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Group classification of a class of equations arising in financial mathematics

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ABSTRACT

We provide a group classification of a class of nonlinearisable evolution partial differential equations which arise in Financial Mathematics. Sixteen different cases are identified for the general problem and another seven for a restricted version. In the cases for which the algebra is suitable we determine the solution to the problem $u(0,x) = U$, where U is a constant. In addition we provide a number of solutions based upon reduction using inequivalent subalgebras.

1. Introduction

In the modelling of the value of an option, a contingent liability or some other financial instrument such as a derivative the end result is often a partial differential equation the solution of which is required to be compatible with some constraint such as a terminal condition, *i.e.* $u(T,x) = U$, where $u(t,x)$ is the value of the option (say) at time t if the underlying asset has value x and that at some time, T, that value is to be U. It is usual for both T and U to be fixed although there are many variations possible. In many cases the partial differential equations derived are linear or trivially linearisable and a surprising number of them have the maximal number of Lie point symmetries for an $(1+1)$ evolution partial differential equation which means that they can be transformed by means of a point transformation to an equation of the form of the classical heat equation, $u_t = u_{xx}$. The use of such equations has grown tremendously since the pioneering works of Black, Scholes and Merton [4,5,25] on the value of an option. The symmetry analysis of the Black–Scholes equation was firstly undertaken by Gasizov and Ibragimov [15].[3] In terms of the Mubarakzyanov classification scheme [26–29] the algebra[4] of the symmetries is $\{A_{3,8} \oplus_s A_{3,1}\} \oplus_s \infty A_1$, where in a more common parlance the first subalgebra is $\mathrm{sl}(2,R)$, the second subalgebra, W_3, is the three-dimensional Heisenberg–Weyl algebra more familiar from quantum mechanics and the third, ∞A_1 is the infinite-dimensional abelian subalgebra generated by the solutions of the linear evolution partial differential equation. It should be emphasised that not all linear or linearisable evolution partial differential equations have this algebra. Heath et al. [16] modelled a problem of risk minimisation with the equation

$$2u_t + 2au_x + b^2u_{xx} - u_x^2 + 2\nu(x) = 0 \tag{1.1}$$

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[3] See also Ibragimov and Wafo Soh [18]. Although this paper precedes in appearance that cited in the main body of the text, the latter was prepared before the former.

[4] We use the standard notations $\oplus$ and $\oplus_s$ to denote direct sum and semidirect sum, respectively. In the instance of the latter the action is to the right.

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in which the ultimate term, $\nu(x)$, depends upon the 'space' variable. It is quite obvious that (1.1) is trivially linearisable. The form of the function $\nu(x)$ has a critical effect upon the number of Lie point symmetries possessed by the equation. In general there are just the two obvious point symmetries, ∂_t and $f(t,x)\exp[u/b^2]\partial_u$, where $f(t,x)$ satisfies (1.1), but for certain functions $\nu(x)$ the number of point symmetries is greater even up to the maximal number admitted by an $(1+1)$ evolution partial differential equation [10].

In this paper we consider the group classification of a class of nonlinear evolution partial differential equations of the form

$$u_t + F(x)u_{xx} + Q(x)u_x^2 + G(x)u_x + P(x)u + C(x) = 0, \tag{1.2}$$

in which the coefficient functions depend upon the variable x which typically represents the value of the underlying asset, for example the price of a stock upon which an option is placed. Special cases of (1.2) are the Black–Scholes–Merton equation, the Longstaff equation [24], the Vasicek equation [41] and the Cox–Ingersoll–Ross equation [9], in which $Q(x)$ is zero. Specific motivation for this more general form is to be found in the paper of Benth and Karlsen [3] who allow for quite general dependence upon the independent variables in their theoretical discussion although the examples treated – the models of Stein and Stein [37] and Heston [17] – are more than somewhat simpler and are indeed linearisable. Here we insist that (1.2) be not linearisable by means of a point transformation. It order to keep the computations as simple as is consistent with this requirement we do make some simplification of (1.2) using equivalence point transformations, thereby not affecting the algebra. In particular we write

$$u(t,x) = U(t,X) + g(x) \quad \text{and} \quad X = h(x), \tag{1.3}$$

where $h'(x) \neq 0$ and transformations of this form constitute a subgroup of the equivalence group of transformations of (1.2). It is a simple calculation by direct substitution into (1.2) to show that

$$h'(x)^2 = \frac{1}{F(x)} \quad \text{and} \quad g'(x) = \frac{1}{4Q(x)}\big(F'(x) - 2G(x)\big). \tag{1.4}$$

Under this transformation (1.2) becomes

$$U_t + U_{XX} + \frac{Q(x)}{F(x)}U_X^2 + P(x)U + C(x) + g(x)P(x) - \frac{G(x)^2}{4Q(x)} + \frac{F'(x)^2}{16Q(x)}$$
$$- \frac{F(x)G'(x)}{2Q(x)} + \frac{F(x)G(x)Q'(x)}{2Q(x)^2} - \frac{F(x)F'(x)Q'(x)}{4Q(x)^2} + \frac{F(x)F''(x)}{4Q(x)} = 0$$

in which the primes denote differentiation with respect to x. This variable may be eliminated by substituting the inverse transformation $x = h^{-1}(X)$.

The equation for which we perform the group classification is then

$$u_t + u_{xx} + Q(x)u_x^2 + P(x)u + C(x) = 0, \tag{1.5}$$

on reversion to lowercase variables and redefinition of the coefficient functions as appropriate. The requirement that (1.5) be not linearisable means that we exclude the cases $Q(x) = 0$ and $Q(x) = \text{const}$, $P(x) = 0$.

In the case that $Q(x) = \text{const}$ and $P(x) = 0$ (1.5) becomes

$$u_t + u_{xx} + \mu u_x^2 + C(x) = 0.$$

When we apply the transformation

$$u = \frac{1}{\mu}\log v + \frac{1}{2\mu}\int f(x)\,\mathrm{d}x + \lambda t,$$

where $4\mu C(x) = -(2f' + f^2 + 4\mu\lambda)$, the equation can be mapped into

$$v_t + v_{xx} + f(x)v_x = 0,$$

which is patently linear. Alternately we may apply the transformation

$$u = \frac{1}{\mu}\log v + \lambda t,$$

under which the equation may be mapped into

$$v_t + v_{xx} + g(x)v = 0,$$

where $\mu C(x) = g(x) - \lambda\mu$, and this again is linear.

The group classification of equations of diffusion type of a structure similar to that of (1.5) has been the subject of a number of studies. This was already investigated by Lie [23] in the case of linear equations when he classified partial differential equations of the second order in two independent variables (see also the more recent treatment by Ovsiannikov [33]). The investigation of nonlinear equations in terms of their symmetries commenced with the study of the nonlinear diffusion equation, $u_t=(A(u)u_x)_x$, by Ovsiannikov in 1959 [32]. Katkov [21] extended the investigation to generalized Burgers equations of the form $u_t=u_{xx}+B(u)u_x$. Then Dorodnitsyn [11] performed the group classification of equations belonging to the class $u_t=(A(u)u_x)_x+C(u)$ while Akhatov, Gazizov and Ibragimov [1] investigated the classification of equations of the form $u_t=G(u_x)u_{xx}$. The Lie point symmetries of the class of constant-coefficient diffusion–convection equations

$$u_t=\big(A(u)u_x\big)_x+B(u)u_x$$

were investigated by several authors [12,31,42], but a complete classification in terms of its groups was presented only recently [35]. The extension to the investigation of the Lie symmetries of the variable-coefficient equation

$$u_t=\big(u^n\big)_{xx}+g(x)u^m+f(x)u^s u_x$$

was reported by Gandarias [14]. Cherniga and Serov [8] generalised these results through their examination of the Lie symmetries of the nonlinear reaction–diffusion equation with convection, $u_t=(A(u)u_x)_x+B(u)u_x+C(u)$. Popovych and Ivanova [35] performed the complete group classification of the variable-coefficient diffusion–convection equation

$$f(x)u_t=\big(g(x)A(u)u_x\big)_x+B(u)u_x.$$

Subsequently the complete classifications of the classes of reaction–diffusion equations $f(x)u_t=(g(x)u^n u_x)_x+h(x)u^m$, $u_t=(D(u)u_x)_x+h(x)u$ and $f(x)u_t=(g(x)A(u)u_x)_x+h(x)B(u)u_x$, has been presented in [38,40], [39] and [19,20], respectively. Although all of the equations mentioned above are particular cases of the more general class,

$$u_t=F(t,x,u,u_x)u_{xx}+G(t,x,u,u_x),$$

classified by Basarab-Horwath, Lahno and Zhdanov [2], the equivalence group of this latter class is essentially wider than those for the subclasses considered and so the results of Basarab-Horwath, Lahno and Zhdanov cannot be used directly in the classification of the symmetries of the equations mentioned above. However, we do note that these results are useful to find additional equivalence transformations for these classes.

We recall that group classification is one of the symmetry tools used to choose relevant models from parametric classes of systems of (partial or ordinary) differential equations. Thus, for example, real models are often constrained with *a priori* requirements to symmetry properties following from physical laws such as the Galilean or special relativistic principles. Moreover differential equations of models could contain parameters or functions which have been found experimentally and so are not strictly fixed. (These parameters and functions are called arbitrary elements.) At the same time mathematical models should be sufficiently simple to be able to be analysed and solved. The symmetry approach allows one to take the following relevancy criterion for choosing parameter values. Differential equations arising from modelling have to admit a symmetry group with certain properties or the most extensive symmetry group from those possible. This directly leads to the necessity to solve problems of group classification.

In the approach used here under solution of the problem of group classification of a class of (systems of) differential equations we understand the realisation of the following algorithm:

1. Determination of the group $G^{\cap}$ of local transformations that are symmetries for all systems from the class.
2. Construction of the group $G^{\sim}$ (the equivalence group) of local transformations which transform the considered class into itself.
3. Description of all possible $G^{\sim}$-inequivalent values of parameters that admit maximal invariance groups wider than $G^{\cap}$.

Above we saw that the essential equation for our study is (1.5). In Section 2 we determine the Lie point symmetries of this equation in general and in the particular case that $C(x)=0$ up to its equivalence group. We follow in Section 3 with a listing of the Lie algebras for each of the possible cases and indicate the relationship between the classes of the general classification and those which remain in the case of the particular classification. The solutions of the cases for which the terminal problem is meaningful are presented in Section 4. We also present the solution of a related problem in which a possible utility function is determined to allow the existence of a symmetry and hence reduction. Since not every problem in Financial Mathematics is a terminal problem, in Section 5 we provide some examples, selected from cases other than those discussed above, of reductions using optimal systems. We conclude the paper with a brief discussion in Section 6.

2. Group classification

Using the direct method [22,36] we find that equivalence group $G^{\sim}$ of class (1.5) comprises the transformations

$$t' = \varepsilon_1^2 t + \varepsilon_3, \qquad x' = \varepsilon_1 x + \varepsilon_4, \qquad u' = \varepsilon_2 u + \varepsilon_5,$$
$$Q' = \varepsilon_2^{-1} Q, \qquad P' = \varepsilon_1^{-2} P, \qquad C' = \varepsilon_1^{-2}\varepsilon_2 C - \varepsilon_1^{-2}\varepsilon_5 P,$$

where the ε_i, $i = 1, \dots, 5$, are constants subject to the constraint $\varepsilon_1\varepsilon_2 \neq 0$.

Here we present a complete classification of Lie symmetries for class (1.5) modulo the equivalence transformations. We search for operators of the form

$$\Gamma = \tau(t,x,u)\partial_u + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$$

which generate one-parameter groups of point symmetry transformations of equations from class (1.5). These operators satisfy the necessary and sufficient criterion of infinitesimal invariance [6,7,30,33], *i.e.* the action of the rth extension (or prolongation) $\Gamma^{(r)}$ of Γ on the (rth-order) differential equation, modulo the differential equation under consideration, results in zero identically. Here we require that

$$\Gamma^{(2)}\big\{u_t + u_{xx} + Q(x)u_x^2 + P(x)u + C(x)\big\} = 0 \tag{2.1}$$

identically, modulo equation (1.5).

After elimination of u_t due to (1.5), Eq. (2.1) becomes an identity in six variables, t, x, u, u_x, u_{xx} and u_{tx}. In fact Eq. (2.1) is a multivariable polynomial in the variables u_x, u_{xx} and u_{tx}. The coefficients of the different powers of these variables must be zero, giving the determining equations for the coefficients τ, ξ and η. Since Eq. (1.5) is an evolution equation which is a polynomial in the pure derivatives of u with respect to x, it can be shown that $\tau = \tau(t)$ and $\xi = \xi(x,t)$. Finally by taking the coefficients of u_{xx}, u_x^2, u_x and the term independent of derivatives in the Lie infinitesimal invariance criterion, (2.1), we obtain the following determining equations for $\tau(t)$, $\xi(t,x)$ and $\eta(t,x,u)$:

$$2\xi_x - \tau_t = 0,$$
$$\eta_{uu} + Q\eta_u - 2Q\xi_x + \xi Q_x + \tau_t Q = 0,$$
$$2\eta_{xu} - \xi_{xx} - \xi_t + 2Q\eta_x = 0,$$
$$\eta_t + \tau_t C + \xi C_x + \xi P_x u + \eta P + \tau_t P u + \eta_{xx} - \eta_u(C + Pu) = 0.$$

From the first two equations we find that

$$\xi = \tfrac{1}{2}x\tau_t + \sigma(t) \quad \text{and} \quad \eta = -\frac{Q_x}{Q}\xi u + \phi(x,t)e^{-Qu} + \psi(x,t).$$

We substitute for ξ and η into the third and fourth determining equations. In the resultant equations the term independent of u in the substituted third determining equation and the coefficient of u in the substituted fourth give, respectively,

$$\phi Q^3 Q_x = 0 \quad \text{and} \quad -2\phi_x Q_x - \phi Q_{xx} + \phi P Q = 0.$$

Since $Q \neq 0$ and, when $Q = \text{const}$, $P \neq 0$, these two equations give $\phi = 0$. Hence

$$\eta = -\frac{Q_x}{Q}\xi u + \psi(x,t).$$

Substitution of the forms of ξ and η in the last two determining equations yields, by taking coefficients of the two powers of u, four identities in the unknown functions $\tau(t)$, $\sigma(t)$, $\psi(x,t)$, $Q(x)$, $P(x)$ and $C(x)$. When we study all possible cases of integration of these identities up to the equivalence group $G^{\sim}$, we find the following possibilities.

1. $\forall Q(x)$, $\forall P(x)$, $\forall C(x)$: $A^{\cap} = \langle \partial_t \rangle$;
2. $\forall Q(x)$, $P(x) = p$, $\forall C(x)$: $\langle \partial_t, e^{-pt}\partial_u \rangle$;
3. $Q(x) = e^{ax}$ $(a \neq 0)$, $P(x) = px$ $(p \neq 0)$, $C(x) = \frac{pxa^4 - c + \frac{1}{2}a^2p^2x^2 + ap^2x}{2a^4 e^{ax}}$:
$$\big\langle \partial_t, e^{pt/a}\big[2a^2\partial_x - \big(2a^3u + pe^{-ax}\big)\partial_u\big]\big\rangle;$$
4. $Q(x) = x^2$, $P(x) = px^{-2} + b$ $(p \neq 0)$, $C(x) = [16c - (4b^2x^4 + 4pbx^2)\log x - b^2x^4 + 4bx^2]/16x^4$:
$$\big\langle \partial_t, e^{bt}\big[8\partial_t + 4bx\partial_x + b(-8u + 2b\log x + b)\partial_u\big]\big\rangle;$$

5. $Q(x)=x^a$ $(a\neq 0,2)$, $P(x)=px^{-2}+b$, $C(x)=\frac{c}{x^{a+2}}+\frac{b(a^3-3a^2+(2+\frac{1}{2}bx^2+p)a+bx^2)}{2a^2(a-2)x^a}$:

$$\left\langle \partial_t,\, \mathrm{e}^{2bt/a}\left[\frac{a}{b}\partial_t+x\partial_x-\left(au+\frac{b}{a(a-2)}x^{2-a}\right)\partial_u\right]\right\rangle;$$

6. $Q(x)=\mathrm{e}^{ax}$, $P(x)=p$, $C(x)=c\mathrm{e}^{-ax}$:

$$\langle \partial_t,\, \mathrm{e}^{-pt}\partial_u,\, \partial_x-au\partial_u\rangle;$$

7. $Q(x)=x^{-2}$, $P(x)=p$, $C(x)=\frac{3px^2}{4}+c\log x$:

$$\left\langle \partial_t,\, \mathrm{e}^{-pt}\partial_u,\, \mathrm{e}^{-pt}\left[\frac{2}{p}\partial_t-x\partial_x-\left(2u-ct-\frac{p}{8}x^4\right)\partial_u\right]\right\rangle;$$

8. $Q(x)=x^2$, $P(x)=p$ $(p\neq 0)$, $C(x)=cx^{-4}+\frac{p}{4}x^{-2}-\frac{p^2}{4}\log x$:

$$\left\langle \partial_t,\, \mathrm{e}^{-pt}\partial_u,\, \mathrm{e}^{pt}\left[\frac{2}{p}\partial_t+x\partial_x-\left(2u-\frac{p}{8}-\frac{p}{2}\log x\right)\partial_u\right]\right\rangle;$$

9. $Q(x)=x^a$ $(a\neq 0,\pm 2)$, $P(x)=p$, $C(x)=\frac{c+2pa(a-1)(a^2-4)x^2+p^2(a+2)^2x^4}{4a^2(a^2-4)x^{2+a}}$:

$$\left\langle \partial_t,\, \mathrm{e}^{-pt}\partial_u,\, \mathrm{e}^{2pt/a}\left[\partial_t+\frac{p}{a}x\partial_x-\left(pu-\frac{p^2}{a^2(2-a)}x^{-a+2}\right)\partial_u\right]\right\rangle;$$

10. $Q(x)=1$, $P(x)=px^{-2}$, $C(x)=c_1-\frac{4c_1^2p^2}{(4c_3-2p-p^2)^2}x^2+c_3x^{-2}\log x$:

$$\left\langle \partial_t,\, \exp\left[\frac{8pc_1t}{(-p^2-2p+4c_3)}\right]\left[\frac{-p^2-2p+4c_3}{4pc_1}\partial_t+x\partial_x-\left(\frac{c_3}{p}+\frac{2c_1px^2}{p^2+2p-4c_3}\right)\partial_u\right]\right\rangle;$$

11. $Q(x)=1$, $P(x)=px^{-2}$, $C(x)=c^2x^2+\frac{p(2+p)}{4x^2}\log x$:

$$\langle \partial_t,\, -2\cos 4ct\,\partial_t+4cx\sin 4ct\,\partial_x+c\big(4cx^2\cos 4ct-(p+2)\sin 4ct\big)\partial_u,$$
$$2\sin 4ct\,\partial_t+4cx\cos 4ct\,\partial_x-c\big(4cx^2\sin 4ct+(p+2)\cos 4ct\big)\partial_u\rangle;$$

12. $Q(x)=1$, $P(x)=px^{-2}$, $C(x)=-c^2x^2+\frac{p(2+p)}{4x^2}\log x$:

$$\langle \partial_t,\, \mathrm{e}^{4ct}\big(2\partial_t+4cx\partial_x+c\big(4cx^2-p-2\big)\partial_u\big),$$
$$\mathrm{e}^{-4ct}\big(2\partial_t-4cx\partial_x+c\big(4cx^2+p+2\big)\partial_u\big)\rangle;$$

13. $Q(x)=1$, $P(x)=px^{-2}$, $C(x)=\frac{p(2+p)}{4x^2}\log x$:

$$\left\langle \partial_t,\, 2t\partial_t+x\partial_x-\frac{p+2}{4}\partial_u,\, t^2\partial_t+tx\partial_x+\frac{x^2-(p+2)t}{4}\partial_u\right\rangle;$$

14. $Q(x)=1$, $P(x)=p$, $C(x)=\frac{p^2-m^2}{16}x^2+c_1x$:

$$\left\langle \partial_t,\, \mathrm{e}^{-pt}\partial_u,\, \mathrm{e}^{(m-p)t/2}\left[\partial_x-\frac{(p^2-m^2)x+8c_1}{4(m+p)}\partial_u\right],\right.$$
$$\left.\mathrm{e}^{-(m+p)t/2}\left[\partial_x+\frac{(p^2-m^2)x+8c_1}{4(m-p)}\partial_u\right]\right\rangle;$$

15. $Q(x)=1$, $P(x)=p$, $C(x)=\frac{p^2+m^2}{16}x^2+c_1x$:

$$\left\langle \partial_t,\, \mathrm{e}^{-pt}\partial_u,\, \mathrm{e}^{-pt/2}\left[\sin\tfrac{1}{2}tm\,\partial_x+\frac{(m\cos\frac{1}{2}tm-p\sin\frac{1}{2}tm)((p^2+m^2)x+8c_1)}{4(m^2+p^2)}\partial_u\right],\right.$$
$$\left.\mathrm{e}^{-pt/2}\left[\cos\tfrac{1}{2}tm\,\partial_x-\frac{(m\sin\frac{1}{2}tm+p\cos\frac{1}{2}tm)((p^2+m^2)x+8c_1)}{4(m^2+p^2)}\partial_u\right]\right\rangle;$$

Table 1
The Lie algebras of the Lie point symmetries of the first set of (sixteen) equations.

Algebra	Equation number
A_1	1
A_2	2, 3, 4, 5, 10
$A_1 \oplus_s A_2$	6
$A_{3,2} \Leftrightarrow A_1 \oplus_s 2A_1$	7
$A_{3,4} \Leftrightarrow E(1,1)$	8
$A_{3,5}$	9
$A_{3,8} \Leftrightarrow \mathrm{sl}(2,R)$	11, 12, 13
$A_1 \oplus_s 3A_1$	14
$A_1 \oplus_s A_{3,1} \Leftrightarrow A_1 \oplus_s W_3$	15, 16

Table 2
The Lie algebras of the Lie point symmetries of the second set of (seven) equations. The numbers in parentheses indicate the source equation out of the 16 listed in Table 1.

Algebra	Equation number
A_1	1 (1)
A_2	2 (2), 3 (3)
$A_{3,5}$	4 (9)
$A_1 \oplus_s A_2$	5 (6)
$A_{3,8} \Leftrightarrow \mathrm{sl}(2,R)$	6 (13)
$A_1 \oplus_s 3A_1$	7 (14)

16. $Q(x)=1,\ P(x)=p,\ C(x)=\frac{p^2}{16}x^2+c_1x$:

$$\left\langle \partial_t, \mathrm{e}^{-pt}\partial_u, \mathrm{e}^{-pt/2}\left[\partial_x - \frac{p^2x+8c_1}{4p}\partial_u\right], \mathrm{e}^{-pt/2}\left[t\partial_x - \frac{(pt-2)(p^2x+8c_1)}{4p^2}\partial_u\right]\right\rangle.$$

The algebras presented are the maximal invariance algebras if and only if the corresponding sets of functions Q, P, C are not $G^{\sim}$-equivalent to functions with most extensive invariance algebras. For example, in case 5 $(a,b,c) \neq (0,0,0)$.

As a special case of the above we consider the subset of equations in the class

$$u_t + u_{xx} + Q(x)u_x^2 + P(x)u = 0,$$

to obtain which we set $C(x)=0$ in (1.5) since the results can be presented in an elegantly compact form. We do not consider the case, $Q(x)=\mathrm{const}$ and $P(x)=0$, since it is linearisable. The results for the class of equations considered are

1. $\forall Q(x),\ \forall P(x)$: $A^{\cap} = \langle\partial_t\rangle$;
2. $\forall Q(x),\ P(x)=p$: $\langle\partial_t, \mathrm{e}^{-pt}\partial_u\rangle$;
3. $Q(x)=x^a,\ P(x)=px^{-2}$: $\langle\partial_t, 2t\partial_t + x\partial_x - au\partial_u\rangle$;
4. $Q(x)=x^a,\ P(x)=0$: $\langle\partial_t, \partial_u, 2t\partial_t + x\partial_x - au\partial_u\rangle$;
5. $Q(x)=\mathrm{e}^{ax},\ P(x)=p$: $\langle\partial_t, \mathrm{e}^{-pt}\partial_u, \partial_x - au\partial_u\rangle$;
6. $Q(x)=1,\ P(x)=-2x^{-2}$: $\langle\partial_t, 2t\partial_t + x\partial_x, t^2\partial_t + tx\partial_x + \frac{x^2}{4}\partial_u\rangle$;
7. $Q(x)=1,\ P(x)=p$: $\langle\partial_t, \partial_x, \mathrm{e}^{-pt}\partial_u, \mathrm{e}^{-pt}(\partial_x - \frac{px}{2}\partial_u)\rangle$.

One notes with some interest that the removal of the function $C(x)$ leads to a considerable reduction in the number of possible cases for which a nontrivial set of symmetries is found.

3. The algebras

We summarise the algebras of the symmetries associated with the different possibilities in Tables 1 and 2.

It is immediately obvious that there is a considerable variety of algebras of the symmetries for the particular cases. Whilst it is true that a number of them are quite simple in the sense of having only one or two elements, the possibilities amongst the three- and four-dimensional algebras show that even an equation of specified form can produce many variations. In the instances of the three-dimensional algebras there are to be found five of the eleven real Lie algebras of dimension three [34]. As we see below, it is not only the dimension of the algebra which is of importance but also the particular algebra.

4. Problems with a terminal condition

In the Introduction we stated that a standard problem in Financial Mathematics is to solve the given differential equation subject to a terminal condition of the form $u(T,x)=U$. Since the equations are independent of t, there is no loss of generality in taking $T=0$. To determine whether such a solution exists one assumes a general symmetry of the form

$$\Gamma=\sum_{i=1}^{n}\alpha_i\Gamma_i, \tag{4.1}$$

where n is the number of Lie point symmetries of the given equation[5] and the α_i, $i=1,\dots,n$, are constants to be determined by the application of Γ to the conditions

$$t=0 \quad\text{and}\quad u(0,x)=U, \tag{4.2}$$

in which the second condition must be satisfied for all values of x. We illustrate the procedure with the sixteenth equation.

We apply Γ, (4.1), with $n=4$ and the symmetries of the sixteenth equation as given above to the conditions $t=0$ and $u(0,x)=U$. From the former it follows immediately that $\alpha_1=0$. In the case of the latter we separate the coefficients of x^1 and x^0 to obtain

$$x^1:\quad -\frac{\alpha_3 p^2}{4p}+\frac{\alpha_4 2p^2}{4p^2}=0$$

and

$$x^0:\quad \alpha_2-\frac{8c_1}{4p}\alpha_3+\frac{8c_1}{4p^2}2\alpha_4=0$$

from which it follows that

$$\alpha_2=0 \quad\text{and}\quad \alpha_3=\frac{2}{p}\alpha_4. \tag{4.3}$$

Hence the symmetry compatible with the terminal condition is

$$\Gamma=\exp[-\tfrac{1}{2}pt]\bigl\{4(pt+2)\partial_x-t\bigl(p^2x+8c_1\bigr)\partial_u\bigr\}. \tag{4.4}$$

The invariants obtained from the associated Lagrange's system are

$$t \quad\text{and}\quad u+\bigl(p^2x^2+16c_1x\bigr)\frac{t}{8(pt+2)}$$

and we make the substitution

$$u=f(t)-\bigl(p^2x^2+16c_1x\bigr)\frac{t}{8(pt+2)} \tag{4.5}$$

into

$$u_t+u_{xx}+u_x^2+pu+\frac{p^2}{16}x^2+c_1x=0 \tag{4.6}$$

to obtain the first-order equation

$$\dot f+pf=\frac{p}{4}-\frac{p}{2(2+pt)}-\frac{4c_1^2t^2}{(2+pt)^2}$$

so that, after we take into consideration the terminal condition, the solution of (4.6) is

$$\begin{aligned} u(t,x)={}&\left[U-\frac{1}{4}+\frac{4c_1^2}{p^3}\right]\exp[-pt]+\frac{1}{4}-\frac{4c_1^2}{p^3}-\bigl(p^2x^2+16c_1x\bigr)\frac{t}{8(pt+2)}\\ &-\exp[-pt]\left\{\left(\frac{p}{2}-\frac{16c_1^2}{p^2}\right)\int_0^t\frac{\exp[ps]}{2+ps}\,\mathrm{d}s+\frac{16c_1^2}{p^2}\right\}\int_0^t\frac{\exp[ps]}{(2+ps)^2}\,\mathrm{d}s. \end{aligned} \tag{4.7}$$

The method of solution for the fifteenth equation is the same.

[5] In the case of a linear equation or of one linearisable by means of a point transformation there exists a subalgebra of infinite dimension comprising solutions to a related linear equation. They are excluded from consideration in a symmetry of the form (4.1). In the present paper such symmetries are precluded by construction.

In the case of the fourteenth equation one has a different algebra and so cannot assume that everything proceeds as above. The mechanics of the calculation are the same and we merely highlight the details. The coefficients, α_1 and α_2, are zero. We find that

$$\alpha_3 = -\frac{p+m}{p-m}\alpha_4$$

so that the single symmetry compatible with the terminal condition is

$$\Gamma = \left(\exp\bigl[-(p+m)t/2\bigr] - \frac{p+m}{p-m}\exp\bigl[-(p-m)t/2\bigr]\right)\partial_x - \frac{(p^2-m^2)x+8c_1}{4(p-m)}\bigl(\exp\bigl[-(p+m)t/2\bigr] - \exp\bigl[-(p-m)t/2\bigr]\bigr)\partial_u. \tag{4.8}$$

The invariants determined from the associated Lagrange's system for (4.8) are

$$t \quad \text{and} \quad u + \frac{(p^2-m^2)x^2+16c_1x}{8(p-m)}\frac{1-\mathrm{e}^{mt}}{1-\frac{p+m}{p-m}\mathrm{e}^{mt}}.$$

The first-order equation resulting from the reduction of

$$u_t + u_{xx} + u_x^2 + pu + \frac{p^2-m^2}{16}x^2 + c_1x = 0$$

is

$$\dot f(t) + pf(t) + \frac{(\mathrm{e}^{mt}-1)[(m-p)^2(m+p) - 16c_1^2 + \mathrm{e}^{mt}((m-p)(m+p)^2 + 16c_1)]}{4(m-p+\mathrm{e}^{mt}(m+p))^2} = 0.$$

This equation is readily solved and so we have another instance of an algebra being commensurate with the terminal condition.

When we apply the same procedure to the other equations, we find that the only one which possesses a symmetry compatible with the terminal condition above is the sixth equation. The symmetry is

$$\Gamma = -\partial_x + a\bigl(u - U\exp[-pt]\bigr)\partial_u$$

for which the invariants are

$$t \quad \text{and} \quad u\exp[ax] - U\exp[ax-pt]$$

so that we make the substitution

$$u(t,x) = U\exp[-pt] + f(t)\exp[-ax]. \tag{4.9}$$

The function $f(t)$ is a solution of the Riccati equation

$$\dot f + a^2f^2 + \bigl(a^2+p\bigr)f + c = 0$$

and is given implicitly by

$$t - t_0 = -\frac{4}{\sqrt{4a^2c-(a^2+p)^2}}\arctan\left\{\frac{2a^2f(t)+a^2+p}{\sqrt{4a^2c-(a^2+p)^2}}\right\},$$

$$t + t_0 = \frac{4}{\sqrt{(a^2+p)^2-4a^2c}}\operatorname{arctanh}\left\{\frac{2a^2f(t)+a^2+p}{\sqrt{(a^2+p)^2-4a^2c}}\right\}$$

or

$$t - t_0 = \frac{2}{2a^2f(t)+a^2+p}$$

depending upon whether

$$4a^2c - \bigl(a^2+p\bigr)^2 \geqslant 0,$$

$$4a^2c - \bigl(a^2+p\bigr)^2 \leqslant 0$$

or

$$4a^2c-\left(a^2+p\right)^2=0.$$

From the terminal condition and (4.9) it is apparent that $f(0)=0$ and hence the constant of integration is given by

$$t_0=\frac{4}{\sqrt{4a^2c-(a^2+p)^2}}\arctan\left\{\frac{a^2+p}{\sqrt{4a^2c-(a^2+p)^2}}\right\},$$
$$t_0=\frac{4}{\sqrt{(a^2+p)^2-4a^2c}}\operatorname{arctanh}\left\{\frac{a^2+p}{\sqrt{(a^2+p)^2-4a^2c}}\right\}$$

or

$$t_0=\frac{2}{a^2+p},$$

respectively.

There is a variation of the problem of the terminal condition which is of the form

$$u(0,x)=g(x), \tag{4.10}$$

where $g(x)$ is more in the nature of a utility function. Usually the utility function is specified in the modelling process, but there is no reason not to examine the possibility of the existence of a suitable symmetry for a utility function which is a consequence of the existence of such a symmetry. We illustrate this with case 5 for which (1.5) becomes

$$u_t+u_{xx}+x^a u_x^2+\left(\frac{p}{x^2}+b\right)u+\frac{c}{x^{a+2}}+\frac{b[a^3-3a^2+a(2+\frac{1}{2}(x^2+p)+bx^2)]}{2a^2x^a(a-2)}=0, \tag{4.11}$$

where a, b and p are constants subject to the constraint that $a\neq 2$. We apply the general symmetry,[6]

$$\Gamma=\alpha_1\partial_t+\alpha_2\exp\left[\frac{2bt}{a}\right]\left\{\frac{a}{b}\partial_t+x\partial_x-\left[au+\frac{b}{a(a-2)}x^{2-a}\right]\partial_u\right\} \tag{4.12}$$

to the condition

$$t=0\quad\text{and}\quad u(0,x)=g(x)$$

and obtain $\alpha_1=-a\alpha_2/b$, where α_2 is arbitrary provided $g(x)$ satisfies the first-order equation

$$xg'+ag=-\frac{b}{a(a-2)}x^{2-a}. \tag{4.13}$$

The solution of (4.13) is

$$g(x)=Kx^{-a}-\frac{b}{2a(a-2)}x^{2-a}, \tag{4.14}$$

where K is a constant of integration, which, we recall, is $u(0,x)$. Under the conditions now determined the general symmetry, (4.12), is

$$\Gamma=\left(\exp[2bt/a]-1\right)\frac{a}{b}\partial_t+\exp[2bt/a]\left\{x\partial_x-\left[au+\frac{b}{a(a-2)}x^{2-a}\right]\partial_u\right\}. \tag{4.15}$$

The two invariants of (4.15) are

$$\frac{x^2}{\exp[2bt/a]-1}\quad\text{and}\quad ux^a+\frac{b}{2a(a-2)}x^2$$

and to perform the reduction of order we write

$$u(t,x)=x^{-a}f(y)-\frac{b}{2a(a-2)}x^{2-a},\quad\text{where } y=\frac{x^2}{\exp[2bt/a]-1}. \tag{4.16}$$

[6] There are different approaches to the determination of a suitable function $g(x)$ 4 (4.10). One could apply the inverse method by fitting an invariant solution of the evolution equation to find the compatible function. Here, for the convenience of the reader, we continue to use the method already applied.

Under the reduction (4.16) Eq. (4.11) becomes

$$4ay^2f''+4ay^2f'^2-4ayff'-\left(2by^2+4a^2y-2ay\right)f'+a^3f^2+\left(a^2+a+p\right)af+ac=0. \tag{4.17}$$

Fortunately it is not necessary to be able to solve (4.17) in full. If we compare the terminal condition, (4.14), with the transformation in (4.16), it is quite apparent that $f=K$, $\forall x$. Then (4.17) becomes

$$a^3K^2+\left(a^2+a+p\right)aK+ac=0$$

with solution

$$K=-\frac{(a^2+a+p)a\pm\sqrt{\{(a^2+a+p)^2a^2-4a^4c\}}}{2a^3}. \tag{4.18}$$

In any financial context one would want $g(x)$ and *a fortiori* $u(t,x)$ to be positive. This does impose some constraint upon acceptable values for the parameters in the model. It is not surprising that the acceptable utility function, $g(x)$, is related in a very direct way through (4.18) to the parameters of the model.

We note quite a variation in the algebraic structures of the different equations. The equations of maximal symmetry have either the Weyl–Heisenberg subalgebra and a single remnant of the $\mathrm{sl}(2,R)$ subalgebra of the classical heat equation or the combination of two abelian subalgebras which has no connection with the classical heat equation. Either is sufficient for the resolution of the problem of the terminal condition. Lesser algebraic structures are not sufficient with the exception of the algebra of the sixth equation.

5. Invariant solutions

Although Lie symmetry analysis does not help to construct the general solution of a system of nonlinear partial differential equations, it often gives an approach to deduce wide classes of solutions being invariant with respect to different subgroups of the Lie symmetry group. Roughly speaking, the main theorem on invariant solutions [33,30] claims that all solutions invariant with respect to an r-parametric group of symmetries of the given n-dimensional system can be obtained by solving a system of differential equation with $n-r$ independent variables. In particular, if $r=n-1$, invariant solutions can be constructed via the solution of a system of ordinary differential equations. Therefore in our two-dimensional case to reduce an equation to an ordinary differential equation we have to use one-dimensional subalgebras of the algebra of maximal symmetry. Invariance with respect to two-dimensional algebras leads to the possibility of reduction of the initial equation to an algebraic one.

As we mentioned, in general any subalgebra (possessing the transversality property) of a Lie symmetry algebra of a partial differential equation corresponds to a class of invariant solutions. Since almost always there exists an infinite number of such subalgebras, in most of cases it is practically impossible to list all invariant solutions. Therefore one needs an effective systematic tool of their classification that gives us an “optimal system” of such solutions, from which we can find all possible invariant solutions. L.V. Ovsiannikov proved [33] that the optimal system of solutions consists of solutions invariant with respect to all proper inequivalent subalgebras of the symmetry algebra. Such a set of inequivalent subalgebras is now called an optimal system of subalgebras. For more detail about construction of optimal sets of subalgebras we refer to [33,30]. Inequivalent subalgebras of low-dimensional Lie algebras were classified in [34].

Below we consider Lie symmetry reductions of two equations from class (1.5).

The equation of case 13,

$$u_t+u_{xx}+u_x^2+px^{-2}u+\frac{p(2+p)}{4x^2}\log x=0,$$

has as its algebra a realization of $\mathrm{sl}(2,\mathbb{R})$ which is spanned by the operators

$$v_1=\partial_t,\qquad v_2=2t\partial_t+x\partial_x-\frac{p+2}{4}\partial_u,\qquad v_3=t^2\partial_t+tx\partial_x+\frac{x^2-(p+2)t}{4}\partial_u.$$

The list of its inequivalent subalgebras is

$$\langle v_1\rangle,\quad \langle v_2\rangle,\quad \langle v_1+v_3\rangle,\quad \langle v_1,v_2\rangle.$$

Reductions with respect to these subalgebras give:

$\langle v_1\rangle$:

$$u=v(x),\qquad v''+\left(v'\right)^2+px^{-2}v+\frac{p(2+p)}{4x^2}\log x=0.$$

$\langle v_2\rangle$:

$$u = v(\omega) - \frac{p+2}{8}\log t, \qquad \omega = t^{-1/2}x,$$
$$v'' + \left(v'\right)^2 - \frac{\omega}{2}v' + p\omega^{-2}v + \frac{p(p+2)}{4}\omega^{-2}\log\omega - \frac{p+2}{8} = 0.$$

$\langle v_1 + v_3\rangle$:

$$u = \frac{tx^2}{4(t^2+1)} - \frac{p+2}{8}\log\left(1+t^2\right) + v(\omega), \qquad \omega = \frac{x}{\sqrt{t^2+1}},$$
$$v'' + \left(v'\right)^2 + p\omega^{-2}v + \frac{p(p+2)}{4}\omega^{-2}\log\omega + \frac{\omega^2}{4} = 0.$$

$\langle v_1, v_2\rangle$: As a special case we have

$$u = -\frac{p+2}{4}\log x + c \quad \text{and} \quad 8p + 12 + p^2 + 16pc = 0$$

in which the invariants are given the same value. Therefore we can eliminate c through $c = -(p+2)(p+6)/16p$ and obtain the particular solution

$$u = -\frac{p+2}{4}\left(\log x + \frac{p+6}{4p}\right).$$

In general the double reduction leads to an Abel's equation of the second kind,

$$\left(w + \frac{p+q}{4}\right)\frac{\mathrm{d}w}{\mathrm{d}s} + w^2 + s = 0,$$

the solution of which is not obvious. Evidently there is a payoff between getting *a* solution and having the general form of the equation.

We now look at the equation of case 8,

$$u_t + u_{xx} + x^2u_x^2 + pu + cx^{-4} + \frac{p}{4}x^{-2} - \frac{p^2}{4}\log x = 0.$$

Its Lie symmetry algebra is a realization of $A_{3,4}$ $(E(1,1))$ spanned by

$$v_1 = \partial_t, \qquad v_2 = \mathrm{e}^{-pt}\partial_u, \qquad v_3 = \mathrm{e}^{pt}\left[\frac{2}{p}\partial_t + x\partial_x - \left(2u - \frac{p}{8} - \frac{p}{2}\log x\right)\partial_u\right]$$

with inequivalent subalgebras

$$\langle v_1\rangle, \quad \langle v_2\rangle, \quad \langle v_3\rangle, \quad \langle v_2 + \varepsilon v_3\rangle, \quad \langle v_1, v_2\rangle, \quad \langle v_1, v_3\rangle, \quad \langle v_2, v_3\rangle,$$

where $\varepsilon = \pm 1$.

Reductions with respect to these subalgebras give

$\langle v_1\rangle$: $u = v(x)$:

$$v'' + x^2\left(v'\right)^2 + pv + \frac{p}{4}x^{-2} - \frac{p^2}{4}\log x = 0.$$

$\langle v_2\rangle$: Does not give reduction.

$\langle v_3\rangle$:

$$u = \mathrm{e}^{-pt}v(\omega) + \frac{4p\log x - p}{16}, \qquad \omega = x\mathrm{e}^{-pt/2},$$
$$v'' + \omega^2\left(v'\right)^2 + c\omega^{-4} = 0.$$

The general solution of this equation has the form

$$v=\begin{cases} C_2-\frac{1}{2}\int x^{-6}(-3x^3+\sqrt{x^6(m^2-18)}\tan\frac{\sqrt{x^6(m^2-18)}\log(C_1x)}{2x^3})\,\mathrm{d}x,\\ C_2+\frac{1}{2}\int x^{-6}(3x^3+\sqrt{x^6(m^2+18)}\tanh\frac{\sqrt{x^6(m^2+18)}\log(C_1x)}{2x^3})\,\mathrm{d}x,\\ -\frac{3+3\sqrt{2}}{4x^2}-\frac{1}{x^2}\Phi(C_1x^{3\sqrt{2}},1,-\frac{1}{3}\sqrt{2})+C_2\end{cases}$$

if $c=(m^2-9)/4$, $-(m^2+9)/4$ or $9/4$, respectively, where Φ is a general Lerch Phi function [13],

$$\Phi(z,a,v)=\sum_{n=0}^{\infty}\frac{z^n}{(v+n)^a}.$$

$\langle v_2+\varepsilon v_3\rangle$:

$$u=\mathrm{e}^{-pt}v(\omega)+\frac{4p\varepsilon\log x-p\varepsilon-8\mathrm{e}^{-2pt}}{16},\qquad \omega=x\mathrm{e}^{-pt/2},$$
$$v''+\omega^{-2}(v')^2+c\omega^{-4}+\frac{p}{2\varepsilon}=0.$$

As was the case with the previous equation, this equation is of Riccati form in v' and is linearised by means of a Riccati transformation. Maple gives the solution as

$$\begin{aligned}v=C_2+\frac{1}{2}\int&\frac{1}{\omega[C_1Y(-\frac{1}{4}\Psi,\frac{\sqrt{c}}{2}\omega^{-2})+J(-\frac{1}{4}\Psi,\frac{\sqrt{c}}{2}\omega^{-2})]}\\ &\times\left[2C_1\sqrt{c}Y\left(1-\frac{1}{4}\Psi,\frac{\sqrt{c}}{2}\omega^{-2}\right)+2\sqrt{c}\omega^2\sqrt{\Psi}J\left(1-\frac{1}{4}\Psi,\frac{\sqrt{c}}{2}\omega^{-2}\right)\right.\\ &\left.+\omega^2(1+\Psi)\left(C_1Y\left(-\frac{1}{4}\Psi,\frac{\sqrt{c}}{2}\omega^{-2}\right)+J\left(-\frac{1}{4}\Psi,\frac{\sqrt{c}}{2}\omega^{-2}\right)\right)\right]\mathrm{d}\omega,\end{aligned}$$

where J and Y are Bessel functions of the first and second kinds, respectively, and $\Psi=\sqrt{(\varepsilon-2p)/\varepsilon}$.

$\langle v_1,v_2\rangle$: Does not give reduction.

$\langle v_1,v_3\rangle$:

$$u=Ax^{-2}+\frac{p(-1+4\log x)}{16},\qquad 4A^2+6A+c=0.$$

As we noted in the previous example, the solution obtained by the elimination of A between these two expressions is particular. If one follows the normal procedure of reduction, the general solution compatible with this subalgebra is found from the solution of an Abel's equation of the second kind. The invariants of v_3 are $x\exp[-pt/2]$ and $(u+p/16-(p/4)\log x)x^2$. We make the change of variables

$$\omega=x\exp[-pt/2]\quad\text{and}\quad u=\mathrm{e}^{pt}v(\omega)-\frac{p}{16}+\frac{p}{4}\log x$$

whereby the equation of case 8 is reduced to the second-order equation

$$v''+\omega^2v'^2+\frac{c}{\omega^4}=0 \tag{5.1}$$

and v_1 becomes $\omega\partial_\omega-2v\partial_v$. Under the further reduction using this symmetry we obtain an Abel's equation of the second kind, namely

$$(s+2r)s'-3s+c+s^2=0,$$

where $r=v\omega^2$ and $s=v'\omega^3$. Alternately we observe that (5.1) is a Riccati equation in v' and so we can express the solution of (5.1) as the quadrature

$$v(\omega)=A_0+\int\frac{3A_2\omega^3+c}{\omega^3(A_1+A_2\omega^3+c\log\omega)}\,\mathrm{d}\omega.$$

The obvious symmetry of (5.1), ∂_v, has its origin in v_2.

$\langle v_2,v_3\rangle$: Does not give reduction.

6. Discussion

Evolution partial differential equations play an important role in the analysis of problems which arise in Financial Mathematics. Whilst many of them are linear and can be related by means of a point transformation to the classical heat equation, maybe with a source/sink term, a significant number are intrinsically nonlinear under point transformations. We have examined a class of such equations and identified the equations, up to equivalence transformations, which have symmetries additional to those obvious by inspection. The algebraic structures displayed by the equations are varied. The class with the algebras of maximal dimension was demonstrated to have symmetries compatible with the standard terminal condition. Only one of the equations with an algebra of lower dimension had this property. However, we were able to provide a solution for a related problem, in which the terminal condition is more in the appearance of a utility function, provided that the required utility function was compatible with the symmetries available. In addition to the general nonlinear equation of a class mentioned above we examined the symmetries of a restricted class. An interesting effect of the imposition of the restriction was a marked reduction in the number of different cases. In particular the case of maximal symmetry with the specific algebra $A_1 \oplus_s A_{3,1}$ (cases 15 and 16) in the unrestricted equation was no longer found although the case in which the three-dimensional subalgebra was abelian (case 14) not only persisted but even maintained the same number of symmetries. This persistence of the number of symmetries is to be found in all of the equations which had a nontrivial algebra in the general case and for which there exists an equivalent equation in the restricted case. We also illustrated a number of invariant solutions for two of the classes not included in the above.

In terms of one of the important problems in Financial Mathematics, *i.e.* the problem of a terminal condition, the number of cases in which the algebra was conducive to determining the solution of that specific problem was *a priori* somewhat disappointing. Evidently the number of symmetries is not completely critical as the three-dimensional algebra of case 6 of the general equation was just as satisfactory as the four-dimensional algebras. One notes the similarity of structure of the algebras concerned. One should emphasize that this is in the context of the problem of a terminal condition.

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