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Generalising Generic Differentiability Properties from Convex to Locally Lipschitz Functions

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David Preiss proved that every locally Lipschitz function on an open subset of a Banach space which has an equivalent norm Gâteaux (Fréchet) differentiable away from the origin is Gâteaux (Fréchet) differentiable on a dense subset of its domain. It is known that every continuous convex function on an open convex subset of such a space is Gâteaux (Fréchet) differentiable on a residual subset of its domain. We show that for a locally Lipschitz function on a separable Banach space (with separable dual) there are residual subsets which if the function were convex would coincide with its set of points of differentiability. These are the sets where the function is fully intermediately differentiable (fully and uniformly intermediately differentiable) and sets where the subdifferential mapping is weak* (norm) lower semi-continuous. We discuss the role of these sets in generating the subdifferential and present a refinement of Preiss' result. © 1994 Academic Press, Inc.

0. INTRODUCTION

In generalising differentiability theory from convex to locally Lipschitz functions on real Banach spaces the principal task is to discern particular differentiability properties for locally Lipschitz functions which have long been considered equivalent for convex functions.

A real valued function ϕ on an open convex subset A of a normed linear space X is convex if

$$\phi(\lambda x + (1 - \lambda)y) \leq \lambda\phi(x) + (1 - \lambda)\phi(y)$$
$$\text{for all } x, y \in A \text{ and } 0 \leq \lambda \leq 1.$$

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A real valued function ψ on an open subset A of a normed linear space X is *locally Lipschitz* if for each $x_0 \in A$ there exists a $K_0 > 0$ and a $\delta_0 > 0$ such that

$$|\psi(x) - \psi(y)| \leq K_0\|x - y\| \qquad \text{for all } x, y \in B(x_0; \delta_0).$$

A continuous convex function is locally Lipschitz.

For a continuous convex function ϕ, the *right-hand derivative*

$$\phi'_+(x)(y) \equiv \lim_{\lambda \to 0+} \frac{\phi(x + \lambda y) - \phi(x)}{\lambda}$$

always exists at every $x \in A$ for all $y \in X$, and at each $x \in A$, $\phi'_+(x)(y)$ is a continuous sublinear function in y. For a locally Lipschitz function ψ, the *generalised Clarke derivative* is

$$\psi^0(x)(y) \equiv \limsup_{\substack{z \to x \\ \lambda \to 0+}} \frac{\psi(z + \lambda y) - \psi(z)}{\lambda}$$

and at each $x \in A$, $\psi^0(x)(y)$ is a continuous sublinear function in y.

For a continuous convex function ϕ, $\phi'_+(x)(y) = \phi^0(x)(y)$ at every $x \in A$ for all $y \in X$.

When studying the differentiability of a continuous convex function ϕ we consider the subdifferential

$$\partial\phi(x) \equiv \{f \in X^* : f(y) \leq \phi'_+(x)(y) \text{ for all } y \in X\}$$

which for each $x \in A$ is a non-empty, weak* compact convex set. For a locally Lipschitz function ψ we consider the *generalised Clarke subdifferential*

$$\partial\psi^0(x) \equiv \{f \in X^* : f(y) \leq \psi^0(x)(y) \text{ for all } y \in X\}$$

which for each $x \in A$ is a non-empty, weak* compact convex set. Clearly, for a continuous convex function ϕ, $\partial\phi(x) = \partial\phi^0(x)$ at every $x \in A$.

A real valued function ψ on an open subset A of a normed linear space X is said to be *Gâteaux differentiable* at $x \in A$ if

$$\psi'(x)(y) \equiv \lim_{\lambda \to 0} \frac{\psi(x + \lambda y) - \psi(x)}{\lambda}$$

exists for all $y \in X$ where $\psi'(x)$ is a continuous linear functional on X, and is said to be *Fréchet differentiable* at x if this limit is approached uniformly for all $y \in X$ where $\|y\| = 1$.

Further, ψ is said to be *strictly differentiable* at $x \in A$ if ψ is Gâteaux differentiable at x and

$$\lim_{\substack{z \to x \\ \lambda \to 0+}} \frac{\psi(z + \lambda y) - \psi(z)}{\lambda} = \psi'(x)(y)$$

for all $y \in X$, and is said to be *uniformly strictly differentiable* at x if this limit is approached uniformly for all $y \in X$ where $\|y\| = 1$.

For convex functions, Gâteaux differentiability and strict differentiability are equivalent and Fréchet differentiability and uniformly strict differentiability are equivalent but for locally Lipschitz functions they are distinct properties.

There are two major theorems on the differentiability of continuous convex functions.

THEOREM 0.1. (Preiss *et al.* [11]). *A continuous convex function on an open convex subset of a Banach space which has an equivalent norm Gâteaux differentiable away from the origin is Gâteaux differentiable on a dense* G_δ *subset of its domain.*

THEOREM 0.2. [9, p. 24]. *A continuous convex function on an open convex subset of a Banach space where every separable subspace has separable dual is Fréchet differentiable on a dense* G_δ *subset of its domain.*

Banach spaces where every continuous convex function on an open subset is Fréchet differentiable on a dense G_δ subset of its domain are called *Asplund spaces* and are characteristed by the property that every separable subspace has separable dual.

Corresponding deep results have been established for locally Lipschitz functions.

THEOREM 0.3. (Preiss, [10]). *A locally Lipschitz function on an open subset of*

(i) *a Banach space which has an equivalent norm Gâteaux differentiable away from the origin is Gâteaux differentiable on a dense subset of its domain,*

(ii) *an Asplund space is Fréchet differentiable on a dense subset of its domain.*

When we compare the locally Lipschitz function theorem with the continuous convex function theorems we notice that we have lost the residual property of the set of points of differentiability. We show that for each locally Lipschitz function on a separable Banach space there is a dense G_δ subset of its domain which if the function were convex would coincide with its set of points of differentiability.

1. Fully Intermediate Differentiability

Appropriate generalisations of the derivative for locally Lipschitz functions are defined by means of the Dini derivative.

For a locally Lipschitz function ψ on an open subset A of a normed linear space X, the *upper Dini derivative* at $x \in A$ in the direction $y \in X$ is

$$\psi^+(x)(y) \equiv \limsup_{\lambda \to 0+} \frac{\psi(x + \lambda y) - \psi(x)}{\lambda}$$

and the *lower Dini derivative* at $x \in A$ in the direction $y \in X$ is

$$\psi^-(x)(y) \equiv \liminf_{\lambda \to 0+} \frac{\psi(x + \lambda y) - \psi(x)}{\lambda}.$$

Clearly, $\psi^-(x)(y) = -(-\psi)^+(x)(y)$ and ψ has a right-hand derivative $\psi'_+(x)(y)$ if and only if $\psi^+(x)(y) = \psi^-(x)(y)$. We notice that for a continuous convex function the Dini derivatives coincide with the right-hand derivative.

We say that ψ is *pseudo-regular* at $x \in A$ if $\psi^+(x)(y) = \psi^0(x)(y)$ for all $y \in X$. Again a continuous convex function is pseudo-regular on its domain.

We say that ψ is *intermediately differentiable* at $x \in A$ if there exists a continuous linear functional f on X such that

$$\psi^-(x)(y) \le f(y) \le \psi^+(x)(y) \qquad \text{for all } y \in X.$$

But further, we say that ψ is *fully intermediately differentiable* at $x \in A$ if both ψ and $(-\psi)$ are pseudo-regular at x. At such a point $x \in A$, $\partial\psi^0(x)$ is the set of all the intermediate derivatives of ψ at x.

We can characterise various differentiability conditions by continuity properties of the Dini derivatives.

THEOREM 1.1. *Given a locally Lipschitz function ψ on an open subset A of a normed linear space X*

(i) *ψ is pseudo-regular at $x \in A$ if and only if $\psi^+(x)(y)$ is upper semi-continuous at x for all $y \in X$,*

(ii) *ψ is fully intermediately differentiable at $x \in A$ if and only if $\psi^+(x)(y)$ is upper semi-continuous and $\psi^-(x)(y)$ is lower semi-continuous at x for all $y \in X$,*

(iii) *ψ is strictly differentiable at $x \in A$ if and only if $\psi^+(x)(y)$ is continuous at x for all $y \in X$,*

(iv) *ψ is uniformly strictly differentiable at $x \in A$ if and only if $\psi^+(x)(y)$ is continuous at x uniformly for all $y \in X$, $\|y\| = 1$.*

Proof. (i) For any given $y \in X$, it is clear that

$$\psi^0(x)(y) \geq \limsup_{z \to x} \psi^+(z)(y).$$

Given $\varepsilon > 0$, in any neighborhood of x there exists $z_0 \in A$ and $z_0 + \lambda_0 y$ where $\lambda_0 > 0$ such that

$$\psi^0(x)(y) - \varepsilon < \frac{\psi(z_0 + \lambda_0 y) - \psi(z_0)}{\lambda_0}.$$

Consider ψ restricted to the interval $[z_0, z_0 + \lambda_0 y]$. Since ψ is locally Lipschitz it follows from Lebesgue's Differentiation Theorem that there exists a $0 \leq \lambda_1 \leq \lambda_0$ such that

$$\psi'(z_0 + \lambda_1 y)(y) \geq \frac{\psi(z_0 + \lambda_0 y) - \psi(z_0)}{\lambda_0}.$$

So

$$\limsup_{z \to x} \psi^+(z)y) \geq \psi^0(x)(y)$$

and

$$\psi^0(x)(y) = \limsup_{x \to x} \psi^+(z)(y).$$

It follows that ψ is pseudo-regular at x if and only if $\psi^+(x)(y)$ is upper semi-continuous at x for all $y \in X$.

(ii) follows from (i) and from the fact that for all $x \in A$ and $y \in X$

$$\psi^-(x)(y) = -(-\psi)^+(x)(y).$$

(iii) If ψ is strictly differentiable at $x \in A$, given $\varepsilon > 0$ and $y \in X$ there exists a $\delta(\varepsilon, y) > 0$ such that

$$\left| \frac{\psi(z + \lambda y) - \psi(z)}{\lambda} - \psi'(x)(y) \right| < \varepsilon \qquad \text{for } \|z - x\| < \delta \text{ and } 0 < \lambda < \delta.$$

Then $|\psi^+(z)(y) - \psi'(x)(y)| \leq \varepsilon$ for $\|z - x\| < \delta$.

Conversely, if $\psi^+(x)(y)$ is continuous at x for all $y \in X$ then from (i),

$$\psi^+(x)(y) = \psi^0(x)(y) \qquad \text{for all } y \in X.$$

Consider ψ restricted to a one-dimenstional affine subspace M through x generated by y. Now $\psi|_M$ is locally Lipschitz and by Lebesgue's Differentiation Theorem is Gâteaux differentiable almost everywhere in M. However, since $\psi^+(x)(y)$ is continuous at x, then $\psi^+(x_n)(y)$ is convergent to $\psi^+(x)(y)$ as $x_n \to x$ in M for points x_n where ψ is Gâteaux differentiable on M. As the pointwise limit of linear functionals, $\psi^+(x)(y)$ is also linear in y on M. Since $\psi^+(x)(y) = \psi^0(x)(y)$ for all $y \in X$, which is sublinear in y, we deduce that $\psi^0(x)(y)$ is linear in y and so ψ is strictly differentiable at x.

(iv) If ψ is uniformly strictly differentiable at $x \in A$, then we can uniformise the agrument in (iii) for all $y \in X$, $\|y\| = 1$ to give the continuity property for the Dini derivative.

Conversely, suppose that ψ is strictly differentiable at $x \in A$ but not uniformly strictly differentiable at x. Then there exists an $r > 0$, $z_n \to x$, $\lambda_n \to 0+$, and $y_n \in X$, $\|y_n\| = 1$ such that

$$\left| \frac{\psi(z_n + \lambda_n y_n) - \psi(z_n)}{\lambda_n} - \psi'(x)(y_n) \right| > r.$$

But then by Lebesgue's Differentiation Theorem there exists a v_n in the interval $[z_n, z_n + \lambda_n y_n]$ such that $|\psi^+(v_n)(y_n) - \psi'(x)(y_n)| \geq r$. That is, $\psi^+(x)(y)$ even though continuous at x is not uniformly so for all $y \in X$, $\|y\| = 1$. ∎

If ψ is Gâteaux differentiable at $x \in A$ then it is intermediately differentiable at x but it is not necessarily fully intermediately differentiable at x. If ψ is strictly differentiable at $x \in A$ then it is fully intermediately differentiable at x.

A continuous convex function is Gâteaux differentiable if and only if it is intermediately differentiable.

For a locally Lipschitz function ψ on an open subset A of a normed linear space X we define approximate upper Dini derivatives which possess desirable continuity properties. Given $x \in A$ and $y \in X$ and $p \in \mathbb{N}$ we write

$$\psi_p^+(x)(y) \equiv \sup_{0<\lambda<1/p} \frac{\psi(x+\lambda y)-\psi(x)}{\lambda}$$

and observe that $\psi^+(x)(y) = \lim_{p\to\infty} \psi_p^+(x)(y)$.

We note that, given $p \in \mathbb{N}$, for $x \in A$, $\psi_p^+(x)(y)$ and $\psi^+(x)(y)$ are continuous functions in y and given $y \in X$, as the supremum of continuous functions, $\psi_p^+(x)(y)$ is lower semi-continuous in x.

The following theorem identifies, for a locally Lipschitz function on a separable Banach space, significant residual subsets of its domain.

THEOREM 1.2. *A locally Lipschitz function ψ on an open subset A of a separable Banach space is*

(i) *pseudo-regular on a dense G_δ subset of A,*

(ii) *fully intermediately differentiable on a dense G_δ subset of A.*

Proof. (i) Given $y \in X$ and $p \in \mathbb{N}$, we have that $\psi_p^+(x)(y)$ is lower semi-continuous on A. So there exists a dense G_δ subset of D_y of A where $\psi_p^+(x)(y)$ is continuous at each $x \in D_y$ for every $p \in \mathbb{N}$. Given $\varepsilon > 0$ and $x \in D_y$ there exists a $p \in \mathbb{N}$ such that

$$\psi_p^+(x)(y) - \psi^+(x)(y) < \frac{\varepsilon}{2}$$

and there exists a $\delta > 0$ such that

$$\psi_p^+(x)(y) + \frac{\varepsilon}{2} > \psi_p^+(z)(y) \qquad \text{for all } z \in A \text{ and } \|z-x\| < \delta.$$

Then

$$\begin{aligned} \psi^+(x)(y) + \varepsilon &> \psi_p^+(z)(y) \\ &\geq \psi^+(z)(y) \qquad \text{for all } z \in A \text{ and } \|z-x\| < \delta. \end{aligned}$$

As is Theorem 1.1(i), $\psi^+(x)(y) = \psi^0(x)(y)$.

Since X is separable there exists a countable dense set $\{y_n\}$ in X and therefore a dense G_δ subset $D \equiv \bigcap_1^\infty D_{y_n}$ of A where for each $x \in D$,

$$\psi^+(x)(y_n) = \psi^0(x)(y_n) \qquad \text{for all } n \in \mathbb{N}.$$

But as both $\psi^+(x)(y)$ and $\psi^0(x)(y)$ are continuous in y we conclude that for every $x \in D$,

$$\psi^+(x)(y) = \psi^0(x)(y) \qquad \text{for all } y \in X.$$

(ii) follows from (i). ∎

The proof of (i) is somewhat simpler than that given in [7, Theorem].

The following is an obvious equivalent formulation of intermediate differentiability.

LEMMA 1.3. *A locally Lipschitz function ψ on an open subset A of a normed linear space X is intermediately differentiable at $x \in A$ if and only if there exists a continuous linear functional f on X and for every $y \in X$ there exists a sequence $\lambda_n \to 0+$ such that*

$$\lim_{n\to\infty} \frac{\psi(x + \lambda_n y) - \psi(x)}{\lambda_n} = f(y).$$

We now consider a generalisation of Fréchet differentiability for locally Lipschitz functions.

A locally Lipschitz function ψ on an open subset A of a normed linear space X is said to be *uniformly intermediately differentiable* at $x \in A$ if there exists a continuous linear functional f on X and a sequence $\lambda_n \to 0+$ such that

$$\lim_{n\to\infty} \frac{\psi(x + \lambda_n y) - \psi(x)}{\lambda_n} = f(y) \qquad \text{for all } y \in X, \|y\| = 1.$$

The same sequence is used for all $y \in X$, $\|y\| = 1$, but the rate of convergence to the limit $f(y)$ need not be uniform.

If ψ is Fréchet differentiable at $x \in A$ then it is uniformly intermediately differentiable but it is not necessarily fully intermediately differentiable at x. If ψ is Fréchet differentiable and strictly differentiable at $x \in A$, then it is fully intermediately differentiable at x, but as the example given in [6, p. 373] shows it is not necessarily uniformly strictly differentiable at x. If ψ is uniformly strictly differentiable at $x \in A$ then it is fully and uniformly intermediately differentiable at x.

A continuous convex function is Fréchet differentiable if and only if it is uniformly intermediately differentiable.

We now work towards a generic extension of Preiss' Theorem 0.3(ii).

THEOREM 1.4. *A locally Lipschitz function ψ on an open subset A of an Asplund space X is uniformly intermediately differentiable on a dense G_δ subset of A.*

Proof. Given $\varepsilon > 0$ consider the set

$$O_\varepsilon \equiv \bigcup \left\{ \text{open sets } G\text{: there exists an } f \in X^* \text{ and a } \delta > 0 \text{ such that} \right.$$

$$\sup_{\delta/2<\lambda<\delta} \left| \frac{\psi(x + \lambda y) - \psi(x)}{\lambda} - f(y) \right| < \varepsilon \text{ for all}$$

$$\left. x \in G \text{ and for all } y \in X, \|y\| = 1 \right\}.$$

Now O_ε is open. We show that O_ε contains the set of points where ψ is Fréchet differentiable. Suppose ψ is Fréchet differentiable at $x \in A$. Now there exists a $K > 0$ and a $\delta_0 > 0$ such that $|\psi(z_1) - \psi(z_2)| \leq K\|z_1 - z_2\|$ for all $z_1, z_2 \leftarrow B(x; \delta_0)$. There exists a $0 < \delta < \delta_0/2$ such that

$$\sup_{0<\lambda<\delta} \left| \frac{\psi(x + \lambda y) - \psi(x)}{\lambda} - \psi'(x)(y) \right| < \frac{\varepsilon}{2} \qquad \text{for all } y \in X, \|y\| = 1$$

so

$$\sup_{\delta/2<\lambda<\delta} \left| \frac{\psi(x + \lambda y) - \psi(x)}{\lambda} - \psi'(x)(y) \right| < \frac{\varepsilon}{2}$$

and

$$\sup_{\delta/2<\lambda<\delta} \left| \frac{\psi(z + \lambda y) - \psi(x)}{\lambda} - \psi'(x)(y) \right| < \frac{\varepsilon}{2} + \frac{4K}{\delta}\|z - x\|$$

$$< \varepsilon \qquad \text{for } \|z - x\| < \min\left(\frac{\varepsilon\delta}{8K}, \delta\right),$$

So $B(x; \min(\varepsilon\delta/8K, \delta)) \subseteq O_\varepsilon$. Since X is Asplund, we have from Preiss' Theorem 0.3(ii) that O_ε is dense and so $\bigcap_{\varepsilon>0} O_\varepsilon$ is a dense G_δ subset of A.

Now if $x \in \bigcap_{\varepsilon>0} O_\varepsilon$ then $x \in O_{1/n}$ for all $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ there exists an $f_n \in X^*$ and $\delta_n > 0$ and we can choose $\delta_n/2 < r_n < \delta_n$ such that

$$\left| \frac{\psi(x + r_n y) - \psi(x)}{r_n} - f_n(y) \right| < \frac{1}{n} \qquad \text{for all } y \in X, \|y\| = 1..$$

Since $\{f_n\}$ is bounded and X is an Asplund space, $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ weak* convergent to some $f \in X^*$. Then

$$\lim_{k\to\infty} \frac{\psi(x + r_{n_k} y) - \psi(x)}{r_{n_k}} = f(y) \qquad \text{for all } y \in X, \|y\| = 1;$$

that is, ψ is uniformly intermediately differentiable at x. ∎

We have a result which follows directly from Theorem 1.4 and which corresponds to Theorem 1.2(ii) for Banach spaces with separable dual.

THEOREM 1.5. *A locally Lipschitz function ψ on an open subset A of a Banach space with separable dual is fully and uniformly intermediately differentiable on a dense G_δ subset of A.*

M. Fabian and D. Preiss [5] actually showed that for a large class of Banach spaces which includes the Asplund spaces, a locally Lipschitz function on an open subset of such a space is intermediately differentiable on a residual subset of its domain. For fully intermediate differentiability, our proof in Theorem 1.2 applies only for the class of separable Banach spaces.

We note that on a finite-dimensional normed linear space although a locally Lipschitz function is Fréchet differentiable wherever it is Gâteaux differentiable and uniformly strictly differentiable whenever it is strictly differentiable [6, p. 379], nevertheless the associated uniformity conditions do not hold automatically for intermediate differentiability.

EXAMPLES 1.6. (i) Consider the locally Lipschitz function ψ on $\mathbb{R}$ defined by

$$\psi(x) = \begin{cases} x \sin\left(\dfrac{x \ln|x|}{|x|}\right), & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Now $\partial\psi^0(0) = [-1, 1]$ and ψ is fully intermediately differentiable at 0, but it has a unique uniformly intermediate derivative of zero at 0.

(ii) The locally Lipschitz function ψ_1 on $\mathbb{R}$ defined by

$$\psi_1(x) = \begin{cases} x \sin(\ln|x|), & x > 0 \\ 0 & x = 0 \\ x \cos(\ln|x|), & x < 0 \end{cases}$$

also has $\partial\psi_1^0(0) = [-1, 1]$ and is fully intermediately differentiable at 0, but is not uniformly intermediately differentiable at 0. ∎

The following example of a Lipschitz function given by R. T. Rockafellar [12, p. 97] illustrates the complexity of the relations between the various sets of points of differentiability.

EXAMPLE 1.7. There exists a measurable set S in $\mathbb{R}$ with the property that, for every non-empty open interval I, the sets $I \cap S$ and $I \cap (\mathbb{R}\backslash S)$ are both of positive measure. Define a function θ on $\mathbb{R}$ by

$$\theta(x) = \begin{cases} +1 & \text{for } x \in S \\ -1 & \text{for } x \in \mathbb{R}\backslash S \end{cases}$$

and a function ψ on $\mathbb{R}$ by

$$\psi(x) = \int_0^x \theta(t)\, dt.$$

Then ψ is globally Lipschitz and nowhere strictly differentiable. The set of points where ψ is differentiable is first category but of full measure. The set of points where ψ is pseudo-regular is residual but is disjoint from the set of points where ψ is differentiable and so is of measure zero. Now the set of points where ψ is intermediately differentiable is residual and of full measure since it contains the points where ψ is differentiable. However, the set of points where ψ is fully intermediately differentiable is residual but of measure zero since it is disjoint from the set of points where ψ is differentiable. ∎

2. Lower Semi-continuity of the Subdifferential Mapping

The differentiability of a continuous convex function ϕ is associated with continuity properties of its subdifferential mapping $x \mapsto \partial\phi(x)$ and there is an extension of this association for a locally Lipschitz function ψ and its Clarke subdifferential mapping $x \mapsto \partial\psi^0(x)$.

Given a topological space A and a normed linear space X, a set-valued mapping Φ from A into subsets of the dual X^* is said to be *weak* (norm) upper semi-continuous* at $t_0 \in A$ if for each weak* (norm) open subset W of X^* where $\Phi(t_0) \subseteq W$, there exists an open neighbourhood U of t_0 such that $\Phi(U) \subseteq W$. If Φ is weak* upper semi-continuous on A and $\Phi(t)$ is weak* compact and convex in X^* for each $t \in A$ then Φ is called a *weak* cusco* on A and Φ is said to be a *minimal weak* cusco* on A if its graph does not contain the graph of any other weak* cusco on A.

For a continuous convex function ϕ on an open convex subset A of a normed linear space X, the subdifferential mapping $x \mapsto \partial\phi(x)$ is a minimal weak* cusco on A [9, p. 100]. For a locally Lipschitz function ψ an open subset A of a normed linear space X the subdifferential mapping $x \mapsto \partial\psi^0(x)$ is a weak* cusco on A but is not necessarily minimal.

The following characterisations of differentiability are well known.

PROPOSITION 2.1. (i) *A continuous convex function ϕ on an open convex subset A of a normed linear space X is Gâteaux (Fréchet) differentiable at $x \in A$ if and only if $\partial\phi(x)$ is a singleton (and the subdifferential mapping $x \mapsto \partial\phi(x)$ is norm upper semi-continuous at x)* [9, p. 18].

(ii) *A locally Lipschitz function ψ on an open subset A of a normed linear space X is strictly (uniformly strictly) differentiable at $x \in A$ if and only if $\partial\psi^0(x)$ is singleton (and the subdifferential mapping $x \mapsto \partial\phi^0(x)$ is norm upper semi-continuous at x)* [6, p. 374].

Given a topological space A and a normed linear space X a set-valued mapping Φ from A into subsets of the dual X^* is said to be *weak* (norm) lower semi-continuous* at $t_0 \in A$ if for each $f \in \Phi(t_0)$ and weak* (norm) open subset W of X^* where $f \in W$ there exists an open neighbourhood U of t_0 such that $\Phi(t) \cap W \neq \emptyset$ for each $t \in U$.

Now if Φ is single-valued and weak* (norm) upper semi-continuous at $t_0 \in A$ then clearly Φ is weak* (norm) lower semi-continuous at t_0. Further, in the special case when Φ is single-valued on a dense subset of A, if Φ is weak* lower semi-continuous at $t_0 \in A$ then Φ is single-valued at t_0. But in general, the two continuity conditions are independent of each other.

Nevertheless, they are related for minimal weak* cuscos. This relation is a consequence of the following property of minimal weak* cuscos.

LEMMA 2.2 *Given a minimal weak* cusco Φ from a topological space A into subsets of the dual X^* of a normed linear space X, if for an open set U in A and a weak* closed convex subset K in X^* we have $\Phi(t) \cap K \neq \emptyset$ for each $t \in U$, then $\Phi(U) \subseteq K$.*

Proof. The set-valued mapping of Φ' from A into subsets of X^* defined by

$$\Phi'(t) = \begin{cases} \Phi(t) \cap K & \text{for } t \in U \\ \Phi(t) & \text{for } t \notin U \end{cases}$$

is a weak* cusco whose graph is contained in that of Φ. But since Φ is minimal we conclude that $\Phi(t) \subseteq K$ for all $t \in U$. ∎

PROPOSITION 2.3. *A minimal weak* cusco Φ from a topological space A into subsets of the dual X^* of a normed linear space X is single-valued and weak* (norm) upper semi-continuous at $t_0 \in A$ if and only if it is weak* (norm) lower semi-continuous at t_0.*

Proof. Consider $f \in \Phi(t_0)$. Given a weak* (norm) open convex set W where $f \in W$, there exists a weak* (norm) open convex set V such that $f \in V \subseteq \overline{V}^{w*} \subseteq W$. Since Φ is weak* (norm) lower semi-continuous at t_0 there exists an open neighbourhood U of t_0 such that

$$\Phi(t) \cap V \neq \emptyset \qquad \text{for each } t \in U.$$

So

$$\Phi(t) \cap \overline{V}^{w*} \neq \emptyset \qquad \text{for each } t \in U.$$

Since Φ is minimal we have by Lemma 2.2 that

$$\Phi(U) \subseteq \overline{V}^{w*}$$

and so

$$\Phi(U) \subseteq W.$$

This implies that Φ is single-valued and weak* (norm) upper semi-continuous at t_0.

The converse is obvious. ∎

So then for a continuous convex function ϕ, Gâteaux (Fréchet) differentiability of ϕ at $x \in A$ is characterised by the weak* (norm) lower semi-continuity of the subdifferential mapping $x \mapsto \partial\phi(x)$ at x. Now for a locally Lipschitz function ψ the set of points where ψ is strictly differentiable (uniformly strictly differentiable) is contained in the set of points where the subdifferential mapping $x \mapsto \partial\psi^0(x)$ is weak* (norm) lower semi-continuous but the converse does not hold generally. So for a locally Lipschitz function ψ we are led to consider the set of points where the subdifferential mapping $x \mapsto \partial\psi^0(x)$ is weak* (norm) lower semi-continuous as a set which generalises the set of points of differentiability.

We consider first locally Lipschitz functions which are pseudo-regular on their domain and show that they possess properties close to those which generate a minimal subdifferential mapping.

We need the following general property which should be compared with Theorem 1.1 (iii).

LEMMA 2.4. *Given a locally Lipschitz function ψ on an open subset A of a normed linear space X, the subdifferential mapping $x \mapsto \partial\psi^0(x)$ is weak* lower semi-continuous at $x \in A$ if and only if $\psi^0(x)(y)$ is continuous at x for all $y \in X$.*

Proof. It is known that given $y \in X$, $\psi^0(x)(y)$ is upper semi-continuous in x [4, p. 26] so it is sufficient to consider lower semi-continuity. Given $y \in X$ and $\varepsilon > 0$ there exists an open neighbourhood U of x such that

$$\partial\psi^0(z) \cap \{f \in X^* : f(y) > \psi^0(x)(y) - \varepsilon\} \neq \varnothing \qquad \text{for all } z \in U.$$

Since

$$\psi^0(z)(y) = \sup\{f(y) : f \in \partial\psi^0(z)\}$$

then

$$\psi^0(z)(y) > \psi^0(x)(y) - \varepsilon \qquad \text{for all } z \in U;$$

that is, $\psi^0(x)(y)$ is lower semi-continuous at x.

Conversely, suppose that the subdifferential mapping $x \mapsto \partial\psi^0(x)$ is not weak* lower semi-continuous at $x \in A$. Then there exists a weak* open set

$$W \equiv \{f \in X^* : f(y_j) > \alpha_j \text{ for all } j \in \{1, 2, \ldots, n\}\}$$

such that $W \cap \partial\psi^0(x) \neq \varnothing$, but there exists a sequence $\{z_k\}$ in A where $z_k \to x$ such that $W \cap \partial\psi^0(z_k) = \varnothing$ for each $k \in \mathbb{N}$. Now for each $k \in \mathbb{N}$, W and $\partial\psi^0(z_k)$ can be separated by a weak* closed hyperplane of the form

$$\{f \in X^* : f(w_k) = \beta_k\} \qquad \text{where } w_k \in \operatorname{co}\{y_1, y_2, \ldots, y_n\}.$$

Since $\operatorname{co}\{y_1, y_2, \ldots, y_n\}$ is compact, the sequence $\{w_k\}$ has a subsequence $\{w_{k_l}\}$ convergent to some $w \in X$. Since for a given $x \in A$, $\psi^0(x)(y)$ is continuous in y, for each $l \in \mathbb{N}$,

$$\psi^0(z_{k_l})(w) \leq \limsup_{l \to \infty} \beta_{k_l}.$$

But given $f \in W \cap \partial\psi^0(x)$ there exists an $r > 0$ such that

$$\limsup_{l \to \infty} \beta_{k_l} < f(w) - r$$

and so $\psi^0(z_{k_l})(w) < \psi^0(x)(w) - r$ for all $l \in \mathbb{N}$ and we conclude that $\psi^0(x)(w)$ is not lower semi-comtinuous at x. ∎

THEOREM 2.5. *Consider a locally Lipschitz function ψ pseudo-regular on an open subset A of a Banach space X. If the subdifferential mapping $x \mapsto \partial\psi^0(x)$ is weak* (norm) lower semi-continuous on a residual subset*

D of A then ψ is strictly differentiable (uniformly strictly differentiable) on D.

Proof. From Lemma 2.4, $\psi^0(x)(y)$ is continuous in x on D for all $y \in X$. Since ψ is pseudo-regular on A, $\psi^+(x)(y)$ is continuous in x on D for all $y \in X$. From Theorem 1.1(iii) we deduce that ψ is strictly differentiable on D. It follows from [2, Theorem 4.7, p. 474] that the subdifferential mapping $x \mapsto \partial\psi^0(x)$ is a minimal weak* cusco and the result for norm lower semi-continuity follows immediately from Propositions 2.1 and 2.3. ∎

Now weak* cuscos associated with separable Banach spaces possess the following lower semi-continuity properties.

THEOREM 2.6. *Consider a Baire space A and a normed linear space X.*

(i) *If X is separable, every locally bounded weak* cusco Φ from A into subsets of X^* is weak* lower semi-continuous on a residual subset of A.*

(ii) *If X^* is separable, every weak* cusco Φ from A into subsets of X^* is norm lower semi-continuous on a residual subset of A.*

Proof. (ii) Since X^* is separable it has a countable base $\{B_n\}$ of open balls. If Φ is not norm lower semi-continuous at $t_0 \in A$ then there exists a B_{n_0} such that $\Phi(t_0) \cap B_{n_0} \neq \emptyset$ but in any neighbourhood U of t_0 there exists a t such that $\Phi(t) \cap B_{n_0} = \emptyset$. Now there exists $\overline{B}_{n_1} \subseteq B_{n_0}$ such that $\Phi(t_0) \cap B_{n_1} \neq \emptyset$. Then $\Phi(t) \cap \overline{B}_{n_1} = \emptyset$. Now $\Phi(t) \subseteq C(\overline{B}_{n_1})$ which is weak* open and since Φ is weak* upper semi-continuous at t there exists an open neighbourhood U' of t where $U' \subseteq U$ such that $\Phi(U') \cap \overline{B}_{n_1} = \emptyset$. Then $\Phi(U') \cap B_{n_1} = \emptyset$. Now given $n \in \mathbb{N}$, the set

$$\{t \in A : \Phi(t) \cap B_n \neq \emptyset \text{ but for every open neighbourhood } U \text{ of } t \text{ there exists an open set } U' \subseteq U \text{ such that } \Phi(U') \cap B_n = \emptyset\}$$

is nowhere dense in A. So the subset of A where Φ is not norm lower semi-continuous is first category in A.

(i) Since Φ is locally bounded on A, for some $n_0 \in \mathbb{N}$ there exists a non-empty open subset U_{n_0} of A such that $\Phi(U_{n_0}) \subseteq n_0 B(X^*)$. Since X is separable the weak* topology on $n_0 B(X^*)$ is separable and metrisable. So there exists a countable base for the weak* topology on $n_0 B(X^*)$. So by an argument similar to that given in (ii) above and using the regularity of the weak* topology we show that the subset D_{n_0} of U_{n_0} where Φ is not weak* lower semi-continuous is first category in U_{n_0}. Now for each $n > n_0$ we may choose an open subset U_n of A such that $U_{n_0} \subseteq U_n \subsetneq U_{n+1}$

and $\Phi(U_n) \subseteq nB(X^*)$; but then $D_{n_0} \subseteq D_n \subseteq D_{n+1}$. But since Φ is locally bounded on A we may choose U_n so that $A = \bigcup_{n=n_0}^{\infty} U_n$ and then $D = \bigcup_{n=n_0}^{\infty} D_n$ is the subset of A where Φ is not weak* lower semi-continuous on A and is first category in A. ▌

From Proposition 2.3 we have the following well-known result.

COROLLARY 2.7. *Consider a Baire space A and a normed linear space X.*

(i) *If X is separable, every locally bounded minimal weak* cusco Φ from A into subsets of X^* is single-valued on a residual subset of A.*

(ii) *If X^* is separable, every minimal weak* cusco Φ from A into subsets of X^* is single-valued and norm upper semi-continuous on a residual subset of A.*

From Corollary 2.7 we deduce differentiability properties for continuous convex functions given originally by S. Mazur [8] for separable Banach spaces and E. Asplund [1] for Banach spaces with separable dual.

It follows from Theorems 2.5 and 2.6(i) that a pseudo-regular locally Lipschitz function on a separable Banach space generates a subdifferential mapping which is a minimal weak* cusco.

We should note that the continuity property given in Theorem 2.6 does not hold generally for locally bounded weak* cuscos from a Baire space into subsets of the dual of a non-separable Asplund space.

EXAMPLE 2.8. Consider the non-separable Hilbert space $l_2(\mathbb{R})$ and the set-valued mapping Φ from $l_2(\mathbb{R})$ into subsets of $l_2(\mathbb{R})$ defined by

$$\Phi(x) = \{\lambda\delta_{\|x\|} : 0 \leq \lambda \leq 1\},$$

where $\delta_{\|x\|} \in l_2(\mathbb{R})$ is defined by

$$\delta_{\|x\|}(\alpha) = \begin{cases} 0, & \alpha \neq \|x\| \\ 1, & \alpha = \|x\|. \end{cases}$$

Clearly, $\Phi(x)$ is weakly compact and convex. Consider $x \in l_2(\mathbb{R})$ and sequence $\{x_n\}$ in $l_2(\mathbb{R})$ convergent to x. If $\|x_n\| = \|x\|$ for all $n \in \mathbb{N}$, then $\Phi(x_n) = \Phi(x)$ for all $n \in \mathbb{N}$. If $\|x_n\| \neq \|x\|$ for each $n \in \mathbb{N}$, given $y \in l_2(\mathbb{R})$ and $0 \leq \lambda_n \leq 1$ we have $(\lambda_n\delta_{\|x_n\|}, y) \to 0$ so $\lambda_n\delta_{\|x_n\|}$ is weakly convergent to $0 \in \Phi(x)$. Therefore, Φ is a weak cusco on $l_2(\mathbb{R})$. However, if $\|x_n\| \neq \|x\|$ for each $n \in \mathbb{N}$, we have $\|\lambda_n\delta_{\|x_n\|} - \delta_{\|x\|}\| \geq 1$, and we conclude that Φ is not norm lower semi-continuous at any point of $l_2(\mathbb{R})$.

Nevertheless, we do have the following norm lower semi-continuity property for certain weak* cuscos which map into subsets of the dual of an Asplund space.

THEOREM 2.9. *Given a locally bounded weak* cusco Φ from a Baire space A into subsets of the dual X^* of an Asplund space X, if Φ is single-valued on a residual subset D of A then Φ is single-valued and norm lower semi-continuous on a residual subset of A.*

Proof. Consider a minimal weak* cusco $\bar{\Phi}$ contained in Φ. Since X is an Asplund space, $\bar{\Phi}$ is single-valued and norm upper semi-continuous on a residual subset D' of A. Now $D \cap D'$ is a residual subset of A and for $t \in D \cap D'$, $\Phi(t) = \bar{\Phi}(t)$ and since $\bar{\Phi}$ is norm upper semi-continuous at t then Φ is norm lower semi-continuous at t. ∎

3. THE RESIDUAL SET GENERALISING THE SET OF POINTS OF DIFFERENTIABILITY

We now turn to our general problem of determining for a locally Lipschitz function on a separable Banach space the appropriate residual subset of the domain which if the function were convex would coincide with its set of points of differentiability.

For a separable Banach space (with separable dual), in Section 1 we identified the set of points where the function is fully intermediately differentiable (full and uniformly intermediately differentiable) as a residual subset of the domain which could be regarded as a set generalising the set of points of Gâteaux (Fréchet) differentiability. In Section 2 we identified the set of points where the subdifferential mapping is weak* (norm) lower semi-continuous as a residual subset of the domain which could be regarded in the same way.

Now in general the set of points where the locally Lipschitz function is fully intermediately differentiable (fully and uniformly intermediately differentiable) is not related by set containment to the set of points where the subdifferential mapping is weak* (norm) lower semi-continuous. In Example 1.7 on $\mathbb{R}$, the locally Lipschitz function ψ is lower semi-continuous at every point of $\mathbb{R}$ but is fully intermediately differentiable on a residual set of measure zero. Examples 1.6 are of locally Lipschitz functions where the subdifferential mapping is a minimal cusco but by Proposition 2.3 is not lower semi-continuous at 0. However, both examples are fully intermediately differentiable at 0 and Example 1.6(i) is also uniformly intermediately differentiable at 0.

An important component of Preiss' Theorem [10] is his showing that for a locally Lipschitz function ψ on an open subset A

(i) of a Banach space with an equivalent norm Gâteaux differentiable away from the origin,

(ii) of an Asplund space,

at any point $x \in A$, the Clarke subdifferential $\partial\psi^0(x)$ is generated by the

(i) Gâteaux derivatives of ψ

(ii) Fréchet derivatives of ψ.

By this we mean that $\partial\psi^0(x)$ is the weak* closed convex hull of the cluster points of $\psi'(x_n)$ for sequences $\{x_n\}$ where ψ is (i) Gâteaux, (ii) Fréchet differentiable and as $\{x_n\}$ converges to x. This generalises the result from finite-dimensional spaces [4, p. 63].

We could ask whether either of our two types of residual subsets of A which could be regarded as generalisations of the set of points of Gâteaux (Fréchet) differentiability generate the Clarke subdifferential in the same way.

For a locally Lipschitz function ψ on an open subset A of a separable Banach space (with separable dual) we could consider as our subset generalising the set of points of Gâteaux (Fréchet) differentiability, the sets of points where both ψ is fully intermediately differentiable (fully and uniformly intermediately differentiable) and the subdifferential mapping of $x \mapsto \partial\psi^0(x)$ is weak* (norm) lower semi-continuous. This is a smaller residual set which coincides with the points of Gâteaux (Fréchet) differentiability for a continuous convex function. In the particular case where the subdifferential mapping $x \mapsto \partial\psi^0(x)$ is a minimal weak* cusco, the set of points where ψ is fully intermediately differentiable (fully and uniformly intermediately differentiable) and the subdifferential mapping $x \mapsto \partial\psi^0(x)$ is weak* (norm) lower semi-continuous is a residual set where ψ is single-valued and so this set generates the subdifferential [3, Corollary 4.2, p. 472]. In this case the set of points where the subdifferential mapping $x \mapsto \partial\psi^0(x)$ is weak* (norm) lower semi-continuous is included in the set of points where ψ is fully intermediately differentiable (fully and uniformly intermediately differentiable) but as Examples 1.6 show, the two sets are not necessarily equal.

The following example given by Borwein [2, Example 6.4(b), p. 77] shows that in general the set of points where the subdifferential mapping is lower semi-continuous does not generate the subdifferential.

EXAMPLE 3.1. Consider E a dense open set in $\mathbb{R}$ not of full measure. Then

$$\psi(x) = \int_0^x \chi_{C(E)}(t)\, dt,$$

where $\psi_{C(E)}$ is the characteristic function on $C(E)$. Then

$$\partial\psi^0(x) = \begin{cases} 0 & \text{for } x \in E \\ [0, 1] & \text{for } x \in C(E). \end{cases}$$

Now the subdifferential mapping $x \mapsto \partial\psi^0(x)$ is lower semi-continuous only on E and it is clear that such a set does not generate the subdifferential at points of $C(E)$.

The problem of determining whether the subdifferential is generated by the set of points where the locally Lipschitz function is fully intermediately differentiable (fully and uniformly intermediately differentiable) is much more difficult. It is clear from Lemma 2.4 that for a locally Lipschitz function ψ on an open subset A of a separable Banach space X (with separable dual), at points $x \in A$ where the subdifferential mapping $x \mapsto \partial\psi^0(x)$ is weak* lower semi-continuous then for each $y \in X$,

$$\psi^0(x)(y) = \limsup_{\substack{z \to x \\ z \in D}} \psi^+(z)(y),$$

where D is the residual set where ψ is fully intermediately differentiable (fully and uniformly intermediately differentiable). Consequently, at such points $x \in A$, $\partial\psi^0(x)$ is generated by the fully intermediate (fully and uniformly intermediate) derivatives of ψ. This implies that Example 1.7 which in so many ways exemplifies pathological behaviour, does have its subdifferential at each point generated by its fully and uniformly intermediate derivatives. However, weak* lower semi-continuity is in general a stronger condition than necessary to guarantee such a generation of the subdifferential.

Nevertheless, we can show that pseudo-regularity in a direction is of significance in generating the subdifferential and this produces a refinement of Preiss' characterisation.

THEOREM 3.2. *For a locally Lipschitz function ψ on an open subset A of a Banach space X, given $x \in A$ and $y \in X$*

$$\psi^0(x)(y) = \limsup_{\substack{z \to x \\ z \in D_y}} \psi^+(z)(y),$$

where

$$D_y \equiv \{z \in A : \psi^+(z)(y) = \psi^0(z)(y)\}.$$

Proof. Given $\varepsilon > 0$, consider the sets

$$F_\varepsilon \equiv \{z \in A : \psi^0(z)(y) > \psi^0(x)(y) - \varepsilon\}$$

and for $p \in \mathbb{N}$,

$$R_\varepsilon^p \equiv \{z \in A : \psi_p^+(z)(y) > \psi^0(x)(y) - \varepsilon\}.$$

It follows from the definition of the generalised Clarke derivative that F_ε and R_ε^p are non-empty. Since $\psi_p^+(z)(y)$ is lower semi-continuous in z,

$$R_\varepsilon \equiv \bigcap_{p \in \mathbb{N}} R_\varepsilon^p \text{ is a } G_\delta \text{ subset of } A.$$

We show that R_ε is dense in F_ε. Suppose not. Then there exists a $z_0 \in F_\varepsilon$ and an $r > 0$ such that $R_\varepsilon \cap F_\varepsilon \cap B(z_0; r) = \emptyset$. Then for all each $z \in B(z_0; r) \cap F_\varepsilon$ there exists a $p \in \mathbb{N}$ such that

$$\psi_p^+(z)(y) \leq \psi^0(x)(y) - \varepsilon$$

so

$$\psi^+(z)(y) \leq \psi^0(x)(y) - \varepsilon.$$

But for all $z \in B(z_0; r) \backslash F_\varepsilon$,

$$\psi^+(z)(y) \leq \psi^0(z)(y) \leq \psi^0(x)(y) - \varepsilon.$$

This would imply that

$$\psi^0(z_0)(y) \equiv \limsup_{z \to z_0} \psi^+(z)(y) \leq \psi^0(x)(y) - \varepsilon$$

contradicting the fact that $z_0 \in F_\varepsilon$. So we conclude that R_ε is dense in F_ε and is a dense G_δ subset of the Baire space $\overline{F}_\varepsilon$.

Now $\psi^+(z)(y)$ on the Baire space $\overline{F}_\varepsilon$ is upper semi-continuous in z on a residual subset P of $\overline{F}_\varepsilon$ [7, Theorem]. So $P \cap R_\varepsilon$ is a residual subset of $\overline{F}_\varepsilon$ where $\psi^+(z)(y)$ is upper semi-continuous in z relative to $\overline{F}_\varepsilon$ and

$$\psi^+(z)(y) \geq \psi^0(x)(y) - \varepsilon.$$

However, at $z_0 \in P \cap R_\varepsilon$, $\psi^+(z)(y)$ is also upper semi-continuous in z relative to A, because if not there exists an $r > 0$ and a sequence $\{z_n\}$ where $z_n \to z_0$ and

$$\begin{aligned}\psi^+(z_n)(y) &\geq \psi^+(z_0)(y) + r \\ &> \psi^0(x)(y) - \varepsilon\end{aligned}$$

and then $z_n \in \overline{F}_\varepsilon$ for all $n \in \mathbb{N}$ and we have contradicted the property we established on $\overline{F}_\varepsilon$. We conclude from Theorem 1.1(i) that

$$\psi^+(z)(y) = \psi^0(z)(y) \qquad \text{for all } z \in P \cap R_\varepsilon. \quad \blacksquare$$

We note that D_y is a dense G_δ subset of A [7, Theorem].

Preiss' Theorem provides spaces where the subdifferential is generated by the derivatives. On such spaces the proof of Theorem 3.2 will extend to the subset of A where the locally Lipschitz function is differentiable. We define the subsets F_ε and R_ε^p on the subset of A where the locally Lipschitz function is differentiable and the upper semi-continuity of $\psi^+(z)(y)$ at $z_0 \in P \cap F_\varepsilon$ relative to A follows because the derivatives generate the subdifferential. So we deduce the following refined version of Preiss' result.

COROLLARY 3.3 *For a locally Lipschitz function ψ on an open subset A of*

(i) *a Banach space X with equivalent norm Gâteaux differentiable away from the origin,*

(ii) *an Asplund space X,*

given $x \in A$ and $y \in X$

$$\psi^0(z)(y) = \limsup_{\substack{z \to x \\ z \in E}} \psi'(z)(y),$$

where

(i) $E \equiv G \cap D_y = \{z \in G : \psi'(z)(y) = \psi^0(z)(y)\}$,

(ii) $E \equiv F \cap D_y = \{z \in F : \psi'(z)(y) = \psi^0(z)(y)\}$

and $G(F)$ is the subset of A where ψ is Gâteaux (Fréchet) differentiable.

It is clear from Example 1.7 that this result cannot be extended to have the subdifferential generated by the strict derivatives, since in that example there are none. However, in that example at each $x \in \mathbb{R}$,

$$\psi^0(x)(y) = +1 \quad \text{and} \quad (-\psi)^0(x)(y) = \psi^0(x)-y) = -1.$$

Further, ψ is differentiable almost everywhere in S with derivative $+1$ and almost everywhere in $\mathbb{R} \backslash S$ with derivative -1.

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