is the stress in the broken fiber and \( \varepsilon_f \) the elastic part of the fibers strain. An additional equation for the creep part of the strain is needed, namely,

\[
\dot{e}_f = C(\sigma_m^2)^\gamma (\varepsilon_f)^{-\delta}.
\]

Combination of (27) and (28) gives

\[
\dot{e}_f = \frac{\sigma_f^2}{E_f} + C(\sigma_m^2)^\gamma (\varepsilon_f)^{-\delta}.
\]

By substitution of \( \varepsilon_f = \partial u_b/\partial x \) from (29) into (30) and use of (23), (26) and (28), the governing system of partial differential equations for the evolution of the stress of a broken fiber

\[
\frac{\partial \sigma_f^2}{\partial t} = \frac{D E_f \varepsilon_f}{4C_m} \frac{\partial \sigma_f^2}{\partial x^2} + \frac{3D B E_f \varepsilon_f}{4} \frac{\partial}{\partial x} \left[ \sigma_m^2 + \frac{\partial \sigma_f^2}{\partial x} \right] - E_f C(\sigma_m^2)^\gamma (\varepsilon_f)^{-\delta} + \frac{E_f \varepsilon_f}{E_f},
\]

\[
is derived. The boundary conditions for \( \sigma_f^2 \) are

\[
\sigma_f^2(t) = \sigma_f(t), \quad \sigma_f^2(L/2, t) = 0,
\]

while the initial conditions for \( \sigma_f^2 \) and \( \varepsilon_f \) are

\[
\sigma_f^2(x, 0) = \begin{cases} \sigma_f(t), & 0 \leq x \leq L/2 - L_f, \\ \sigma_f(0), & L/2 - L_f \leq x \leq L/2, \end{cases}
\]

and

\[
\varepsilon_f(x, 0) = \dot{e}_f(0) - \sigma_f^2(0)/E_f,
\]

respectively.

Similar shear-lag models for matrix shear stress relaxation in metallic matrix composites were developed by Du and McMeeking (1995) and in polymeric matrix composites by Lagoudas et al. (1989) and Mason et al. (1992) (all these works assume elastic fibers).

We perform numerical experiments to obtain the stress in the fiber when the unit cell shown in Fig. 3 is subjected to a constant overall strain (see Appendix B for details on the numerical integration). In this case, the response is computed from (30) with \( \dot{e}_f = 0 \). These results represent the stress in a broken fiber in a relaxation test. In a relaxation test in which the application of the load was instantaneous (elastic behavior assumed for both fibers and matrix) fibers break upon loading only and not thereafter. Thus, Fig. 4 shows the stress relaxation behavior at a break starting at the instant of loading. Moreover, in the same figure the stress recovery segment can be seen. The stress recovery length \( L_f \) is defined as the distance from the break at which the tensile stress just attains the value of the remote fiber stress (at \( x = 0 \), increases as time increases. Within this distance the fiber stress decreases, i.e. the fiber relaxes. Beyond the stress recovery segment, the stress in the fiber is almost uniform and constant at its value at the cell end (at \( x = 0 \), a value that relaxes with time. Thus, as long as the stress recovery segment has not reached the cell end, the behavior of the solution is insensitive to the fiber length. This behavior has been already confirmed for the case of elastic fibers (Du and McMeeking, 1995). An important feature of the results, already pointed out by Du and McMeeking (1995), is that the gradient of stress within the stress recovery length is only weakly dependent on position. This means that the shear stress in the matrix, which is proportional to the stress gradient, can be approximated as uniform. In Fig. 5(a) the time evolution of the “uniform” matrix shear
stress is computed from (30) and compared to that for the “limit” case of elastic fibers (Du and McMeeking, 1995). It is found that the shear stress relaxes in a shorter time when the fibers creep. This situation is reversed for constant applied loads, as we show in the next section.

3.2.2. Model with matrix shear stress relaxation at constant overall applied stress

At constant applied load, fibers fail upon initial loading and then randomly as the stress acting on the unbroken fibers increases. The stress recovery segments in the broken fibers $L_f$ extend with time. Thus, in order to predict the creep response of the composite, we combine system (7) with Eq. (10) to obtain

$$\ddot{\varepsilon} = \frac{B \left\{ \sigma_0 - f \sigma_f \left[ 1 - \frac{t}{\tau} \left( \frac{\sigma}{\tau_e} \right)^m \right]^n \right\}}{(1 - f)^n \left\{ 1 + \frac{f}{\tau_e} \left[ 1 - (m + 1) \left( \frac{\sigma}{\tau_e} \right)^m \right] \right\}} + \frac{1 - f}{\tau_m} \left\{ E_f C \left[ 1 - (m + 1) \left( \frac{\sigma}{\tau_e} \right)^m \right] \sigma_f (\varepsilon - \sigma_f)^{-\mu} + \sigma_f \left( \frac{\sigma}{\tau_e} \right)^m \frac{\tau}{\tau_e} \right\},$$

(34a)

$$\dot{\sigma}_f = E_f \dot{\varepsilon} - E_f C \sigma_f^\mu \left( \varepsilon - \frac{\sigma_f}{E_f} \right)^{-\mu}.$$

(34b)

The above system predicts the creep response of the composite once $L_f$ and $L_f$ are given as functions of time. These two functions at time $t$ are clearly now averaged quantities over fibers broken earlier than $t$.

The determination of $L_f$ and $L_f$ proceeds as in Du and McMeeking (1995). By incrementing system (34) from $t$ to $t + \Delta \tau$, the fiber stress (away from the breaks) increases from $\sigma_f$ at $t$ to $\sigma_f + \Delta \sigma_f$ at $t + \Delta \tau$ and the recovery distance from $L_f$ to $L_f + \Delta L_f$. The evolution

![Fig. 4. Stress in a broken fiber along its length estimated by (30) for a composite subjected to a constant strain $E_f/S_0 = 0.175$. The stress is normalized by its remote decreasing value at $x = 0$. The values of the parameters are $f = 0.25$, $L = 1500$, $m = 5$, $n = 2$, $E = 1.34$, $\nu = 3$, $\mu = 2$, $t_0 = 0.0084$ and $\eta = 4.1875 \times 10^{-6}$.

Fig. 5. (a), (b) and (c): Evolution of the “uniform” matrix shear stress within the stress recovery length estimated for composites with parameters $f = 0.25$, $L = 5000$, $m = 5$, $n = 2$, $E = 1.34$, $\nu = 3$, $\mu = 2$, $t_0 = 0.0084$ and $\eta = 4.1875 \times 10^{-6}$. (a) Composites subjected to constant strain $E_f/S_0 = 0.175$ with creeping (30) (dashed line) and non-creeping fibers ((31) in Du and McMeeking (1995))(dashed line). (b) Composites subjected to constant load $\sigma_0/S_0 = 0.15$ with creeping (30) (solid line) and non-creeping fibers (model in Du and McMeeking (1995))(dashed line). (c) Creep (solid line) and relaxation (dashed line) problem for a composite with initial stress (which remains constant for the creep problem) $\sigma_0/S_0 = 0.15$. (d): Evolution of matrix normal stress for composites with relaxing (solid line) and constant matrix shear stress (dashed line) subjected to constant load $\sigma_0/S_0 = 0.142$. The values of the parameters are $f = 0.2$, $L = 5000$, $m = 5$, $n = 2$, $E = 1.34$, $t_0 = 0.0084$, $\nu = 3$, $\mu = 2$ and $\eta = 4.1875 \times 10^{-6}$.](image-url)
of the averaged fiber stress on the already broken fibers (averaged in a sense that will be made precise below) can be followed by system (30) and equals $\dot{\sigma}_f^m$ at time $t + \Delta t$. The recovery distance depends on $\dot{\sigma}_f^m$ and on the fibers that are newly broken in this increment of time and thus their number must be computed: According to the Curtin model, at the beginning of the increment, at imposed stress $\sigma_f$, the number of breaks in the distance $2(L_f + \Delta L_f)$ is

$$n = \frac{2(L_f + \Delta L_f)}{L_0} \left( \frac{\sigma_f}{S_0} \right)^m. \quad (35)$$

The number of breaks within the length $2(L_f + \Delta L_f)$ at imposed stress $\sigma_f + \Delta \sigma_f$ is

$$n + \Delta n = \frac{2(L_f + \Delta L_f)}{L_0} \left( \frac{\sigma_f + \Delta \sigma_f}{S_0} \right)^m. \quad (36)$$

The time-independent fiber failure stochastics within the framework of the averaged fiber stress on the already broken fibers (averaged in a sense that will be made precise below) can be followed by system (30) and equals $\dot{\sigma}_f^m$ at time $t + \Delta t$. The recovery distance depends on $\dot{\sigma}_f^m$ and on the fibers that are newly broken in this increment of time and thus their number must be computed: According to the Curtin model, at the beginning of the increment $t + \Delta t$ in the existing broken fibers can thus be computed as

$$\dot{\sigma}_f^m(x)_{\text{new}} = \min \left\{ \frac{4\tau_0(0.5L - x)}{D} ; \dot{\sigma}_f^m(x) \right\} \frac{n}{n + \Delta n}$$

$$+ \min \left\{ \frac{4\tau_0(0.5L - x)}{D} ; \sigma_f + \Delta \sigma_f \right\} \frac{\Delta n}{n + \Delta n} \quad (37)$$

4. Creep response – discussion

In this paper, the most comprehensive models for creep rupture studied by Du and McMeeking (1995) have been extended to include the primary creep of ceramic fibers observed in operational temperatures for ceramic matrix composites. The first one is an extension of the Curtin–McLean rupture model that is based on time-independent fiber failure stochastics within the framework of global load sharing for elastic fibers and includes the effect of the creep matrix. The second model takes additionally into account the effect of broken fiber stress relaxation due to matrix/fiber shear stress interaction. To do so, a cylindrical cell containing a broken fiber is considered, and a bilinear approximation of the fiber stress distribution in the broken fiber is employed to derive the evolution of the stress recovery segments.

Material parameters have been varied to assess the sensitivity of the overall creep behavior to the variation of the individual parameter. It is found that a low fiber volume fraction $f$, a low modulus ratio $E_f/E_m$, a low creep exponent $n$, a high characteristic fiber stress $\sigma_0$, a low matrix creep constant $B$, and a low Weibull modulus of the fibers $m$, all increase the composite creep rate and decrease its lifetime (if finite). Moreover, the models described herein, the Curtin model for the fast-fracture tensile behavior of composites and the models in Du and McMeeking (1995), all predict better ultimate behavior for the composite for high ratio $\tau_0/S_c$ (Fig. 6). It should be noted, however, that in Du and McMeeking
Evolution of the strain for composites with creeping (solid line) and non-creeping fibers (dashed line). The curves of the strain versus time for creeping fibers estimated for a composite subjected to constant applied load $\sigma_0/S_0 = 0.2$. The stresses are normalized by their initial values. The values of the parameters are $f = 0.3$, $L = 5000$, $m = 5$, $n = 2$, $E = 1.34$, $\nu = 3$, $\mu = 2$, $\tau_0 = 0.0084$ and $\eta = 4 \times 10^{-4}$.

Moreover, from the set of problems analyzed, it is found that creep rate, creep life and overall strain at rupture, are all sensitive to the creep-compliance of the fibers. The creep-compliance of the fibers increases as the parameters $A$, $I$ and $p$ of the constitutive law (3) increase. High fiber creep-compliance drives composites to creep faster, leading however to greater lifetimes and greater overall strains at rupture. The faster creep rate for more compliant fibers is self-evident, while the greater lifetimes observed are attributed to two different factors that alter the distribution of stresses on the fibers and matrix with time: (i) The more compliant the fibers, the slower the matrix stress relaxation – since the matrix must deform with a rate compatible with the more creep-resistant fibers – and therefore the less the load carried by the fibers. Hence, the stress in the fibers increases slower and fewer fibers fail. The inverse situation, load transfer from fibers to a more creep-resistant matrix is possible but it is not addressed in the present paper: (ii) The more compliant the fibers, the less significant the stress relaxation in broken fibers and thus the less the recovery length increase. As a result, the load carrying capacity of the broken fibers decreases more slowly and so does the load carried by the unbroken ones. Hence again, fewer fibers fail. The basic assumptions of the shear-lag model used for the time evolution of the recovery length are those of Hedgepeth (1961), for which it is perhaps instructive to note that the recovery length scales as $\sqrt{E/G}$ if both constituents are assumed elastic ($E$ is the Young modulus of the fibers and $G$ the effective shear modulus of the matrix). The greater overall strains at rupture is the net outcome of the greater lifetimes and faster creep rates. For example, if the fiber primary creep is ignored then creep rate, as well as lifetime and strain at rupture, are all underestimated even for short-term creep (Fig. 9(a) and (b)). The second factor, i.e. the extent of fiber stress relaxation, becomes important only at long times (for low stress levels), since the shear stress relaxes very slowly in comparison with the axial normal stress in the matrix (Fig. 7). While for low stress levels the inclusion of fiber stress relaxation is essential.
for the prediction of the lifetime and rupture strain (Fig. 8), for intermediate and high stress levels the effect may as well be ignored (Fig. 9).

The fact that high fiber creep-compliance is desirable (in the absence of any explicit creep-damage mechanism) in terms of composite lifetime but not in terms of overall strain is considered of importance for effective composite design and optimization. For example, for some uses (e.g. gas turbines) an engineer may only permit 0.1% strain in a lifetime of 20 years while for other uses (e.g. some jet engines) 1% in 1000 h may be allowable.

Appendix A

Here we use the principle of superposition in time, which was proposed in general for non-agging phenomena by Boltzmann (1874), in order to prove that the viscous part of relation (2) is actually a generalization for the case of a time-dependent fiber stress of (3) (valid for constant fiber stress). Since in a fiber-reinforced composite there is load redistribution due to fiber failure, it is relation (2) that should be used for the study of failure of fiber-reinforced composites and not relation (3) (as it has been erroneous in the case many works in the literature).

Integration of (3) gives

$$\varepsilon_f(t) = \frac{A}{p + 1} \sigma_f t^{p+1},$$

(A.1)

for $\varepsilon_f(0) = 0$. Introducing the transformations

$$\varepsilon^c_f = (\varepsilon_f)^{1/(p+1)}, \quad \sigma = (\sigma_f)^{1/(p+1)} \quad \text{and} \quad A = [A(p + 1)]^{1/(p+1)},$$

the above equation reads as

$$\varepsilon^c_f(t) = \tilde{A} \tilde{\sigma} t.$$

The linearity of equation (A.2) with respect to the stress history first implies that, at constant axial stress $\sigma$, applied at time $t_i$, the corresponding transformed strain $\varepsilon^c_f(t)$ at any time $t > t_i$ may be written as

$$\varepsilon^c_f(t) = \tilde{A} \tilde{\sigma} (t - t_i).$$

(A.3)

The linearity implies more the principle of superposition (in time). Every stress history can be decomposed into infinitesimal steps, $d\sigma_f(t_i)$ applied at various times $t_i$ less than the current time $t$. According to (A.3), each step history produces an infinitesimal strain history $d\varepsilon^c_f(t_i) = (t - t_i)d\sigma_f(t_i)$. Summing all the infinitesimal contributions, one obtains

$$\varepsilon^c_f(t) = \tilde{A} \tilde{\sigma} \int_0^t (t - t_i)d\sigma_f(t_i).$$

(A.4)

where it was assumed that $\varepsilon_f^c(0) = 0$. Differentiation with respect to time now gives

$$\dot{\varepsilon}_f^c(t) = \tilde{A} \tilde{\sigma} \dot{\sigma}_f(t),$$

(A.5)

where it was assumed that $\sigma_f(0) = 0$, or

$$\varepsilon^c_f = (p + 1)^{1/(p+1)} \varepsilon_f^{1/(p+1)}.$$  \hspace{1cm} (A.6)

This equation is not only a consequence of the principle of superposition but an equivalent alternative statement. Indeed, (3) follows from (A.6). Assumptions $\varepsilon_f^c(0) = 0$ and $\sigma_f(0) = 0$ are justified by the initial conditions imposed on the composite as it is made precise in Section 2.

From the above analysis it is deduced that the fibers follow the constitutive relation

$$\dot{\varepsilon}_f = \frac{\sigma_f}{E_f} + C \left( \varepsilon_f - \frac{\sigma_f}{E_f} \right)^{-\mu},$$

(A.7)

where $C = (p + 1)^{(p+1)/p}A^{(p+1)/p}$, $v = 1/(p + 1)$ and $\mu = -p/(p + 1)$, if the mechanical analogue of a Maxwell unit consisting of a linear spring ($\dot{\varepsilon}_f = \sigma_f/E_f$) connected in series ($\varepsilon_f = \varepsilon_f^c + \varepsilon_f^d$) with a nonlinear dashpot (A.6) is adopted.

Appendix B

In this appendix we describe the finite difference scheme that was employed to solve (30) of the main text for the case of constant displacement loading ($\dot{x} = 0$). A similar procedure can be applied for the case of constant uniaxial applied stress.

Introducing the dimensionless variables

$$\tilde{x} = \frac{E_f E}{S_c}, \quad \tilde{\sigma} = \frac{\sigma}{S_c}, \quad \tilde{t} = \frac{tBE_f}{S_c}, \quad \tilde{E}_f = \frac{E_f}{E_m},$$

$$\tilde{\eta} = \frac{CE_f^2 \varepsilon^{n-1}}{B}, \quad \tilde{D} = \frac{D}{T}, \quad \tilde{L}_f = \frac{L_f}{L}, \quad \text{and} \quad \tilde{x} = \frac{x}{L},$$

(B.1)

Eq. (30) becomes

$$\frac{\partial \varepsilon^c_f}{\partial \tilde{t}} + \frac{1 + \nu}{2} \frac{\partial^2 \tilde{\sigma}_f^c}{\partial \tilde{t}^2} \phi \frac{\partial^2 \tilde{\sigma}_f^c}{\partial \tilde{x}^2} + 3 \tilde{D}^2 \frac{\partial \tilde{\sigma}_f^c}{\partial \tilde{t}} \phi \frac{\partial^2 \tilde{\sigma}_f^c}{\partial \tilde{x}^2} - \tilde{\eta} (\tilde{\sigma}_f^c)^m (\tilde{\sigma}_f^c)^n,$$

(B.2a)

$$\frac{\partial \tilde{\sigma}_f^c}{\partial \tilde{t}} = \tilde{\eta} (\tilde{\sigma}_f^c)^m (\tilde{\sigma}_f^c)^n,$$

(B.2b)

where $\phi = \frac{\sqrt{\nu}}{\sqrt{\nu} + 1}$, while the initial and boundary conditions read as

$$\tilde{\sigma}_f^c(0, \tilde{t}) = \tilde{\sigma}_f(\tilde{t}), \quad \tilde{\sigma}_f^c(1/2, \tilde{t}) = 0,$$

(B.3)

$$\tilde{\sigma}_f^c(\tilde{x}, 0) = \begin{cases} \tilde{\sigma}_f(0), & \text{for } 0 \leq \tilde{x} \leq 1/2 - \tilde{L}_f, \\ \frac{\tilde{\sigma}_f(0)}{2\tilde{L}_f} (1 - 2\tilde{x}), & \text{for } 1/2 - \tilde{L}_f < \tilde{x} \leq 1/2, \end{cases}$$

(B.4)

and

$$\tilde{\sigma}_f^c(\tilde{x}, 0) = 0.$$  \hspace{1cm} (B.5)

To ease the exposition of the numerical scheme for the solution of (B.2) we introduce

$$z = (\tilde{\varepsilon}_f^c)^{1/(p+1)}, \quad u = \tilde{\sigma}_f^c,$$

and write $\tilde{x} = \frac{1 + \nu}{2} \tilde{D}^2 \tilde{E}_f \phi$, $\beta = \frac{\tilde{D}^2 \tilde{E}_f \phi}{2}$ Then, (B.2) becomes

$$\frac{\partial u}{\partial \tilde{t}} = z \frac{\partial^2 u}{\partial \tilde{t}^2} + \beta \frac{\partial^2 u}{\partial \tilde{x}^2} \left[ p(\tilde{x}, \tilde{t}, u) \frac{\partial u}{\partial \tilde{t}} - \tilde{\eta} u^2 z \frac{\partial z}{\partial \tilde{t}} \right],$$

(B.6a)

$$\frac{\partial \tilde{z}}{\partial \tilde{t}} = \tilde{\eta} u \tilde{z},$$

(B.6b)

where

$$p(\tilde{x}, \tilde{t}, u) = \left( \tilde{\sigma}_f^c \right)^m \left( 3 \tilde{D}^2 \frac{\partial \tilde{\sigma}_f^c}{\partial \tilde{t}} \phi \frac{\partial \tilde{\sigma}_f^c}{\partial \tilde{x}^2} \right)^n \mu.$$  \hspace{1cm} (B.7)

The initial and boundary conditions B.3, B.4 and B.5 for the new dependent variables $u$ and $z$ now read

$$\tilde{u}(0, \tilde{t}) = \tilde{\sigma}_f(\tilde{t}), \quad \tilde{u}(1/2, \tilde{t}) = 0,$$

(B.8)

$$\tilde{u}(\tilde{x}, 0) = \begin{cases} \tilde{\sigma}_f(0), & \text{for } 0 \leq \tilde{x} \leq 1/2 - \tilde{L}_f, \\ \frac{\tilde{\sigma}_f(0)}{2\tilde{L}_f} (1 - 2\tilde{x}), & \text{for } 1/2 - \tilde{L}_f < \tilde{x} \leq 1/2, \end{cases}$$

(B.9)

$$\tilde{z}(\tilde{x}, 0) = 0.$$  \hspace{1cm} (B.10)

respectively. We employ a finite difference scheme for the numerical solution of this coupled system of a partial differential equation and an ordinary differential equation. For $J \in \mathbb{N}$, $J \geq 1$ we denote by $\tilde{x}_i = h, j = 0, 1, \ldots, J + 1, \text{the points of a regular partition of the interval } [0,1/2] \text{ of mesh size } h = \frac{1}{J}$. We denote the time step by $\Delta \tilde{t}$ and
write $i^k = -\Delta t^k$ for $k \geq 0$. Our numerical scheme produces approximations $(u_j^k, z^k)$ of the exact solution $(u, z)$ at the points $(x_j, t^k)$, $j = 0, 1, \ldots, J + 1$, $k \geq 0$ as follows:

$$
\frac{u_{j+1}^k - u_j^k}{\Delta t} = \beta \left[ \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{(\Delta x)^2} \right],
$$

$$
\frac{u_{j+1}^k - u_j^k}{\Delta t} = \frac{\alpha}{\Delta t} \left[ \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{(\Delta x)^2} \right] + \frac{\beta}{(\Delta t)^2} \left[ \frac{\eta(\Delta x)^2}{(\Delta x)^2} \right],
$$

$$
0 \leq j \leq J + 1, \quad k \geq 0. \tag{B.11}
$$

Eq. (B.12) is a modified Crank–Nicolson method for (B.6a) and is equivalent to a symmetric tridiagonal system of linear equations. The coefficient matrix changes with time but the overall cost of solving this system of equations remains $O(J)$.

References


