Existence of Regular Solutions for Nonlinear Signorini's Problems

M. Biroli

Istituto di Matematica del Politecnico di Milano

AND

U. Mosco

Département de Mathématiques, Université de Paris Sud, Orsay, France
and Istituto di Matematica dell' Università di Roma

Received March 11, 1983; revised December 5, 1983

0. Introduction

The aim of this paper is to prove the existence of a Lipschitz solution for nonlinear obstacle problems with quadratic growth in the gradient and Signorini's boundary conditions. For the linear Signorini's problem, existence and regularity results have been given by Brézis [5], and by Hanouzet and Joly [13]; in particular, the last two authors apply a dual estimation technique, consisting of estimating the conormal derivative of the solution as a measure on the boundary. For the nonlinear Signorini's problem of the type considered in this paper, no existence results for weak solutions are known; the only results available are due to Frehse [10], and refer to the interior regularity of arbitrary bounded weak solutions.

To obtain the existence of a (Lipschitz) weak solution in the present nonlinear quadratic case, global estimates up to the boundary are needed. Let us remark incidentally that $C^\alpha$ regularity up to the boundary of the weak solution was previously obtained by da Veiga [2], da Veiga and Conti [3], by Giaquinta and Modica [15], using a direct variational approach; however, they only allow a sublinear growth in the gradient. In this case the existence of weak solutions can be obtained by standard methods. Our proof of the existence of a weak solution, in the case of quadratic growth in the gradient, relies on an a priori estimate up to the boundary of the Lipschitz norm for $C^1 \cap H^2$ arbitrary solution $v$, which is obtained in Section 3. We at first estimate the derivatives of $v$ in the interior of the domain and its tangential derivatives on the boundary, adapting the differential quotient technique of Frehse to the present situation in which
the nonlinear term $H(\cdot, u, p)$ is only assumed to be continuous in $(u, p)$ (see (1.2)). The conormal derivative on the boundary is then estimated by adapting the linear dual estimate of Hanouzet and Joly to the quadratic nonlinear operator of our case, which is done in Section 2. The existence of a Lipschitz weak solution $u$ is then obtained by a suitable approximation technique (see Sect. 4).

To investigate further regularity properties of such a solution, we can use the $C^{1,\alpha}$ regularity result given by Caffarelli for the linear case; however, we apply this result to the present case in a slightly generalized form, see Section 5, in which the obstacle is not assumed to belong to $H^{2,\infty}$ but only to $C^{1,\delta}$. Let us finally remark that recently a $C^{1,\alpha}$ regularity result for the solution of Signorini's problem with smooth obstacles has been obtained by Kinderlehrer [17]; in this paper $H$ is supposed to be 0, but the coefficients $a_{ij}$ (see (1.1)) can depend on the gradient.

1. Results

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, with a smooth boundary $\Gamma$. Let $a_{ij} \in L^{\infty}(\Omega)$, $i, j = 1, \ldots, N$ be functions such that for all $\xi \in \mathbb{R}^N$,

$$\sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \geq v|\xi|^2 \text{ a.e. in } \Omega, \quad v > 0. \quad (1.1)$$

We define the operator $A: H^1(\Omega) \to (H^1(\Omega))^\prime$ by setting

$$\langle Au, v \rangle = \sum_{i,j=1}^{N} \int_{\Omega} a_{ij}(x) \frac{\partial u}{\partial x_j}(x) \frac{\partial v}{\partial x_i}(x) \, dx \quad (1.2)$$

for arbitrary $u, v \in H^1(\Omega)$ and the operator $L: H^1(\Omega) \to H^{-1}(\Omega)$, formally written as

$$L = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial}{\partial x_j} \right),$$

by restricting $v \in H^{1,1}_{0}(\Omega)$ in the identity above.

We recall that $H^1(\Omega)$ denotes the Sobolev space of all functions, which are square integrable in $\Omega$ together with their first-order distribution derivatives, $H^1_0(\Omega)$ the subspace of all functions of $H^1(\Omega)$ vanishing on $\Gamma$, $(H^1(\Omega))^\prime$ the dual space of $H^1(\Omega)$ and $H^{-1}(\Omega)$ the dual of $H^1_0(\Omega)$. Let $H(x, u, p)$ be a given function of $x \in \Omega$, $u \in \mathbb{R}$, $p \in R^N$, which is measurable in $x$ for fixed $(u, p) \in R \times R^N$, continuous in $(u, p)$ for a.e. $x \in \Omega$ and such that

$$|H(x, u, p)| \leq K(1 + |p|^2)$$
for a.e. \( x \in \Omega \), all \((u, p) \in \mathbb{R} \times \mathbb{R}^N\) with \(|u| \leq C\), the constant \( K \) above possibly depending on \( C \).

Now let us suppose that a measurable function \( \psi \) is given, such that
\[
K^\psi \cap L^\infty(\Omega) \neq \emptyset,
\]
where
\[
K^\psi = \{ v \in H^1(\Omega); v \leq \psi \text{ a.e. in } \Omega \}.
\]
and let us consider the following variational inequality:
\[
\langle Au, v-u \rangle + \int_\Omega H(x, u(x), Du(x))(v(x)-u(x)) \, dx \geq 0 \quad \forall v \in K^\psi \cap L^\infty(\Omega), \quad u \in K^\psi \cap L^\infty(\Omega).
\]
(1.3)

In order to prove the existence of a Lipschitz solution \( u \) of (1.3) we make the following additional assumptions on the coefficients of \( L \),
\[
a_{ij} \in H^{1,\infty}(\Omega) \quad i, j = 1, \ldots, N,
\]
(1.4)
and we assume furthermore that
\[
\psi \in H^{1,\infty}(\Omega)
\]
(1.5)
and that there exists a Lipschitz subsolution \( \Phi \) of the problem, that is a function \( \Phi \) such that
\[
\Phi \in H^{1,\infty}(\Omega), \quad \Phi \leq \psi \text{ a.e. in } \Omega
\]
(1.6)
such that
\[
A\Phi + H(\cdot, \Phi, D\Phi) \leq 0 \quad \text{in } H^1(\Omega)'.
\]
(1.7)

Then we have

**Theorem 1.** Under the assumptions (1.1), (1.2), (1.4)-(1.7), there exists a solution \( u \in H^{1,\infty}(\Omega) \) of problem (1.3), satisfying the additional condition \( u \geq \Phi \text{ a.e. in } \Omega \).

**Remark 1.** We observe that (1.6), (1.7) hold if there exists \( K \leq \min_{\Omega} \psi \), such that \( H(x, K, 0) \leq 0 \). If
\[
H(x, u, p) u \geq -K_1 + \lambda |u|^2 - K_2 |p|^2, \quad \lambda > 0,
\]
such a \( K \) exists and depends only on \( \lambda, K_1 \). The assumptions (1.6), (1.7) (existence of a subsolution \( \Phi \) of (1.3) have some connection with similar assumptions considered in [10, 8]).
Remark 2. The problem with two obstacles $\psi_1, \psi_2 \in H^{1,\infty}(\Omega)$, $\psi_1 < \psi_2$ in $\Omega$, can be dealt with by combining the methods of this paper with those of [19]; in this case the assumption (1.6) can be dropped.

Remark 3. By the same methods used here one can prove the existence of a locally Lipschitz solution of the Dirichlet problem, without differentiability assumptions on $H(x, u, p)$; for regular $H(x, u, p)$, the local Lipschitz continuity of weak solutions has been proved by Frehse [10].

In Section 2 we give the proof of a general dual estimate and in Section 3 we obtain an a priori estimate of the Lipschitz norm of regular solutions. Both results are used in the proof of Theorem 1, given in Section 4. Finally, in Section 5, we consider further regularity results.

We wish to thank Frehse for stimulating discussions on the topic of this paper.

2. Dual Inequalities

Let $V = H^1(\Omega) \cap L^\infty(\Omega)$, $V_0 = H^1_0(\Omega) \cap L^\infty(\Omega)$. We first give a topological definition of the conormal derivative associated with $A$, by means of a Green's formula for topological duals.

**Lemma 1.** Let $u \in V$, $Lu \in B'$, where $V_0 \subset V \subset B$ and $D(\Omega)$ is dense in $B$. There exists a unique $F \in B'$ and a unique $\gamma_0 u \in (H^{1/2}(\Gamma) \cap L^\infty(\Gamma))'$ such that

$$\langle Au, v \rangle_{V', V} = \langle F, v \rangle_{B', B} + \langle \gamma_0 u, \gamma_0 v \rangle_{\Gamma}$$

$\forall v \in V$ ($\gamma_0 = \text{trace operator}$, $\langle , \rangle_{\Gamma}$ is the duality product between $H^{1/2}(\Gamma) \cap L^\infty(\Gamma)$ and its dual).

For $\Phi \in H^{1/2}(\Gamma) \cap L^\infty(\Gamma)$, let us set

$$\mathcal{L}(\Phi) = \langle Au, v \rangle_{V', V} - \langle Lu, v \rangle_{B', B}, \quad \gamma_0 v = \Phi, \quad v \in V.$$  

We observe that $\mathcal{L}(\Phi)$ depends only on $\Phi$. We will prove now that $\mathcal{L}(\Phi)$ is linear continuous on $H^{1/2}(\Gamma) \cap L^\infty(\Gamma)$. It is sufficient to select a linear continuous extension $\Phi \to \Phi$ from $H^{1/2}(\Gamma) \cap L^\infty(\Gamma)$ to $H^1(\Omega) \cap L^\infty(\Omega) = V$. Such an extension is given by

$$-A\Phi = 0, \quad \gamma_0 \Phi = \Phi.$$

Then,

$$\mathcal{L}(\Phi) = \langle \sigma, \Phi \rangle_{\Gamma},$$
where $\sigma \in (H^{1/2}(\Gamma) \cap L^\infty(\Gamma))'$. As to the uniqueness, let $F_1, F_2 \in \mathcal{B}'$ and $\sigma_1, \sigma_2 \in (H^{1/2}(\Gamma) \cap L^\infty(\Gamma))$, be such that

$$
\langle Au, v \rangle_{V', V} = \langle F_1, v \rangle_{B', B} + \langle \sigma_1, v \rangle_B = \langle F_2, v \rangle_{B', B} + \langle \sigma_2, v \rangle_B.
$$

Then

$$
\langle F_1 - F_2, v \rangle_{B', B} = 0 \quad \forall v \in \mathcal{D}(\Omega) \Rightarrow F_1 = F_2,
$$

$$
\langle \sigma_1 - \sigma_2, v \rangle_B = 0 \quad \forall v \in (H^{1/2}(\Gamma) \cap L^\infty(\Gamma)),
$$

that implies $\sigma_1 = \sigma_2$.

**Remark 1.** If $B = L^p(\Omega)$, $p > 1$, $a_{ij} \in C(\bar{\Omega})$, $u \in C^1(\bar{\Omega})$, then $\gamma_{a} u$ is the conormal derivative according to the usual definition.

We give now a Green’s formula in the order duals. Let $V^*(V_0^*)$ be the order dual of $V(V_0)$, $V^*_+(V_0^+_*)$ be the set of nonnegative vectors in $V^*(V_0^*)$, $\rho: (H^1(\Omega) \cap L^\infty(\Omega)), (-H^1_0(\Omega) \cap L^\infty(\Omega))$ be the transpose of the injection of $H^1_0(\Omega) \cap L^\infty(\Omega)$ into $H^1(\Omega) \cap L^\infty(\Omega)$.

Let

$$
\Theta^+ = \{ f \in (V_0^*)_+, \exists F \in V^*_+ \text{ with } \rho F = f \}.
$$

As in [12, 13], we define for $f^+ \in \Theta^+$ a positive continuous minimal extension $\pi f^+$ by

$$
\langle \pi f^+, v \rangle_{V', V} = \sup_{\substack{u \in V_0^+ \\cap \Theta \subseteq u \subseteq v}} \langle f^+, u \rangle_{V_0^*, V_0} \quad \forall v \in V, v \geq 0.
$$

Let $\Theta = \Theta^+ - \Theta^+$ and, for each $f \in \Theta$, let

$$
\pi f = \pi f^+ - \pi f^-.
$$

Let us denote by $\pi \Theta$ the subspace of $V^*$ generated as above.

Now we list some proprieties illustrating the connections between the order duals and the space $\Theta$:

1. $V_0^* - \Theta \neq \phi$.
2. $\rho$ is a Riesz homomorphism from $V^*$ onto $\Theta$, i.e.,

$$
\rho F^* = (\rho F)^* \quad F \in V^* \quad \rho V^* = \Theta.
$$

3. $\pi$ is a Riesz homomorphism from $\Theta$ into $V^*$ and $\rho \pi \Theta = \Theta$;
(4) \( \pi \Theta \) is a band in \( V^* \), i.e.,

(a) \( f \in \Theta, G \in V^*, |G| \leq |f| \Rightarrow G \in \pi \Theta, \)

(b) \( \pi \Theta \) contains the upper bound of each its subsets bounded in \( V^* \).

(5) \( V^* = \pi \Theta \oplus \gamma_0(H^{1/2}(\Gamma) \cap L^\infty(\Gamma))^* \), where \( \oplus \) denotes the ordered direct sum.

For the proof of (1) we can use the same argument of [13, Ex. 1.4]. The proofs of (2)–(5) are analogous to those given in [13, Propositions 1.2, 1.4 and Theorem 1; 14, Appendix] since those proofs only use the structure of \( V \) and \( V_0 \) as vector lattices and the structure of \( V^*, V^*_0 \) as complete vector lattices (for the latter property see [18, pp. 212, 228]).

We only give the proof of (5). We observe, that, \( V^* \) being a complete vector lattice and \( \pi \Theta \) a band in \( V^* \), we have \( V^* = \pi \Theta + \pi \Theta^\perp \), where \( \pi \Theta^\perp \) is the band of all elements in \( V^* \) disjoint from \( \pi \Theta \) (i.e., \( z \in \pi \Theta^\perp \Leftrightarrow \inf(|z|, |x|) = 0 \ \forall x \in \pi \Theta, \ [18, p. 210] \)). By the definition of \( \pi \Theta, \pi \Theta^\perp \) is constituted by the elements in \( V^* \), vanishing on \( V_0^* \).

Since \( \gamma_0 \) is a Riesz isomorphism of \( V/V_0 \) onto \( H^{1/2}(\Gamma) \cap L^\infty(\Gamma) \), the transpose \( ^t \gamma_0 \) is a Riesz isomorphism between \( (H^{1/2}(\Gamma) \cap L^\infty(\Gamma))^* \) and \( \pi \Theta^\perp \).

**Lemma 2.** Let \( u \in V, Au \in V^* \); then there exists a unique \( \Phi_u \in (H^{1/2}(\Gamma) \cap L^\infty(\Gamma))^* \), such that

\[
Au = \pi Lu + ^t \gamma_0 \Phi_u,
\]

and, if \( Lu \in L^1(\Omega), \Phi_u = \gamma_a u. \)

By (2), we have \( \pi \rho Au \in \pi \Theta \). We consider now

\[
Au - \pi \rho Au.
\]

We can easily see, by the definition of \( \rho \), that \( \rho Au - Lu \in \Theta \). By the definition of \( \pi \) (2.1) vanishes on \( V_0 \), hence (2.1) is in \( \pi \Theta^\perp \). Thus from (5), we have

\[
Au - \pi Lu = ^t \gamma_0 \Phi_u,
\]

where \( \Phi_u \in (H^{1/2}(\Gamma) \cap L^\infty(\Gamma))^* \).

From (2.2) we have

\[
\langle Au, v \rangle_{V^*} = \langle \pi Lu, v \rangle_{V^*} + \langle \Phi_u, \gamma_0 v \rangle_{\Gamma}.
\]

If \( Lu \in L^1(\Omega) \), we have, by the definition of \( \pi \),

\[
\langle \pi Lu, v \rangle_{V^*} = \int_{\Omega} Lu \ dx = \langle Lu, v \rangle_{L^1, L^\infty},
\]
NONLINEAR SIGNORINI'S PROBLEMS

then

\[ \langle Au, v \rangle_{V', V} = \langle Lu, v \rangle_{L^1, L^\infty} + \langle \Phi_u, \gamma_0 v \rangle_{\Gamma} \quad (2.4) \]

From (2.4) and Lemma 1 we have

\[ \Phi_u = \gamma_a u. \]

**THEOREM 2.** Let \( u \) be a solution of \((1.5)\) and suppose \( A\psi \in V^* \), \( L\psi \in L^1(\Omega) \); we have \( L \psi \in L^1(\Omega) \) and

\[ 0 \wedge (L\psi + H(\cdot, \psi, D\psi)) \leq Lu + H(\cdot, u, Du) \leq 0 \]

in \( L^1(\Omega) \), where \( \wedge \) denotes the infimum in the almost everywhere sense,

\[ (\gamma_a \psi) \wedge 0 \leq \gamma_a u \leq 0 \quad \text{in} \quad (H^{1/2}(\Gamma) \cap L^\infty(\Gamma))^*, \]

where \( \Gamma \) denotes the infimum in the order dual of \( H^{1/2}(\Gamma) \cap L^\infty(\Gamma) \).

By the same methods of [10], we find

\[ 0 \wedge (A\psi + H(\cdot, \psi, D\psi)) \leq Au + H(\cdot, u, Du) \leq 0 \quad \text{in} \quad V^*. \quad (2.5) \]

From (2.5), being \( L^1(\Omega) \subset V^* \), we have \( Au \in V^* \). We decompose now the inequality (2.5) on \( \pi \Theta \) and \( \pi \Theta^\perp \) using Lemma 2, and we obtain

\[ 0 \wedge (L\psi + H(\cdot, \psi, D\psi)) \leq Lu + H(\cdot, u, Du) \leq 0 \]

\[ 0 \wedge \Phi_\psi \leq \Phi_u \leq 0. \]

Since \( L\psi \in L^1(\Omega) \), we then find \( Lu \in L^1(\Omega) \), \( \gamma_a \psi = \Phi_\psi \), \( \gamma_a u = \psi_u \) and

\[ 0 \leq \gamma_a \psi \leq \gamma_a u \leq 0. \]

3. **AN ESTIMATE OF THE LIPSCHITZ NORM OF SOLUTIONS IN** \( C^1 \cap H^2 \)

Let \( u \in C^1(\Omega) \cap H^2(\Omega) \) be a solution of \((1.5)\), \( a_0 \in H^{1, \infty}(\Omega) \), \( \psi \in H^{2, \infty}(\psi) \).

**LEMMA 3.** We have, \( x_0 \in \Omega \),

\[ \int_{\Omega \cap B(\delta x_0)} |\nabla u(x)|^2 \frac{|x - x_0|^2}{\delta^2} \, dx \leq \omega(R), \quad (3.1) \]

where \( \omega(R) \to 0 \) for \( R \to 0 \) depends only on \( \|u\|_{L^\infty} \).

The proof is the same as that of Lemma 1.1 in [9]. At points \( x_0 \in \Gamma \) an
additional reflection argument is needed, to prolongate the \( a'_y \)'s and to define the Green function of the problem.

Now let us fix \( x_0 \in \Omega \). We consider the regularised Green function \( G_\rho, \rho > 0 \), and the coefficients \( a_y \), as done in [9]. Let \( \zeta \in C_0^\infty(\Omega), \zeta \geq 0, \zeta = 1 \) on a neighbourhood of \( x_0 \).

We denote
\[
D_k^h \omega(x) = \frac{\omega(x + he_k) - \omega(x)}{h}, \quad k = 1, \ldots, N, \quad h > 0.
\]

We observe now that
\[
u_h^+ = u + \epsilon h D_k^{-h} (\zeta^2 G_\rho (h D_k^h u - hM) + ) \in K^\psi
\]
\((M = \|\psi\|_{H^{1,\infty} + 1}) \) for \( \epsilon \leq \epsilon_0(\rho, \zeta) \) and \( 0 < h \leq h_0 \). By choosing \( u_h^+ \) as test function, we have
\[
\sum_{i,j = 1}^N (a_y D_j u, D_k^{-h} [D_j (\zeta^2 G_\rho (D_k^h u - M) + )])_{L^2} + (H(\cdot, u, Du), D_k^{-h} [D_j (\zeta^2 G_\rho (D_k^h u - M) + )])_{L^2} \geq 0.
\]

Since \( a_y \in H^{1,\infty}(\Omega) \) and \( u \in C^1(\overline{\Omega}) \cap H^2(\Omega) \), by taking the limit as \( h \to 0 \) we find
\[
\sum_{i,j = 1}^N (D_k (a_y D_j u), D_k (\zeta^2 G_\rho (D_k^h u - M) + ))_{L^2} - H(\cdot, u, Du), D_k (\zeta^2 G_\rho (D_k^h u - M) + ))_{L^2} \leq 0.
\]

Consider now the first term; by the same methods of [9] we have
\[
\sum_{i,j = 1}^N (D_k (a_y D_j u), D_k (\zeta^2 G_\rho (D_k^h u - M) + ))_{L^2} \geq \frac{\nu}{2} \int_{\Omega} |DD_k u|_+ G_\rho \zeta^2 dx + |B_\mu|^{-1} \int_{B_\rho} \zeta^2 (D_k u - M)_+ dx - K_1 \|\zeta (D_k u - M) + \|_{L^\infty} - K_1,
\]
where
\[
|DD_k u|_+ = |DD_k u| \quad \text{if} \quad D_k u \geq M
\]
\[
= 0 \quad \text{otherwise}.
\]

(3.2)
For the second term we have

\[
\left| \int_{\Omega} H(x, u, Du) D_k(\xi^2 G_\rho(D_k u - M)_+) \, dx \right| \leq k_2 \int_{\Omega} (1 + |Du|^2)(\xi^2 G_\rho |D_k(D_k u)|_+ + 
\xi^2 \nabla G_\rho \cdot (D_k u - M)_+ + 2\xi \nabla \xi \cdot G_\rho(D_k u - M)_+ \, dx
\]

\[
\leq K_{\text{w}} \int_{\Omega} (1 + |Du|^4) \xi^2 G_\rho \, dx
\]

\[
+ \varepsilon_0 \int_{\Omega} |D(D_k u)|_+ \xi^2 G_\rho \, dx
\]

\[
+ K_2 \|\xi(D_k u - M)_+\|_{L^\infty} \int_{\Omega} (1 + |Du|^2) \nabla G_\rho \, dx
\]

\[
+ K_3 \|\xi(D_k u - M)_+\|_{L^\infty} \int_{\Omega} (1 + |Du|^2) G_\rho \, dx
\]

(We denote now by \( \int_* \) the integration on supp(\( \xi \)))

\[
\leq K'_\text{w}(1 + \|\xi Du\|_{L^\infty}) \int_* (1 + |Du|^2) G_\rho \, dx
\]

\[
\varepsilon_0 \int_{\Omega} |D(D_k u)|_+ \xi^2 G_\rho \, dx
\]

\[
+ K_2'(1 + \|\xi Du\|_{L^\infty}) \int_* (1 + |Du|) |D G_\rho| \, dx
\]

\[
+ K_3' \|\xi Du\|_{L^\infty}.
\]

(3.3)

From (3.2), (3.3), and Lemma 3 we have

\[
\int_{\Omega} |D(D_k u)|_+^2 \xi^2 G_\rho \, dx
\]

\[
+ K_4' |B_\rho|^{-1} \int_{B_\rho} \xi^2 |(D_k u - M)_+|^2 \, dx
\]

\[
- 4\varepsilon_1 \|\xi Du\|_{L^\infty}^2 \leq K.
\]

(3.4)

We consider now the case \( x_0 \in \Gamma \); in this case we can suppose \( \Gamma \subset \{x_N = 0\} \) in a neighbourhood \( \tilde{N} \) of \( x_0 \), \( \tilde{N} \cap \Omega \subset \{x_N > 0\} \), \( a_{i,N} = 0, i \neq N \) on \( \Gamma \cap N \) [11], and we define the Green function in \( x_0 \) by a reflection on
the \( a_{ij} \). By the same methods used above we have (3.4) for \( k = 1, \ldots, N - 1 \); for the case \( k = N \) it is sufficient to observe that from dual inequalities

\[ |D_N u| \leq \| \psi \|_{H_{1,x}} \quad \text{on} \quad \Gamma \cap \overline{N}; \]

then for \( 0 < h \leq h_0 \), \((D_k^h u - M)_+ = 0\) in a neighbourhood of \( \Gamma \cap \overline{N} \); then we have (3.4) for \( k = N \), with \( \text{supp}(\zeta) \subset \overline{N} \) (in this case \( B_\rho = \{ |x - x_0| \leq \rho \} \cap \{ x_N > 0 \} \). From (3.4) we have

\[ K_4' \left| B_\rho \right|^{-1} \int_{B_\rho} \zeta^2 |(D_k u - M)_+|^2 \, dx \]

\[ \leq 4\varepsilon_1 \| \zeta \|_{L^2}^2 \frac{\| D u \|_{L^\infty}}{K}; \]

then passing to the limit as \( \rho \to 0 \),

\[ K_4' \left| (D_k u - M)_+ \right|^2 (x_0) \leq 4\varepsilon_1 \| D u \|_{L^\infty}^2 + K. \]

Analogously we have

\[ K_4'' \left| (D_k u + M)_- \right|^2 (x_0) \leq 4\varepsilon_1 \| D u \|_{L^\infty}^2 + K; \]

then

\[ K_3 \left| D_k u \right|^2 (x_0) \leq 4\varepsilon_1 \| D u \|_{L^\infty}^2 + K + M^2. \]

Summing up for \( k = 1, \ldots, N \) we have

\[ K_3' \left| D u \right|^2 (x_0) \leq 4\varepsilon_1 \| D u \|_{L^\infty}^2 + K + M. \]

By choosing \( x_0 \) such that \( |D u| (x_0) = \text{Max}_{x \in \Omega} |D u| \) we obtain finally

\[ \| D u \|_{L^\infty} \leq K_0', \]

and we observe that \( K_0' \) depends only on \( \| a_{ij} \|_{H^{1,x}}, \| \psi \|_{H^{1,x}}, \) and \( \| u \|_{L^\infty} \).

**Proposition 1.** Let \( \psi \in H^{2,\infty}, \ a_{ij} \in H^{1,\infty}(\Omega) \) and let \( u \in C^1(\overline{\Omega}) \cap H^2(\Omega) \) be a solution of (1.5), then

\[ |D u| \leq K_0' \quad \text{in} \quad \Omega, \]

where \( K_0' \) depends only on \( \| \psi \|_{H^{1,x}}, \| a_{ij} \|_{H^{1,x}}, \) and \( \| u \|_{L^\infty} \).

4. **Proof of Theorem 1.** We first suppose that for all \( C > 0 \) there exists \( K = K_C > 0 \) such that

\[ |H(x, u, p) - H(x, v, q)| \leq K_C(|u - v| + |p - q|), \quad (4.1) \]
if \(|u|, |v|, |p|, |q| \leq C\). For arbitrary \(m > 0\) we set
\[
\varepsilon_m(t) = \begin{cases} 
\ell, & |\ell| < m, \\
m, & t > m, \\
-m, & |t| < -m.
\end{cases}
\]
\(\omega_m(p) = \{\tau_m(p_i)\} \quad \forall p \in \mathbb{R}^N,\)
\(H_m(x, u, p) = \tau_m(H(x, \tau_m(u), \omega_m(p))).\)

Let \(\psi_m\) be a sequence in \(H^{2, \infty}(\Omega)\) such that
\[
\lim_{m \to \infty} \psi_m = \psi \quad \text{in} \quad H^{1, p}(\Omega) \quad \forall 1 < p < +\infty,
\]
\[\|\psi_m\|_{H^{1, p}} \leq C, \quad \psi_m \geq \psi.\]

We observe that there exists \(C_m > 0\) such that
\[B_m = A + H_m(\cdot, \cdot, \cdot) + C_m: H^1(\Omega) \to (H^1(\Omega)),\]
is uniformly monotone, i.e.,
\[\langle B_m v_1 - B_m v_2, v_1 - v_2 \rangle \geq \delta \|v_1 - v_2\|_{L^1}^2, \quad \delta > 0.\]

Consider now the variational inequality
\[
\langle B_m \tilde{u}_m, v - \tilde{u}_m \rangle \geq \langle C_m z, v - \tilde{u}_m \rangle \quad \forall v \in K^{\psi_m}, \quad \tilde{u}_m \in K^{\psi_m}. \tag{4.2_m}
\]

**Lemma 4.** Let \(\tilde{u}_m\) be the solution of \((4.2_m)\), with \(z \geq \Phi\). Then,
\[\Phi \leq \tilde{u}_m \quad \forall m \geq m_0.\]

We observe that there exists \(m_0 > 0\) such that for \(m \geq m_0\)
\[H_m(x, \Phi(x), D\Phi(x)) = H(x, \Phi(x), D\Phi(x)). \tag{4.3}
\]

Let \(v = \tilde{u}_m + (\tilde{u}_m - \Phi)^-\); we have
\[
\langle B_m \tilde{u}_m, (\tilde{u}_m - \Phi)^- \rangle \geq C_m \langle z, (\tilde{u}_m - \Phi)^- \rangle,
\]
then
\[\delta \|(\tilde{u}_m - \Phi)^-\|_{L^1}^2 \leq B_m \Phi, -(u_m - \Phi)^- \rangle \leq C_m \langle z, -(u_m - \Phi)^- \rangle.\]
From (4.3) we have, for $m \geq m_0$,
\[ \delta \| (\tilde{u}_m - \Phi) \|^2_{H^1} \leq C_m < \Phi, - (\tilde{u}_m - \Phi) > \leq 0, \]
thus $\tilde{u}_m \geq \Phi$.

We denote now $u_m = S_m(z), C = \{ v \in L^\infty(\Omega), \Phi \leq v \leq \sup_{x \in \Omega} \psi_m \}, m \geq m_0$.

We observe that $S_m: C \rightarrow C$ and $S_m$ is continuous for the $L^\infty$-norm. Since $H_m(\cdot, \cdot, \cdot)$ is bounded, by using the dual inequalities and the result of [8], we obtain, that $S_m(C)$ is a bounded set in $C^1(\bar{\Omega})$, then there exists a fixed point $u_m \in C^1(\bar{\Omega})$ of $S_m$ in $C$. We remark that $u_m$ is a solution of the variational inequality,
\[ \langle Au_m, v - u_m \rangle + \int_{\Omega} H_m(x, u_m(x), Du_m(x)(v(x) - u_m(x))) \, dx \geq 0 \]
\[ \forall v \in K^{\psi_m}, \quad u_m \in K^{\psi_m}. \] (4.4m)

We know that $u_m \in C^1(\bar{\Omega})$ and it is uniformly bounded with respect to $m$. Hence, from Proposition 1,
\[ \| u_m \|_{H^p} \leq K'_0, \] (4.5)
where $K'_0$ does not depend on $m$. Therefore, we can easily prove that $\lim_{m \rightarrow \infty} \mu_m - u$ in $H^0_0(\Omega)$ then
\[ \lim_{m \rightarrow \infty} u_m = u \quad \text{in} \quad H^{1,p}(\Omega), \quad 1 < p < +\infty, \] (4.6)
\[ \| u \|_{H^{1,p}} \leq K'_0. \]
Passing to the limit in (4.4m), we have that $u$ is a solution of (1.5). We observe finally that $K'_0$ does not depend on $K_e$ in (4.1).

We consider now the general case. There exists a sequence $\{ H_n(x, u, p) \}$ such that
\[ |H_n(x, u, p)| \leq K''_c + K_1 |p|^2 \]
\[ |H_n(x, u, p) - H_n(x, v, q)| \leq K'_c(|u - v| + |p - q|), \]
\[ \lim_{n \rightarrow +\infty} H_n(x, u, p) = H(x, u, p), \]
for almost all $x \in \Omega$, uniformly on bounded sets of $R^{N+1}$ and moreover
\[ H_n(x, u, p) \leq H(x, u, p) \]
for $|u|, |p| \leq \| \Phi \|_{H^{1,\infty}} + 1$. We have
\[ A\Phi + H_n(\cdot, \Phi, D\Phi) \leq 0 \quad \forall n. \]
Consider the variational inequality

\[
\langle Au_n, v - u_n \rangle + \int_{\Omega} H_n(x, u_n(x), Du_n(x))(v(x) - u_n(x)) \, dx \geq 0
\]
\[\forall v \in K^\psi, \quad u_n \in K^\psi; \quad (4.7_n)\]

(4.7_n) has a solution \( u_n \) such that

\[
\Phi \leq u_n \leq \psi \quad \text{and} \quad \|u_n\|_{H^{1,p}(\Omega)} \leq K_0',
\]

then we can suppose, as before,

\[
\lim_{n \to \infty} u_n = u \quad \text{in} \quad H^{1,p}(\Omega), \quad 1 < p < +\infty,
\]

\[
\|u_n\|_{H^{1,p}(\Omega)} \leq K_0'.
\]

Passing to the limit in (4.7_n), we have that \( u \) is a solution of (1.5).

5. Further Regularity Results

From Theorem 1 we can easily deduce the following results.

**Corollary 1.** Under the assumptions of Theorem 1, if \( \psi \in H^2(\Omega) \cap H^{1,\infty}(\Omega) \), there exists a solution \( u \subset H^2(\Omega) \cap H^{1,\infty}(\Omega) \) of problem (1.3), satisfying the additional condition \( u \geq \Phi \) a.e. in \( \Omega \).

Since \( \psi \in H^{1,\infty}(\Omega) \), there exists a solution \( u \geq \Phi \) of (1.3), \( u \subset H^{1,\infty} \), hence \( H(\cdot, u(\cdot), Du(\cdot)) \subset L^\infty(\Omega) \).

Being \( \psi \in H^2(\Omega) \), we have [5] \( u \in H^2(\Omega) \).

**Corollary 2.** Under the assumptions of Theorem 1, if \( a_{ij} \in H^{2,\infty}(\Omega) \) and \( \psi \in C^{1,\alpha}(\Omega) \) \( \alpha_0 \leq \alpha < 1 \), \( \alpha_0 \in (0, 1) \) suitable, then there exists a solution \( u \in C^{1,\beta}(\Omega) \), \( \beta \in (0, 1) \), of (1.3) satisfying the additional condition \( u \geq \Phi \) a.e. in \( \Omega \).

To prove Corollary 2 we first give an improvement of the result of [7], relative to the linear case.

Consider the variational inequality

\[
\langle Au, v - u \rangle + \int_{\Omega} f(x)(v(x) - u(x)) \, dx, \quad v \in K^\psi; \quad u \in K^\psi, \quad (5.1)
\]

where \( f \in L^\infty(\Omega) \).
PROPOSITION 2. If \( a_{ij} \in H^{2,\alpha}(\Omega) \), \( \psi \in C^{1,\alpha}(\overline{\Omega}) \), \( \alpha_0 \leq \alpha < 1 \), \( \alpha_0 \) suitable. Then any bounded solution \( u \) of (5.1) belongs to \( C^{1,\beta}(\overline{\Omega}) \).

We first prove an approximation lemma.

**Lemma 5.** Let \( \psi \) be in \( C^{1,\alpha}(\overline{\Omega}) \), \( \alpha \in (0, 1) \). Then there exists a sequence \( \{ \psi_n \} \subset C^{\infty}(\overline{\Omega}) \) such that

\[
\| \psi_n \|_{C^\alpha} \leq C n^{1-\alpha} \| \psi \|_{C^{1,\alpha}} \tag{5.2}
\]

\[
\| \psi_n - \psi \|_{L^\infty} \leq C n^{- (1 - \alpha)} \| \psi \|_{C^{1,\alpha}}. \tag{5.3}
\]

We can prove the result in the case \( \Omega = \mathbb{R}^N \), \( \psi \in C^{1,\alpha}(\mathbb{R}^N \cap H^{1,\alpha}(\mathbb{R}^N)) \). Let \( \rho \) be a function in \( C^{\infty}(\mathbb{R}^N) \) with support in the ball \( B_1 \) and \( \int \rho(y) \, dy = 1 \); we denote

\[ \rho_n(y) = n^{-N} \rho(ny). \]

We define

\[
\psi_n^+(x) = \int \rho_n(y) \psi(x + y) \, dy
\]

\[
\psi_n^-(x) = \int \rho_n(y) \psi(x - y) \, dy
\]

\[
\psi_n = \frac{1}{2} (\psi_n^+ + \psi_n^-).
\]

We have

\[
|\psi_n(x) - \psi(x)| = \left| \int \rho_n(y) \frac{\psi(x + y) - 2\psi(x) + \psi(x - y)}{2} \, dy \right|
\]

\[
\leq C n^{-(1 + \alpha)} \| \psi \|_{C^{1,\alpha}}.
\]

Moreover we have

\[
|D^2 \psi_n(x)| = \left| \int D\rho_n(y) \left( \frac{D\psi_n(x + y) + D\psi_n(x - y)}{2} \right) \, dy \right|
\]

\[= \left| \int D\rho_n(y) \left( \frac{D\psi_n(x + y) + D\psi_n(x - y)}{2} - (D\psi_n)_{B_1(0)} \right) \, dy \right|
\]

\[
\leq C n^{1-\alpha} \| \psi \|_{C^{1,\alpha}}.
\]

We introduce the convex set

\[ \tilde{K}^\psi = \{ v \in H^1(\Omega), v \leq \psi \text{ a.e. in } \Gamma \}. \]
We consider the linear variational inequality

\[ \langle Au + \lambda u, v - u \rangle \geq 0, \quad \forall v \in \tilde{K}^\psi, \quad u \in \tilde{K}^\psi, \quad \lambda > 0. \]  

(5.4)

If \( \psi \in C^2(\Omega) \), we have \( u \in C^{1,\gamma}(\Omega), \) \( \gamma \in (0, 1) \) suitable [7].

Now let \( \psi \) be in \( C^{1,\gamma}(\Omega) \), \( \{\psi_n\} \) be the sequence considered in Lemma 5 and \( u_n \) the solution of (5.4) relative to \( \psi_n \).

We have, [7, 18],

\[ \|u_n - u\|_{L^\infty} \leq Cn^{-(1 + \gamma)}, \]

\[ \|u_n\|_{C^{\gamma}} \leq Cn^{1 - \gamma}, \]

then

\[ |u(x + h) - 2u(x) + u(x - h)| \leq Cn^{1 - \gamma} |h|^{1 + \gamma} + 2Cn^{-(1 - \gamma)}. \]

By choosing a suitable \( n \), we have

\[ |u(x + h) - 2u(x) + u(x - h)| \leq C|h|^{(1 + \gamma)/(1 + \alpha)}. \]

Also we have easily \( u \in H^{1,\infty}(\Omega) \), then \( u \in C^{1,\delta}(\Omega) \) for \( \delta = \alpha_0 - 2/(1 + \gamma) - 1 \), where \( \delta = (1 + \gamma)/(1 + \alpha) - 1 \) ([6, Corollary 3.4.9 p. 202] proving for the boundary at first that the tangential derivatives are Hölder continuous and then proving the Hölder continuity of the normal derivative).

Consider now the variational inequality (5.1). If \( \psi \in H^{2,\infty}(\Omega) \), we have from the dual inequalities \( Lu \in L^p(\Omega) \) and

\[ (f - Lu)(u - \psi) = 0, \quad \gamma \neq u(u - \psi) = 0. \]

The function \( u \) is also the solution of the variational inequality

\[ \langle Au + \lambda u, v - u \rangle \geq \int_{\Omega} g(x)(v(x) - u(x)) \, dx, \quad \forall v \in \tilde{K}^\psi, \quad u \in \tilde{K}^\psi, \]  

(5.5)

where \((-) g = Lu + \lambda u \in L^p(\Omega)\), then \( u \in C^{1,\delta}(\Omega) \), \( \delta > 0 \).

For \( \psi \in C^{1,\gamma}(\Omega) \), we consider the sequence \( \{\psi_n\} \) of Lemma 5 and the problems

\[ \langle Au_n + u_n, v - u_n \rangle \leq \int_{\Omega} \tilde{f}(x)(v(x) - u_n(x)) \, dx, \quad \forall v \in K^\psi_n, \quad u_n \in K^\psi_n, \]  

(5.6a)

\[ Au + u, v - u \geq \int_{\Omega} \tilde{f}(x)(v(x) - u(x)) \, dx, \quad \forall v \in K^\psi, \quad u \in K^\psi, \]  

(5.6b)

where \( \tilde{f} = f + \lambda u \); a regularization on \( \psi \) as above now gives the result of Proposition 2.
We observe that, from Theorem 1 there exists a solution \( u \in H^{1,\infty}(\Omega) \) of (1.3) with \( u \geq \Phi \); then \( H(\cdot, u, Du) \in L^\infty(\Omega) \) and, by Proposition 2, \( u \in C^{1,\beta}(\Omega) \).

REFERENCES


