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Periodic Solutions of Generalized Liénard Equations

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I. COUNTEREXAMPLE

Consider the equation

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0. \tag{1}$$

It is equivalent to the differential system

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= -f(x, v)v - g(x). \end{aligned} \tag{2}$$

For the differential system (2), P. J. Ponzo and N. Wax in [1] have shown a result about the existence of periodic solutions as follows. (See [1, Theorem 1])

Let (H1) (a) $f(0, 0) < 0$,

(b) $f(a, v) = f(b, v) = 0,\ a < 0,\ b > 0,\ \forall v$,

(c) $f(x, v) > 0,\ x < a,\ x > b$,

(d) $f(x, v)$ is locally Lipschitz in x and v,

(e) for $v \geqslant 0$ and every fixed $x < a$, $vf(x, v)$ is a strictly increasing function of v, with $\lim_{v \to +\infty} vf(x, v) = +\infty$;

(H2) (a) $xg(x) > 0,\ x \neq 0$,

(b) $g(x)$ is locally Lipschitz,

(c) for $G(x) \triangleq \int_0^x g(\xi)\, d\xi,\ G(\pm\infty) = +\infty$;

(H3) $u_\angle^*(x) = \max_{s \leqslant x} u_\angle(s)$ exists for every $x < a$, where

$$u_\angle(x) f(x, u_\angle(x)) + g(x) = 0, \qquad x < a.$$

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Then Eq. (2) has at least one non-zero periodic solution if (H1)–(H3) are satisfied.

These conditions, however, are not sufficient. In this paper, we shall give an example such that the conditions (H1)–(H3) are satisfied, but there is no non-zero periodic solution. Moreover, we shall give some sufficient conditions for the existence of periodic solutions.

EXAMPLE 1. Consider the equation

$$\ddot{x} + (\dot{x}^2 + 1)(x^2 - 4)\dot{x} + x = 0. \tag{3}$$

It is equivalent to the system of ordinary differential equations in the plane

$$\begin{aligned} \dot{x} &= v. \\ \dot{v} &= -(x^2 - 4)(v^2 + 1)v - x, \end{aligned} \tag{4}$$

where

$$f(x, v) = (x^2 - 4)(v^2 + 1), \qquad g(x) = x.$$

We now prove that the conditions (H1)–(H3) are satisfied.

In fact, we have

(H1) (a) $f(0, 0) = (-4) \cdot 1 = -4 < 0$,

(b) let $a = -2$, $b = 2$, then

$$f(a, v) = (4 - 4)(v^2 + 1) = 0, \qquad \forall v,$$

$$f(b, v) = (4 - 4)(v^2 + 1) = 0, \qquad \forall v,$$

(c) $f(x, v) > 0$, $x < a$, $x > b$,

(d) $f(x, v) = (x^2 - 4)(v^2 + 1)$ is locally Lipschitz in x and v,

(e) for $v \geqslant 0$ and every fixed $x < a$, $vf(x, v) = (x^2 - 4)(v^3 + v)$ is a strictly increasing function of v, with $\lim_{v \to +\infty} vf(x, v) = \lim_{v \to +\infty} (x^2 - 4)(v^3 + v) = +\infty$;

(H2) (a) $xg(x) = x^2 > 0$, $x \neq 0$,

(b) $g(x) = x$ is locally Lipschitz,

(c) for $G(x) \triangleq \int_0^x g(\xi)\, d\xi = \int_0^x \xi\, d\xi = \frac{1}{2} x^2$, then $G(\pm\infty) = +\infty$;

(H3) for $v \geqslant 0$ and every fixed $x < a$, $vf(x, v)$ is a strictly increasing function of v, with $\lim_{v \to +\infty} vf(x, v) = +\infty$.

Thus $\exists v_x > 0$ such that $v_x f(x, v_x) > -g(x)$. Moreover, $0 \cdot f(x, 0) = 0 < -g(x)$, and $vf(x, v)$ and $g(x)$ are both continuous functions, thus there exists a unique continuous function $u_\angle(x) > 0$ such that $(x^2 - 4)(u_\angle^2(x) + 1)\, u_\angle(x) + x = 0$, $x < a$.

Because $\lim_{x \to -\infty} (-x/x^2 - 4) = 0$, then $\exists k > 0$, $\nexists x < -k \Rightarrow u_{\angle}(x) \leqslant 1$. Hence $u^*_{\angle}(x) = \max_{s \leqslant x} u_{\angle}(s)$ exists for every $x < a$.

Thus, the conditions (H1)–(H3) are satisfied in the system (4).

We now prove that there does not exist a non-zero periodic solution of the system (4).

We make a curve passing through the point $A(1, \sqrt{3})$

$$v = \frac{0.9^{9/8}\sqrt{3}}{(1.9 - x)^{9/8}}, \qquad 1 \leqslant x < 1.9. \tag{5}$$

Thus

$$\frac{dv}{dx} = \frac{\frac{9}{8} \cdot 0.9^{9/8}\sqrt{3}}{(1.9 - x)^{17/8}}.$$

On the curve (5), the slope of trajectories of the system (4) is

$$\begin{aligned}\left.\frac{dv}{dx}\right|_{(4)} &= (4 - x^2)(v^2 + 1) - \frac{x}{v} \\ &= (4 - x^2) + (4 - x^2) \cdot 3 \cdot 0.9^{9/4}(1.9 - x)^{-9/4} \\ &\quad - 0.9^{-9/8} \cdot 3^{-1/2} \cdot x(1.9 - x)^{9/8}.\end{aligned}$$

We now compare the $(dv/dx)|_{(4)}$ with the slope of the curve (5). Let

$$\begin{aligned} q(x) &= (4 - x^2)(1 + 3 \cdot 0.9^{9/4} \cdot (1.9 - x)^{-9/4}) \\ &\quad - 0.9^{-9/8} \cdot 3^{-1/2} \cdot x(1.9 - x)^{9/8} \\ &\quad - \tfrac{9}{8} \cdot 0.9^{9/8} \cdot \sqrt{3}\,(1.9 - x)^{-17/8} \\ &= \frac{\left(\begin{array}{l}(4 - x^2)(1.9 - x)^{9/4} + 3 \cdot 0.9^{9/4}(4 - x^2) \\ \qquad - 0.9^{-9/8} \cdot 3^{-1/2} \cdot x(1.9 - x)^{27/8} - \frac{9}{8} \cdot 0.9^{9/8}\sqrt{3}\,(1.9 - x)^{1/8}\end{array}\right)}{(1.9 - x)^{9/4}} \\ &\qquad (1 \leqslant x < 1.9). \end{aligned}$$

For $1 \leqslant x < 1.9$, $(1.9 - x)^{9/4} > 0$,

$$2 + x > 0.9^{-9/8} \cdot 3^{-1/2} \cdot x$$

$$(2 - x)(1.9 - x)^{18/8} > (1.9 - x)^{27/8},$$

thus

$$(4 - x^2)(1.9 - x)^{9/4} > 0.9^{-9/8} \cdot 3^{-1/2} \cdot x(1.9 - x)^{27/8}.$$

We prove the following inequality

$$\frac{3\cdot 0.9^{9/4}\cdot(4-x^2)}{\frac{9}{8}\cdot 0.9^{9/8}\cdot\sqrt{3}\,(1.9-x)^{1/8}} = \frac{0.8\sqrt{3}\cdot 0.9^{1/8}(4-x^2)}{(1.9-x)^{1/8}} \geqslant 1, \qquad 1\leqslant x<1.9,$$

that is

$$0.8\sqrt{3}\cdot 0.9^{1/8}(4-x^2)\geqslant(1.9-x)^{1/8}.$$

It is equivalent to the inequality

$$0.8^8\cdot 3^4\cdot 0.9(4-x^2)^8\geqslant 1.9-x, \qquad 1\leqslant x<1.9.$$

Let

$$y=0.9\cdot 0.8^8\cdot 3^4(4-x^2)^8-(1.9-x) = h_1(x)-h_2(x),$$

thus

$$\frac{dy}{dx}=-16\cdot 0.9\cdot 0.8^8\cdot 3^4 x(4-x^2)^7+1,$$

$$\frac{d^2y}{dx^2}=16\cdot 0.9\cdot 0.8^8\cdot 3^4(4-x^2)^6\cdot(15x^2-4)>0, \qquad 1\leqslant x<1.9.$$

Moreover

$$\left.\frac{dy}{dx}\right|_{x=1}<0,$$

$$\left.\frac{dy}{dx}\right|_{x=1.9}>0,$$

so, there exists a unique solution x_0 of the equation

$$\frac{dy}{dx}=0, \qquad (1\leqslant x<1.9). \tag{6}$$

Let

$x_1=1.89$, then $y(x_1)\approx 0.003746296$,

$x_2=1.889$, then $y(x_2)\approx 0.003748056$,

$x_3=1.8895$, then $y(x_3)\approx 0.003739536$,

thus $x_0\in(1.889, 1.89)$.

For $x \in [1, 1.9)$, $h_1(x)$ is a decreasing function, thus

$$\begin{aligned} y(x_0) &= h_1(x_0) - h_2(x_0) \\ &\geqslant [h_1(x_1) - h_2(x_1)] - [h_2(x_0) - h_2(x_1)] \\ &\geqslant 0.00374629 - 0.001 > 0. \end{aligned}$$

Then

$$y(x) > 0, \qquad 1 \leqslant x < 1.9.$$

It follows that

$$0.8\sqrt{3}\, 0.9^{1/8}(4 - x^2) > (1.9 - x)^{1/8},$$

thus

$$q(x) > 0, \qquad \text{when} \quad 1 \leqslant x < 1.9.$$

By the definition of $q(x)$, we obtain

$$\left.\frac{dv}{dx}\right|_{(4)} - \frac{dv}{dx} > 0.$$

Hence, we have

$$\left.\frac{dv}{dx}\right|_{(4)} > \frac{dv}{dx} > 0.$$

Note that for the system (4), $dx/dt = v > 0$ in the half plane $v > 0$. So, the trajectory of the system (4) through any point of the curve (5) passes for increasing t through this curve from below to above. Then, it follows from $dx/dt > 0$ that the positive semi-trajectory $f(A, R^+)$ lies above the curve (5). Since the curve (5) starting at A approaches monotonically to infinity as $x \to 1.9$ and there are no singular points in the strip $1 \leqslant x < 1.9$, it follows that $f(A, R^+)$ approaches to infinity as $t \to +\infty$.

Let $\lambda(x, v) = v^2 + x^2$, thus

$$\begin{aligned} \left.\frac{d\lambda}{dt}\right|_{(4)} &= 2(v \cdot \dot{v} + x \cdot \dot{x}) \\ &= 2[-xv - (x^2 - 4)(v^2 + 1)v^2 + xv] \\ &= 2(4 - x^2)(v^2 + 1)v^2. \end{aligned}$$

Let B_0 denote the region $x^2 + v^2 \leqslant 4$. Note that $(d\lambda/dt)|_{(4)} \geqslant 0$ in B_0, and the set $B = \{(x, v) \in B_0 \mid (d\lambda/dt)|_{(4)} = 0\}$ does not contain any whole non-zero trajectory of the system (4). By the tangent curve method of Poincaré, it follows that there are no closed trajectories or singular closed trajectories of the system (4) in B_0 (see [3]). Since the point $A(1, \sqrt{3})$ is a strict exit point of the region B_0 and $(d\lambda/dt)|_{(4)} \geqslant 0$, the negative semi-trajectory $f(A, R^-)$ must stay in B_0.

Therefore the α-limit set of $f(A, R^-)$ must be contained in B_0. By the Poincaré–Bendixson theory, it is either a single critical point, or a single closed trajectory, or a connected set which is the union of whole trajectories, some of which are critical points and the others non-closed trajectories tending to critical points both as $t \to -\infty$ and as $t \to +\infty$. As stated above, there are no closed trajectories or singular closed trajectories of the system (4) in B_0, so, $f(A, R^-)$ must tend to the origin.

Thus every trajectory of the system (4) is such that one side of it tends to the origin and the other side approaches to infinity.

Hence there does not exist a non-zero periodic solution of the system (4).

II. The Existence of Periodic Solutions

In this paper, we shall give a sufficient condition about existence of periodic solutions.

As first we assume

(a) $f(0, 0) < 0$,

(b) $a < 0, b > 0, f(x, v) \geqslant 0$ for $x < a, x > b, \forall v$.

For $\forall x \in [a, b]$ and $\forall v, f(x, v) \geqslant -M$, where M is a positive constant,

(c) $xg(x) > 0, x \neq 0$. For $G(x) \triangleq \int_0^x g(\xi)\, d\xi$, $G(\pm\infty) = +\infty$,

(d) $f(x, v)$ is locally Lipschitz in x and v, $g(x)$ is locally Lipschitz.

(e) for $v \geqslant 0$ and every fixed $x < a$, $vf(x, v)$ is a strictly increasing function of v, and $\exists v_x > 0$ such that $v_x f(x, v_x) \geqslant -g(x)$. $U^*_\angle(x) = \max_{s \leqslant x} u_\angle(s)$ exists for every $x < a$, where

$$u_\angle(x) \cdot f(x, u_\angle(x)) + g(x) = 0.$$

THEOREM. *System* (2) *has at least one non-zero periodic solution if* (a)–(e) *are satisfied.*

Proof. By the conditions, existence and uniqueness of solutions of initial value problem are valid.

Let $\lambda(x, v) = v^2/2 + G(x)$, thus

$$\left.\frac{d\lambda}{dt}\right|_{(2)} = v \cdot \dot{v} + g(x) \cdot \dot{x}$$
$$= -f(x, v)v^2 - g(x)v + g(x)v$$
$$= -f(x, v)v^2.$$

By the conditions (a) and (d), for $0 < c_0 \ll 1$, every point (x, v) on the closed curve $v^2/2 + G(x) = c_0$, we have $f(x, v) < 0$, thus

$$\left.\frac{d\lambda}{dt}\right|_{(2)} > 0, \qquad v \neq 0.$$

Then the oval $v/2 + G(x) = c_0$ serves as an inner bound for the annulus. (See Fig. 1)

We define

$$\alpha = \min_{x \leqslant a-1} \left\{ \frac{u_{\angle}^{*2}(x)}{2} + G(x) \right\}.$$

Because $G(x) \to +\infty$, as $x \to -\infty$, then

$$\left\{ \frac{u_{\angle}^{*2}(x)}{2} + G(x) \right\} \to +\infty, \qquad \text{as} \quad x \to -\infty.$$

Thus $\exists x_0 \in (-\infty, a-1]$ such that $\alpha = u_{\angle}^{*2}(x_0)/2 + G(x_0)$.

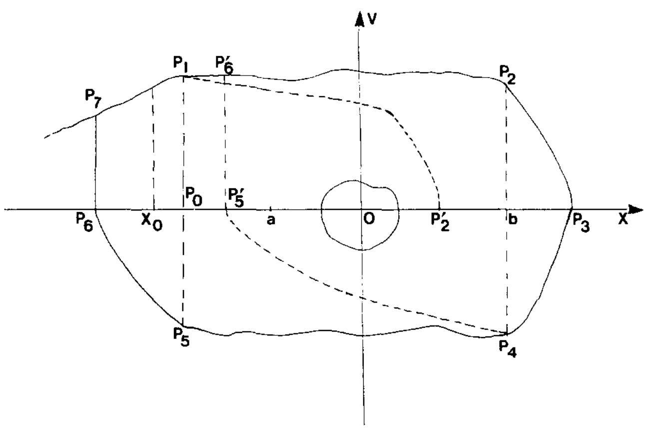

FIG. 1. The phase plane annulus described in the proof of the theorem.

Now define Γ, in $x \leqslant a-1$ as

$$\Gamma: v = \begin{cases} u_L^*(x) & \text{for} \quad x \leqslant x_0, \\ \{2[\alpha - G(x)]\}^{1/2} & \text{for} \quad x_0 \leqslant x \leqslant a-1. \end{cases}$$

We start our outer bound at the point p_1 at $x(p_1) = a-1$, $v(p_1) = \sqrt{2[\alpha - G(a-1)]}$. We consider the unique solution of Eq. (2) passing through p_1. One of three possibilities must hold:

(1) it will intersect the x axis at p_2' on the line segment $\overline{ob}$; or

(2) it will intersect the line $x = b$ at point p_2, where $x(p_2) = b$, $v(p_2) > 0$; or

(3) it will stay in the region D: $v > 0$, $a - 1 \leqslant x < b$.

At first we consider the case (3). Because there is no critical point in the region D, and origin 0 is a repellar, then the positive semi-trajectory $\gamma^+(p_1)$ is not bounded. Moreover, in the region D, we have $dx/dt = v > 0$, thus the positive semi-trajectory $\gamma^+(p_1)$ has a vertical asymptotic line. That is $\exists x_1 \in (a-1, b]$ such that on the positive semi-trajectory $\gamma^+(p_1)$ we have $\lim_{x \to x_1 - 0} dv/dx = +\infty$. Because $dv/dx = -f(x, v) - g(x)/v$, and for $x \in [a-1, b]$, $g(x)$ is a bounded function, so on the $\gamma^+(p_1)$ we have $\lim_{x \to x_1 - 0} f(x, v) = -\infty$, but this contradicts the condition (b).

We now consider the case (2). The total oval going through p_2

$$\frac{v^2}{2} + G(x) = \frac{v^2(p_2)}{2} + G(b)$$

will intersect the x axis at p_3, and the line $x = b$ again at p_4. One has that $x(p_3) = B$, $v(p_3) = 0$, where

$$G(B) = \frac{v^2(p_2)}{2} + G(b),$$

$$x(p_4) = b, \qquad v(p_4) = -v(p_2).$$

Proceeding as before, the trajectory of Eq. (2) passing through the point p_4 is such that either

(1′) it will intersect the x axis at p_5' on the line segment $\overline{p_0 0}$ where $p_0(a-1, 0)$; or

(2′) it will intersect the line $x = a-1$ at p_5, where $x(p_5) = a-1$, $v(p_5) < 0$.

At first we consider the case (1′), the vertical segment p_5' to p_6', a point on the arc $f(p_1; 0, t_0)$, where $f(p_1, t_0) = p_2$. Completes the outer bound.

We consider the case (2′). The oval going through p_5

$$\frac{v^2}{2}+G(x)=\frac{v^2(p_5)}{2}+G(a-1)$$

will intersect the x axis at p_6. One has that $x(p_6)=A$, $v(p_6)=0$, where

$$G(A)=\frac{v^2(p_5)}{2}+G(a-1).$$

The vertical segment p_6 to p_7, a point on the Γ, completes the outer bound.

To the case (1), proceeding as before, completes the outer bound.

This completes the proof of the theorem.

III. An Example

Example 2. Prove equation

$$\ddot{x}+(x^2\dot{x}^2+x^2-1)\dot{x}+x=0 \tag{7}$$

has at least one non-zero periodic solution.

Proof. Equation (7) is equivalent to the differential system

$$\begin{aligned} \dot{x}&=v,\\ \dot{v}&=-(x^2v^2+x^2-1)v-x. \end{aligned} \tag{8}$$

where $f(x,v)=x^2v^2+x^2-1$, $g(x)=x$.

Then we have

(a) $f(0,0)=0+0-1=-1<0$,

(b) let $a=-1$, $b=1$, then $f(x,v)>0$, for $x<a$, $x>b$, $\forall v$. Let $M=1$, for $\forall x\in[a,b]$, $\forall v$,

$$f(x,v)=x^2v^2+x^2-1\geqslant -1=-M,$$

(c) $xg(x)=x^2>0$, $x\neq 0$; $G(x)=\int_0^x g(\xi)\,d\xi=\int_0^x \xi\,d\xi=\frac{1}{2}x^2$, then $G(\pm\infty)=+\infty$,

(d) $g(x)=x$ is locally Lipschitz, and $f(x,v)=x^2v^2+x^2-1$ is locally Lipschitz in x and v,

(e) for $v\geqslant 0$ and every fixed $x<a$, $vf(x,v)=(x^2v^2+x^2-1)v$ is a strictly increasing function of v, and $\exists v_x=1$ such that

$$v_xf(x,v_x)=x^2+x^2-1>x^2>-x=-g(x).$$

Then there exists a unique continuous function $u_{\angle}(x) \leqslant 1$ such that

$$u_{\angle}(x)\, f(x, u_{\angle}(x)) + g(x) = 0.$$

Thus $u^*_{\angle}(x) = \max_{s \leqslant x} u_{\angle}(s)$ exists for every $x < a$.

Thus the conditions (a)–(e) are satisfied.

Hence by the theorem, Eq. (8) has at least one non-zero periodic solution.

So Eq. (7) has at least one non-zero periodic solution.

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