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Regular polyhedra related to projective linear groups

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Abstract

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For each odd prime p, there is a regular polyhedron Π_p of type $\{3, p\}$ with $\frac{1}{2}(p^2-1)$ vertices whose rotation group is $\mathrm{PSL}(2, p)$; its complete group is $\mathrm{PSL}(2, p) \times Z_2$ or $\mathrm{PGL}(2, p)$ as $p \equiv 1$ or $3 \pmod 4$. If $p \equiv 1 \pmod 4$, then the group of Π_p contains a central involution, and identification of antipodal vertices under this involution yields another regular polyhedron $\Pi_p/2$ of type $\{3, p\}$ with $\frac{1}{4}(p^2-1)$ vertices and group $\mathrm{PSL}(2, p)$. Realizations of the polyhedra in euclidean spaces are briefly described.

1. Introduction

While many examples of regular polyhedra are known (see the tables in [3], and the later systematic investigations in [5–7]), most of them are sporadic, rather than falling into infinite families. In this paper, we shall describe two more closely related infinite families (see also Section 8).

With hindsight, the first of these families was (almost) available in the existing literature, in that the symmetry groups had been thoroughly described (see Section 2 below). What we shall do is give a model for the polyhedra, which establishes their existence and provides a combinatorial description. The second family is derived from the first by identifying antipodal vertices.

The discovery of these families resulted from the author's problems with finding geometric realizations of the Klein polyhedron $\{3, 7\}_8$. While good combinatorial descriptions were available (see, for example, [8, Fig. 4]), these turned out not to be suitable for our purpose. In constructing a new model for $\{3, 7\}_8$, we stumbled across a whole family.

Our attention has recently been drawn (by the referee and others) to [9–10], where some of the same polyhedra have been constructed. However, there the construction proceeds directly from the group, and there is no geometric model to provide extra geometric intuition.

2. The group

As is well known, and easy to prove, the matrices

$$R_0=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\qquad R_1=\begin{bmatrix}1 & 0\\ 1 & -1\end{bmatrix},\qquad R_2=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},$$

lying in the group $\mathrm{GL}(2,p)$ of all invertible 2×2 matrices over the field $\mathrm{GF}(p)$ with an odd prime number p of elements, generate the subgroup consisting of all such matrices with determinant ± 1. We give this subgroup the temporary notation $\mathrm{EL}(2,p)$ (where EL stands for equilinear). $\mathrm{EL}(2,p)$ has order $2p(p^2-1)$, and factoring out by the subgroup $H=\{\pm I\}$ ($I=I_2$ being the identity matrix), we obtain a group $\Gamma_p=\mathrm{EL}(2,p)/H$ of order $p(p^2-1)$, whose generators, corresponding to R_0, R_1 and R_2, we shall denote by the same letters. Indeed, it is convenient to write the elements of Γ_p as matrices, if we always bear in mind that we identity the matrices A and $-A$.

The group Γ_p has a structure which depends on the congruence class of p modulo 4. First, suppose that $p\equiv 1 \pmod 4$. Then -1 has a square root, κ say, in $\mathrm{GF}(p)$, and the element

$$K=\begin{bmatrix}\kappa & 0\\ 0 & \kappa\end{bmatrix}=\kappa I$$

in Γ_p satisfies $K^2=I$. Now, in $\mathrm{EL}(2,p)$, $\det K=-1$, and it is clear that every element in Γ_p can be written in the form $K^rA=AK^r$, with $r=0, 1$ and $A\in \mathrm{SL}(2,p)/H=\mathrm{PSL}(2,p)$. Thus $\Gamma_p\cong \mathrm{PSL}(2,p)\times Z_2$, with $Z_2=\langle K\rangle$.

In the other case $p\equiv 3 \pmod 4$, we observe that every nonzero element of $\mathrm{GF}(p)$ is of the form $\pm\lambda^2$ for some $\lambda\in\mathrm{GF}(p)$. Hence every element of $\mathrm{GL}(2,p)$ is expressible in the form

$$\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}A=A\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}$$

for some $A\in\mathrm{EL}(2,p)$, and it follows at once that $\Gamma_p\cong\mathrm{PGL}(2,p)$.

Relations for Γ_p in terms of the generators R_j are easily obtained. Indeed, each R_j is involutory:

$$R_j^2=I,\quad j=0, 1, 2,$$

and if we write

$$S=R_1R_2=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix},\qquad T=R_2R_0=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\ (=R_0R_2)$$

we have

$$S^p = T^2 = (ST)^3 = I.$$

In fact, S and T generate the subgroup PSL$(2, p)$ of Γ_p, and so comparing with [3, pp. 94–95], we see that the set of defining relations for Γ_p is completed by adjoining either of the extra relations

$$(S^4TS^{(p+1)/2}T)^2 = I,$$
$$(S^2TS^{(p+1)/2}T)^3 = I,$$

due to Sunday and Mennicke respectively.

Summarizing the above, we have the following theorem.

Theorem 1. *The group* $\Gamma_p = \mathrm{EL}(2, p)/\{\pm I\}$ *has generators* R_0, R_1, R_2, *with corresponding relations*

$$R_j^2 = (R_0R_1)^3 = (R_1R_2)^p = (R_0R_2)^2 = I$$

together with either of the two relations

$$((R_1R_2)^4R_0R_2(R_1R_2)^{(p+1)/2}R_0R_2)^2 = I,$$
$$((R_1R_2)^2R_0R_2(R_1R_2)^{(p+1)/2}R_0R_2)^3 = I.$$

Moreover, $\Gamma_p \cong \mathrm{PSL}(2, p) \times Z_2$ *or* $\mathrm{PGL}(2, p)$ *as* $p \equiv 1$ *or* $3 \pmod 4$.

We see from Theorem 1 that Γ_p is of a suitable form to be the (automorphism) group of a regular polyhedron of type $\{3, p\}$. Strictly speaking, we ought to verify that Γ_p satisfies the *intersection property* (see, for example, [5, Section 2]): if $I, J \subseteq \{0, 1, 2\}$, then $\langle R_i \mid i \in I\rangle \cap \langle R_i \mid i \in J\rangle = \langle R_i \mid i \in I \cap J\rangle$. (This is not an artificial condition, since it is the group-theoretic equivalent of desirable connectness properties of the lattice of faces of the polyhedron.) This can be proved directly, but in fact our geometric construction will make the intersection property obvious, since we can regard the vertices, edges and faces of our polyhedra as cells of simplicial complexes.

3. The polyhedron

The group EL$(2, p)$ acts transitively on the $p^2 - 1$ nonzero points of the affine plane over GF(p). If we identify the two points x and $-x$ ($x \neq 0$) of this plane, we obtain a set V of $\frac{1}{2}(p^2 - 1)$ points on which Γ_p acts transitively. We shall express the points of V as row vectors, writing x or $-x$ as convenient; Γ_p thus acts on points of V on the right.

The two involutions (or *reflexions,* as we shall usually call them) R_1 and R_2 in Γ_p leave fixed the point $(1, 0)$. This will be the initial vertex of our polyhedron, which we shall denote Π_p, to which we shall apply Wythoff's construction (see [2] or [4]).

Now $(1,0)R_0=(0,1)$, and the images of $(0,1)$ under the subgroup $\langle R_1, R_2\rangle$ (or under $\langle S\rangle$, with $S=R_1R_2$) are all points $(\alpha, 1)$ with $\alpha\in \mathrm{GF}(p)$; these are the p vertices adjacent to $(1,0)$.

Further, the remaining image of $(1,0)$ under the subgroup $\langle R_0, R_1\rangle$ is $(1,1)$, so the three vertices $(1,0)$, $(0,1)$ and $(1,1)$ give the initial triangular face of Π_p.

The complete structure of Π_p is now easily determined. Let $(\alpha, \beta)\in V$ be any vertex. The elements of Γ_p which take $(1,0)$ into (α, β) are those of the form

$$\pm\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}$$

with $\alpha\delta-\beta\gamma=\pm1$, and a typical such element takes $(0,1)$ into (γ, δ). Observing also that the two vertices adjacent to both $(1,0)$ and $(0,1)$ are $(1,1)$ and $(1,-1)\,(=(-1,1))$, we see that we have

Theorem 2. *There is an orientable regular polyhedron Π_p of type $\{3,p\}$ with $\frac{1}{2}(p^2-1)$ vertices and group Γ_p. The vertices of Π_p can be identified with V; then two vertices $x=(\alpha, \beta)$ and $y=(\gamma, \delta)$ determine an edge if and only if $\alpha\delta-\beta\gamma=\pm1$, and if they do, the remaining vertices of the two triangles containing that edge are $x\pm y$.*

The notation $\{3, p\}$ means, as usual, that the faces of Π_p are triangles, meeting p at each vertex. That Π_p is regular, meaning that its group Γ_p is transitive on its *flags*, or triples consisting of a face, an edge of that face, and a vertex of that edge, follows from the construction of Π_p above. Finally, the orientability of Π_p is a consequence of the fact that each of the relations in the presentation of Γ_p in Theorem 1 involves an even number of the generators R_j, so that the identifications required to obtain Π_p from its universal cover $\{3,p\}$ (the sphere for $p=3$ and 5, and the hyperbolic plane for $p\geqslant 7$) preserve orientation.

Let us expand somewhat the rather brief description of Π_p given in Theorem 2. We see that each of the $\frac{1}{2}(p-1)$ vertices $(\alpha, 0)=(-\alpha, 0)$ with $\alpha\in\mathrm{GF}(p)\setminus\{0\}$ is left fixed by $\langle R_1, R_2\rangle$. The vertices adjacent to $(\alpha, 0)$ are all (γ, α^{-1}), with $\gamma\in\mathrm{GF}(p)$. The $\frac{1}{2}(p^2-1)$ vertices of Π_p thus fall into $\frac{1}{2}(p-1)$ *patches*, each consisting of a central vertex $(\alpha, 0)$ and its p adjacent vertices forming the p triangles which surround it. These disjoint patches are then joined by a *web* of intermediate triangles (except in case $p=3$, when the web reduces to a single triangle).

If we were not able to do so already, we can now easily identify the first few polyhedra Π_p. The correspondence of symmetry groups merely serves to confirm our identifications.

Π_3 is the regular tetrahedron $\{3,3\}$.
Π_5 is the regular icosahedron $\{3,5\}$.
Π_7 is the Klein polyhedron $\{3,7\}_8$

(the significance of the suffix is explained below; see also [1] or [3]).

4. The Petrie polygon

Of considerable interest for any regular polyhedron is the length of its Petrie polygon, each two successive edges, but no three, of which are edges of a face. One reason for this interest is that the Petrie polygons, which are r-gons, say, of a regular polyhedron of type $\{p, q\}$ themselves form the faces of a regular polyhedron of type $\{r, q\}$ (the *Petrie dual*), whose Petrie polygons are the p-gonal faces of the original polyhedron. With ordinary duality (taking a $\{p, q\}$ into a $\{q, p\}$) as well, Petrie duality enables up to five regular polyhedra to be associated with a given one.

If the Petrie polygons of Π_p are h-gons, then h is the order (period) of

$$R_0R_1R_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

in Γ_p. Writing f_n for the nth Fibonacci number, with $f_0 = 0$ and $f_1 = 1$ (the sequence can be regarded as extending to negative n), an easy induction argument shows that

$$(R_0R_1R_2)^n = \begin{bmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{bmatrix}.$$

Since $R_0R_1R_2$ has determinant -1, and hence reverses orientation, we see that

$$f_{n-1}f_{n+1} - f_n^2 = (-1)^n,$$

and we thus deduce

Theorem 3. *The length of the Petrie polygon of Π_p is the smallest even integer h such that $f_h \equiv 0 \pmod p$.*

The Petrie polygon is a regular h-gon, whose group order $2h$ must divide the order $p(p^2-1)$ of the group Γ_p of Π_p. In fact, it is a known property of Fibonacci numbers that, with one sole exception ($p = 5$, $h = 10$), h is always a divisor of $p-1$ or $p+1$ (see, for example, [1]).

For the notation $\{3, 7\}_8$ used above, we recall (see [1, 3]) that a regular polyhedron of type $\{p, q\}$ is denoted $\{p, q\}_r$ if it is determined by the length r of its Petrie polygon. Its Petrie dual is then $\{r, q\}_p$ (although in a few cases the suffix p is redundant).

5. Isomorphic polyhedra with the same vertices

As is well known, the regular star (Kepler–Poinsot) polyhedron $\{3, \frac{5}{2}\}$—the great icosahedron—is isomorphic to the icosahedron $\{3, 5\}$, and has the same vertices. An analogous phenomenon occurs for all our polyhedra $\{3, p\}$ (except for $p = 3$), as we shall now see.

As usual, we take $(1,0)$ as our initial vertex. For any $\nu \neq 0$, the vertices (α, ν) $(\alpha \in \mathrm{GF}(p))$ are permuted among themselves by the reflexions R_1 and R_2 which leave $(1,0)$ fixed. So, ignoring the vertices $(\gamma, 0)$, and recalling that we can interchange ν and $-\nu$, we see that there are $\frac{1}{2}(p-1)$ possible choices (including the original choice $\nu = 1$) for the vertex figure of a regular polyhedron $\{3, p\}$ isomorphic to our original one.

We could work out directly which other pairs of vertices x and y were now to be joined by edges (equivalent to that joining $(1,0)$ and $(0,\nu)$, say), but there is an easier method. Write

$$N = \begin{bmatrix} 1 & 0 \\ 0 & \nu \end{bmatrix}.$$

Now $N \notin \mathrm{EL}(2,p)$ unless $\nu = \pm 1$. However, we can let N act on $\mathrm{EL}(2,p)$, and hence on Γ_p, by conjugation, to yield an automorphism of Γ_p. We then obtain new generating reflexions $N^{-1}R_jN$ $(j = 0, 1, 2)$, and a new regular polyhedron, whose vertex corresponding to the vertex x of Π_p is xN. In particular, $(1,0)$ corresponds to itself, but $(\alpha, 1)$ is taken to (α, ν), as required. The new rule for joining vertices (α, β) and (γ, δ) by edges is $\alpha\delta - \beta\gamma = \pm\nu$.

However, there is another way of obtaining such isomorphic polyhedra, this time using the same generating reflexions R_0, R_1, R_2. For we have seen that each vertex $(\mu, 0)$ is also fixed by R_1, R_2, so we can use $(\mu, 0)$ instead of $(1,0)$ as our initial vertex. Geometrically, we can think of this as conjugating Γ_p (trivially) by

$$M = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix},$$

so that the vertex x of $\{3, p\}$ is taken into $xM = \mu x$. For an obvious reason, we shall call this kind of operation on $\{3, p\}$ a *dilatation*.

In fact, these two constructions are not as different as they might at first sight appear. Consider N as above. If, first, $p \equiv 3 \pmod 4$, then $\nu = \pm\mu^2$ for some $\mu \in \mathrm{GF}(p)$, $\mu \neq 0$. We can then write

$$N = \begin{bmatrix} \mu^{-1} & 0 \\ 0 & \pm\mu \end{bmatrix} M,$$

with M as above; the first term of the product already lies in Γ_p, and so acts as an inner automorphism. Thus we have

Theorem 4. *If $p \equiv 3 \pmod 4$, then the $\frac{1}{2}(p-1)$ isomorphic polyhedra with the same vertices as Π_p are all obtained by dilatation.*

On the other hand, if $p \equiv 1 \pmod 4$, then -1 is a square in $\mathrm{GF}(p)$, and so the above trick will not work. Let λ be a fixed nonsquare in $\mathrm{GF}(p)$. Then we can always express a nonzero $\nu \in \mathrm{GF}(p)$ in the form

$$\nu = \mu^2 \quad \text{or} \quad \nu = \mu^2\lambda,$$

and hence N is of the form

$$N = \begin{bmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{bmatrix} M \quad \text{or} \quad \begin{bmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{bmatrix} LM,$$

where

$$L = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix},$$

and M is again as above. Hence the following theorem can be stated.

Theorem 5. *If $p \equiv 1 \pmod 4$, then the isomorphic polyhedra with the same vertices as Π_p fall into two classes of $\frac{1}{4}(p-1)$ each under dilatation.*

6. Another class of regular polyhedra

We saw in Section 2 that, if $p \equiv 1 \pmod 4$, then the group Γ_p contains a central involution

$$K = \begin{bmatrix} \kappa & 0 \\ 0 & \kappa \end{bmatrix},$$

where $\kappa^2 = -1$. This acts as a central symmetry on Π_p. We now obtain a further class of polyhedra of type $\{3, p\}$, which we shall denote here by $\Pi_p/2$, by identifying corresponding opposite vertices, and factoring out Γ_p by K. Since $\det K = -1$, we see that K reverses orientation, and so

Theorem 6. *If p is a prime satisfying $p \equiv 1 \pmod 4$, then there is a non-orientable regular polyhedron $\Pi_p/2$ of type $\{3, p\}$ with $\frac{1}{4}(p^2-1)$ vertices and complete symmetry group* $\mathrm{PSL}(2, p)$.

Again, we can identify the first two examples.

$\Pi_5/2$ is the hemi-icosahedron $\{3, 5\}_5$.

$\Pi_{13}/2$ is the regular polyhedron $\{3, 13\}_7$.

For most of the first few examples (six exceptions for $p \leqslant 241$, the first being $p = 29$, $h = 14$), the involution K is a power of the element $R_0R_1R_2$, giving the rotation of a Petrie polygon (of length h); in that case, $K = (R_0R_1R_2)^{h/2}$. The circumstances governing this are easily explained.

Theorem 7. *Let the prime p satisfy $p \equiv 1 \pmod 4$. If the first number r such that $f_r \equiv 0 \pmod p$ is odd, then the Petrie polyhedron of $\Pi_p/2$ has half the length of that of Π_p. Otherwise, their Petrie polygons have the same length.*

If $f_r \equiv 0 \pmod p$, then $f_{r-1} \equiv f_{r+1} \pmod p$ (and conversely), so that

$$(R_0R_1R_2)^r = \begin{bmatrix} f_{r-1} & 0 \\ 0 & f_{r+1} \end{bmatrix}$$

is diagonal. But $f_{r-1}f_{r+1} \equiv (-1)^r \pmod p$ shows that in this case

$$(R_0R_1R_2)^r = K^r,$$

from which Theorem 7 follows.

As with the polyhedra Π_p, we shall have isomorphic regular polyhedra with the same vertices as $\Pi_p/2$. Indeed, the same analysis as that of Section 5 carries over, except that we must set the dilatation K equal to the identity. But this has the effect of identifying the two polyhedra, in which $(1, 0)$ is joined by an edge to $(0, v)$ and $(0, \kappa v)$ $(v \neq 0)$, respectively. We now see that the crucial point here is whether or not κ itself is a square. Now, if we let θ be a multiplicative generator of the cyclic group $\mathrm{GF}(p)\setminus\{0\}$, we can take

$$\kappa = \theta^{(p-1)/4},$$

so that κ is a square precisely when $\frac{1}{4}(p-1)$ is even. Thus the following theorem can be stated.

Theorem 8. *Let p be a prime with $p \equiv 1 \pmod 4$. Then the $\frac{1}{4}(p-1)$ isomorphic polyhedra with the same vertices as $\Pi_p/2$ fall into two classes under dilatation if $p \equiv 1 \pmod 8$, and into a single class if $p \equiv 5 \pmod 8$.*

7. Realizations

So far, we have treated our new regular polyhedra purely combinatorially. But much of the attraction of the theory of regular polytopes comes from their high degree of symmetry in some euclidean space. It is that aspect which we shall now discuss.

The theory of realizations of regular polytopes is treated in [4], and we have no intention of repeating all the details here. What we do need follows.

A *realization* of Π_p is given by a mapping of the set of its vertices into some euclidean space E (which determines the corresponding edges and faces), such that each of its symmetries is induced by an isometry of E. Since Π_p is finite, we may always take these isometries to be orthogonal mappings. (Note that we do not insist that the realization mapping be one to one, so that distinct vertices of Π_p may coincide in the realization.)

If $(u_1, u_2, \ldots) \subseteq L$, $(v_1, v_2, \ldots) \subseteq M$ are corresponding vertices of realizations of Π_p, and if λ is a real number, then $(w_1, w_2, \ldots)$ with $w_i = (u_i, v_i) \in L \times M$ or $\lambda u_i \in L$ are also vertices of realizations, and under these operations of *blending* and *scalar multiplication* respectively, the space of classes under congruence of realizations takes on the structure of a closed convex cone, whose dimension is the number of equivalence classes under its automorphism group of *diagonals,* or pairs of distinct vertices. (See below for more details.)

A realization is also determined by an orthogonal representation G of Γ_p, together with an initial choice of a vertex in the Wythoff space W_G, which is the space of fixed points of the subgroup G^* of G corresponding to $\Gamma^* = \langle R_1, R_2 \rangle$. Of most interest are the irreducible representations G; for these, we write w_G for the dimension of W_G, and d_G for the degree of G (dimension of the space on which G acts).

We reproduce without proof the various relationships between these numbers, and the numbers of cosets and double cosets of Γ^* in Γ_p.

Theorem 9. *In the following sums G ranges over the irreducible orthogonal representations of Γ_p:*

(a) $\sum_G w_G d_G = \operatorname{card}\{\Gamma^*\sigma \mid \sigma \in \Gamma_p \setminus \Gamma^*\}$;

(b) $\sum_G \frac{1}{2} w_G(w_G + 1) = \operatorname{card}\{\Gamma^*\sigma\Gamma^* \cup \Gamma^*\sigma^{-1}\Gamma^* \mid \sigma \in \Gamma_p \setminus \Gamma^*\}$;

(c) $\sum_G w_G^2 = \operatorname{card}\{\Gamma^*\sigma\Gamma^* \mid \sigma \in \Gamma_p \setminus \Gamma^*\}$.

In fact, the first sum is one less than the number of vertices (namely, $\frac{1}{2}(p^2 - 3)$), the second is the dimension of the cone of all realizations, and the third is the dimension of the Wythoff space of the simplex realization, where the $\frac{1}{2}(p^2 - 1)$ vertices are those of a regular simplex.

We shall now sketch the details of the pure (unblended) realizations, which are associated with the irreducible representations G for which $w_G > 0$. The crucial observation is that the $p + 1$ disjoint sets of vertices $\{\lambda x \mid \lambda \in \mathrm{GF}(p) \setminus \{0\}\}$, with x a given vertex, must form the vertices of regular $\frac{1}{2}(p - 1)$-gons, possibly skew, and possibly degenerate. From this, it is a short step to concluding that we need only consider the components under blending in the case where these polygons are planar and lie in orthogonal complementary subspaces.

In every case, the polygon can degenerate to a point, and the vertices collapse to those of a regular p-simplex, giving

$$w_G = 1, \qquad d_G = p.$$

In case $p \equiv 1 \pmod 4$, the polygon can degenerate to a line segment; the corresponding realization has two pure components, each with

$$w_G = 1, \qquad d_G = \tfrac{1}{2}(p + 1).$$

Finally, for each k with $1 \leqslant k < \frac{1}{4}(p - 1)$, the polygons are of type $\{\frac{1}{2}(p - 1)/k\}$ (degenerate if $(\frac{1}{2}(p - 1), k) > 1$); the corresponding pure realizations form a single family, with

$$w_G = 2, \qquad d_G = p + 1.$$

The faithful realizations, for which the $\frac{1}{2}(p^2 - 1)$ vertices are distinct, are obviously of particular interest. For $p=3$, we just have the tetrahedron itself, and for $p=5$ we have the icosohedron and great icosahedron in three dimensional space. For $p \geqslant 7$, the faithful realizations all come from the last cases discussed, with $(\frac{1}{2}(p-1), k)=1$. Thus, for example, $\{3,7\}_8$ has a single 2-parameter family of faithful realizations in E^8.

Since, for $p \equiv 1 \pmod 4$, every realization of $\Pi_p/2$ is also a realization of Π_p, we can find these realizations in the list above. The first case always occurs, as does the second when $p \equiv 1 \pmod 8$, while the cases of the third that are relevant are those for which k is even. So, for example, $\{3,13\}_7$ has a single 2-parameter family of faithful realizations in E^{14}.

8. Final remark

While this paper was reaching the end of its final draft, we realized that the matrices R_0, R_1 and R_2 generate a group, if their elements are regarded as members of the ring $\mathbb{Z}_q$ of integers modulo q, for any natural number $q \geqslant 3$. We can also factor out by the subgroup $H=\{\pm I\}$ as before, to obtain the automorphism group of a regular polyhedron Π_q of type $\{3, q\}$ with

$$\tfrac{1}{2}q^2 \prod_{p|q}\left(1-\frac{1}{p^2}\right)$$

vertices (the product extends over the distinct prime divisors p of q). We propose to discuss these polyhedra in more detail in a future note.

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