General convergence analysis for two-step projection methods and applications to variational problems

Ram U. Verma

Department of Theoretical & Applied Mathematics, The University of Akron, Akron, OH 44325, USA

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Abstract

First a general model for two-step projection methods is introduced and second it has been applied to the approximation solvability of a system of nonlinear variational inequality problems in a Hilbert space setting. Let $H$ be a real Hilbert space and $K$ be a nonempty closed convex subset of $H$. For arbitrarily chosen initial points $x^0, y^0 \in K$, compute sequences $\{x_k\}$ and $\{y_k\}$ such that

$$x^{k+1} = (1 - a^k)x^k + a^k P_K[y^k - \rho T(y^k)] \quad \text{for } \rho > 0$$

$$y^k = (1 - b^k)x^k + b^k P_K[x^k - \eta T(x^k)] \quad \text{for } \eta > 0,$$

where $T : K \rightarrow H$ is a nonlinear mapping on $K$, $P_K$ is the projection of $H$ onto $K$, and $0 \leq a^k, b^k \leq 1$. The two-step model is applied to some variational inequality problems.

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1. Introduction

Projection/projection type methods have played a significant role in the numerical resolution of variational inequalities based on their convergence analyses. However, the convergence analysis does
require some sort of strong monotonicity besides the Lipschitz continuity. There have been some recent developments where convergence analysis for projection/projection type methods under somewhat weaker conditions such as cocoercivity [1,2] and partial relaxed monotonicity [3] is achieved. Recently, the author [4] introduced a two-step model for nonlinear variational inequalities and discussed the approximation solvability of this model based on the convergence analysis of a two-step projection method in a Hilbert space setting. The two-step projection/projection type methods contain several known as well as new projection methods as special cases, while some have been applied to problems arising, especially from complementarity, computational mathematics, convex quadratic programming, and other variational problems. Later, Nie et al. [5] investigated using the two-step model the approximation solvability of a system of nonlinear variational inequalities involving a combination of strongly monotonic and pseudocontractive mappings. Two-step models for nonlinear variational inequality problems are relatively more challenging than the usual variational inequality problems and their corresponding solvability.

Here in this paper, we intend to introduce the general two-step model for projection methods, which reduces to the two-step model applied in [4] and then apply it to the approximation solvability of a two-step strongly monotonic nonlinear variational inequality in a Hilbert space setting. The obtained results complement results of Verma [4], Nie et al. [5] and others. For more detailed accounts on general variational inequality problems and related iterative procedures, we refer to [6–13].

Let $H$ be a real Hilbert space with the inner product $⟨·, ·⟩$ and norm $∥·∥$. Let $T : K → H$ be any mapping on $K$ and $K$ be a closed convex subset of $H$. We consider a system of two nonlinear variational inequality (abbreviated as SNVI) problems as follows: determine elements $x^*, y^* ∈ K$ such that

$$\langle ρT(y^*) + x^* - y^*, x - x^*⟩ ≥ 0 \quad ∀ x ∈ K \text{ and for } ρ > 0$$

(1)

$$\langle ηT(x^*) + y^* - x^*, x - y^*⟩ ≥ 0 \quad ∀ x ∈ K \text{ and for } η > 0.$$  

(2)

The SNVI (1) and (2) problem is equivalent to the following projection formulas

$$x^* = P_K[y^* - ρT(y^*)] \quad \text{for } ρ > 0$$

(3)

$$y^* = P_K[x^* - ηT(x^*)] \quad \text{for } η > 0.$$  

(4)

where $P_K$ is the projection of $H$ onto $K$.

We note that for $η = 0$ the SNVI (1) and (2) problem reduces to the NVI problem: determine an element $x^* ∈ K$ such that

$$⟨T(x^*), x - x^*⟩ ≥ 0 \quad ∀ x ∈ K.$$  

(5)

Let $K$ be a closed convex cone of $H$. The SNVI (1) and (2) problem is equivalent to a system of nonlinear complementarities (abbreviated as SNC): find the elements $x^*, y^* ∈ K$ such that $T(x^*) ∈ K^*$, $T(y^*) ∈ K^*$ and,

$$\langle ρT(y^*) + x^* - y^*, x^*⟩ = 0 \quad \text{for } ρ > 0$$

(6)

$$\langle ηT(x^*) + y^* - x^*, y^*⟩ = 0 \quad \text{for } η > 0.$$  

(7)

where $K^*$ is a polar cone to $K$ defined by

$$K^* = \{ f ∈ H : ⟨f, x⟩ ≥ 0 \quad ∀ x ∈ K \}.$$  

Now we need to recall the following auxiliary result, most commonly used in the context of approximation solvability of nonlinear variational inequality problems based on iterative procedures.
Lemma 1.1 ([8]). For an element \( z \in H \), we have
\[
x \in K \quad \text{and} \quad \langle x - z, y - x \rangle \geq 0 \quad \forall \ y \in K \quad \text{if and only if} \quad x = P_K(z).
\]

A mapping \( T : H \to H \) is called monotonic if for each \( x, y \in H \), we have
\[
\langle T(x) - T(y), x - y \rangle \geq 0.
\]

A mapping \( T : H \to H \) is called \( r \)-strongly monotonic if for each \( x, y \in H \), we have
\[
\langle T(x) - T(y), x - y \rangle \geq r \|x - y\|^2 \quad \text{for a constant} \ r > 0.
\]

This implies that
\[
\|T(x) - T(y)\| \geq r \|x - y\|,
\]
that is, \( T \) is \( r \)-expansive, and when \( r = 1 \), it is expansive.

Example 1.1. Consider a mapping \( T : R^n \to R^n \) defined by
\[
T(x) = cI(x) + v,
\]
where \( c > 0, x, v \in R^n \) with \( v \) fixed, and \( I \) is the \( n \times n \) identity matrix. Then \( T \) is \( r \)-strongly monotonic for \( 0 < r < c \). For \( x, y \in R^n \), we have
\[
\langle T(x) - T(y), x - y \rangle = c(x - y, x - y)
= c\|x - y\|^2 \geq r\|x - y\|^2 \quad \text{for} \ 0 < r \leq c.
\]

This is an example of a non-strongly monotonic mapping in \([-1, 1]\).

Example 1.2. Consider a mapping \( T \) defined by
\[
T(x) = x^3 \quad \text{for} \ x \in [-1, 1].
\]
The mapping \( T \) is not strongly monotonic.

The mapping \( T \) is called \( s \)-Lipschitz continuous (or Lipschitzian) if there exists a constant \( s \geq 0 \) such that
\[
\|T(x) - T(y)\| \leq s\|x - y\| \quad \forall \ x, y \in H.
\]

\( T \) is called \( \mu \)-cocoercive [1] if for each \( x, y \in H \), we have
\[
\langle T(x) - T(y), x - y \rangle \geq \mu\|T(x) - T(y)\|^2 \quad \text{for a constant} \ \mu > 0.
\]

Clearly, every \( \mu \)-cocoercive mapping \( T \) is \((1/\mu)\)-Lipschitz continuous.

Example 1.3. Consider a mapping \( T : H \to H \) on a Hilbert space \( H \), which is nonexpansive. Then \( I - T \) is \((1/2)\)-cocoercive. For any \( x, y \in H \), we have
\[
\|(I - T)(x) - (I - T)(y)\|^2 = \langle (I - T)(x) - (I - T)(y), (I - T)(x) - (I - T)(y) \rangle
\leq 2\|x - y\|^2 - \langle x - y, T(x) - T(y) \rangle
= 2\langle x - y, (I - T)(x) - (I - T)(y) \rangle,
\]
that is, \( I - T \) is \((1/2)\)-cocoercive.
We can easily see that the following implications on monotonicity, strong monotonicity and expansiveness hold:

\[
\text{strong monotonicity} \Rightarrow \text{monotonicity} \Downarrow \text{expansiveness}
\]

\(T\) is called relaxed \(\gamma\)-cocoercive if there exists a constant \(\gamma > 0\) such that

\[
\langle T(x) - T(y), x - y \rangle \geq (-\gamma)\|T(x) - T(y)\|^2 \quad \forall \ x, y \in H.
\]

2. Projection methods

This section deals with an introduction of general two-step models for projection methods and its special forms that can be applied to the convergence analysis for projection methods in the context of the approximation solvability of the SNVI (1) and (2) problem.

**Algorithm 2.1.** For arbitrarily chosen initial points \(x^0, y^0 \in K\), compute the sequences \(\{x^k\}\) and \(\{y^k\}\) such that

\[
x^{k+1} = (1 - a^k)x^k + a^k P_K[y^k - \rho T(y^k)],
\]

\[
y^k = (1 - b^k)x^k + b^k P_K[x^k - \eta T(x^k)],
\]

where \(P_K\) is the projection of \(H\) onto \(K\), \(\rho, \eta > 0\) are constants, and

\[
0 \leq a^k, b^k \leq 1 \quad \text{for } k \geq 0.
\]

For \(\eta = 0\) and \(b^k = 1\) in Algorithm 2.1, we arrive at

**Algorithm 2.2.** For an arbitrarily chosen initial point \(x^0 \in K\), compute the sequence \(\{x^k\}\) such that

\[
x^{k+1} = (1 - a^k)x^k + a^k P_K[x^k - \rho T(x^k)],
\]

where

\[
0 \leq a^k \leq 1.
\]

For \(b^k = 1\) in Algorithm 2.1, we get

**Algorithm 2.3.** For arbitrarily chosen initial points \(x^0, y^0 \in K\), sequences \(\{x^k\}\) and \(\{y^k\}\) are generated by

\[
x^{k+1} = (1 - a^k)x^k + a^k P_K[y^k - \rho T(y^k)],
\]

\[
y^k = P_K[x^k - \eta T(x^k)],
\]

where

\[
0 \leq a^k \leq 1 \quad \text{for } k \geq 0.
\]

3. Applications

We now present, based on Algorithm 2.1, the approximation-solvability of the SNVI (1) and (2) problem involving strongly \(r\)-monotonic and \(\mu\)-Lipschitz continuous mappings in a Hilbert space setting.
Theorem 3.1. Let $H$ be a real Hilbert space and $K$ a nonempty closed convex subset of $H$. Let $T : K \rightarrow H$ be strongly $r$-monotonic and $\mu$-Lipschitz continuous. Suppose that $x^*, y^* \in K$ form a solution to the SNVI (1) and (2) problem, the sequences $\{x^k\}$ and $\{y^k\}$ are generated by Algorithm 2.1 and

$$0 \leq a^k, b^k \leq 1 \quad \text{and} \quad \sum_{k=0}^{\infty} a^k b^k = \infty.$$ 

Then sequences $\{x^k\}$ and $\{y^k\}$, respectively, converge to $x^*$ and $y^*$ for

$$0 < \rho < 2r/\mu^2 \quad \text{and} \quad 0 < \eta < 2r/\mu^2.$$

Proof. Since $x^*$ and $y^*$ form a solution to the SNVI (1) and (2) problem, it follows that

$$x^* = P_K[y^* - \rho T(y^*)],$$

$$y^* = P_K[x^* - \eta T(x^*)].$$

Applying Algorithm 2.1, we have

$$\|x^{k+1} - x^*\| = \|(1 - a^k)x^k + a^k P_K[y^k - \rho T(y^*)] - (1 - a^k)x^* - a^k P_K[y^* - \rho T(y^*)]\|$$

$$\leq (1 - a^k)\|x^k - x^*\| + a^k\|P_K[y^k - \rho T(y^*)] - P_K[y^* - \rho T(y^*)]\|$$

$$\leq (1 - a^k)\|x^k - x^*\| + a^k\|y^k - y^* - \rho [T(y^*) - T(y^*)]\|.$$  \(8\)

Since $T$ is strongly $r$-monotonic and $\mu$-Lipschitz continuous, we have

$$\|y^k - y^* - \rho T(y^*)\|^2$$

$$= \|y^k - y^*\|^2 - 2\rho (T(y^k) - T(y^*), y^k - y^*) + \rho^2\|T(y^k) - T(y^*)\|^2$$

$$\leq \|y^k - y^*\|^2 - 2\rho r\|y^k - y^*\|^2 + (\rho^2\mu^2)\|y^k - y^*\|^2$$

$$\leq \|y^k - y^*\|^2 + (\rho\mu)^2\|y^k - y^*\|^2 - 2\rho r\|y^k - y^*\|^2$$

$$= [1 - 2\rho r + (\rho\mu)^2]\|y^k - y^*\|^2.$$  \(9\)

As a result, in light of (8), we have

$$\|x^{k+1} - x^*\| \leq (1 - a^k)\|x^k - x^*\| + a^k\|y^k - y^*\|$$

$$\leq (1 - a^k)\|x^k - x^*\| + a^k\|y^k - y^*\|,$$  \(9\)

where $\theta = [1 - 2\rho r + (\rho\mu)^2]^{1/2} < 1$.

Similarly, we have

$$\|y^k - y^*\| = \|(1 - b^k)(x^k - x^*) + b^k P_K[x^k - \eta T(x^k)] - P_K[x^* - \eta T(x^*)]\|^2$$

$$\leq (1 - b^k)\|x^k - x^*\| + b^k\|x^k - x^* - \eta [T(x^k) - T(x^*)]\|$$

$$\leq (1 - b^k)\|x^k - x^*\| + b^k[1 - 2\eta r + (\eta\mu)^2]^{1/2}\|x^k - x^*\|$$

$$= (1 - b^k)\|x^k - x^*\| + b^k\sigma\|x^k - x^*\|.$$  \(10\)

where $\sigma = [1 - 2\eta r + (\eta\mu)^2]^{1/2} < 1$.

It follows from (9) and (10) that

$$\|x^{k+1} - x^*\| \leq (1 - a^k)\|x^k - x^*\|$$

$$+ a^k[(1 - b^k) + b^k\sigma]\|x^k - x^*\|$$  \(11\)
\[
\begin{align*}
&= \left[1 - (1 - \sigma)a^kb^k\right]\|x^k - x^*\| \\
&\leq \prod_{j=0}^{k}[1 - (1 - \sigma)a^jb^j]\|x^0 - x^*\|, \quad (12)
\end{align*}
\]

where \( \sigma = [1 - 2\eta r + (\eta\mu)^2]^{1/2} < 1. \)

Since \( \sigma < 1 \) and \( \sum_{k=0}^{\infty} a^kb^k \) is divergent, it implies in light of [10] that

\[
\lim_{k \to \infty} \prod_{j=0}^{k}[1 - (1 - \theta)a^jb^j] = 0.
\]

Hence, the sequence \( \{x^k\} \) converges to \( x^* \) by (12), and the sequence \( \{y^k\} \) converges to \( y^* \) by (10) for

\[
0 < \eta < 2r/\mu^2 \quad \text{and} \quad 0 < \eta < 2r/\mu^2.
\]

This concludes the proof. \(\square\)

**Theorem 3.2** ([4]). Let \( H \) be a real Hilbert space and \( K \) a nonempty closed convex subset of \( H \). Let \( T : K \to H \) be strongly \( r \)-monotonic and \( \mu \)-Lipschitz continuous. If \( x^*, y^* \in K \) form a solution to SNVI (1) and (2), if Algorithm 2.3 generates sequences \( \{x^k\} \) and \( \{y^k\} \), and if

\[
0 \leq a^k \leq 1 \quad \text{and} \quad \sum_{k=0}^{\infty} a^k = \infty,
\]

then sequences \( \{x^k\} \) and \( \{y^k\} \), respectively, converge to \( x^* \) and \( y^* \) for

\[
0 < \rho < 2r/\mu^2 \quad \text{and} \quad 0 < \eta < 2r/\mu^2.
\]

**Theorem 3.3.** Let \( H \) be a real Hilbert space and \( K \) be a nonempty closed convex subset of \( H \). Let \( T : K \to H \) be strongly \( r \)-monotonic and \( \mu \)-Lipschitz continuous. Suppose that \( x^* \in K \) is a solution to the NVI (5) problem, and the sequence \( \{x^k\} \) is generated by Algorithm 2.2. Then sequence \( \{x^k\} \) converges to \( x^* \) for

\[
0 < \rho < 2r/\mu^2.
\]

**References**


