On the number of distinct prime factors of an odd perfect number

Graeme L. Cohen *, Ronald M. Sorli

Department of Mathematical Sciences, University of Technology, Sydney, NSW 2007, Australia

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Abstract

It is not known whether or not there exists an odd perfect number. We describe an algorithmic approach for showing that if there is an odd perfect number then it has \( t \) distinct prime factors, and we discuss its application towards showing that \( t \geq 9 \).

Keywords: Odd perfect number; Abundant; Deficient; Fermat

1. Introduction

An integer \( n \) is perfect if \( \sigma(n) = 2n \), where \( \sigma \) is the positive divisor sum function. It is not known whether or not there exists an odd perfect number. In these notes, we shall describe an algorithmic approach for showing that if there is an odd perfect number then it has \( t \) distinct prime factors, and we shall discuss its application towards showing that \( t \geq 9 \).

This work is an updated version of research carried out some years ago and reported in Cohen [5] and subsequently Cohen and Sorli [7]. It has not been otherwise published, because of the difficulty of implementing the algorithm to obtain new results. Further computational work on this topic is one theme of the second author’s PhD thesis, supervised by the first author.

The following is a brief history of the problem. It is very easy to see that \( t \geq 4 \). Sylvester [30], and also Dickson [9] and Kanold [22], have shown that \( t \geq 5 \). There have

* Corresponding author.
E-mail addresses: g.cohen@maths.uts.edu.au, graeme.cohen@uts.edu.au (G.L. Cohen),
rons@maths.uts.edu.au (R.M. Sorli).

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been at least five published proofs that \( t \geq 6 \). Kühnel [26] and Webber [31] proved this around the same time, no doubt unaware that Gradstein [11] had done the same much earlier. Kishore [23,24], as part of a wider doctoral study, gave two proofs. (A sixth proof, using recently derived results and illustrating the algorithm, is included here as Appendix B. This will be discussed below.) Proofs that \( t \geq 7 \) were given by Pomerance [27] and Robbins [29], independently, each being a PhD thesis. Finally, Chein [3], also as a PhD thesis, and Hagis [13] have shown that \( t \geq 8 \). With the further assumption that \( 3 \nmid n \), Hagis [15] and Kishore [25] have shown independently that \( t \geq 11 \).

Ten years ago, an algorithm was developed by Brent and Cohen [1,2] for showing that, if there is an odd perfect number \( n \), then \( n > K \), and it was applied in [2] with \( K = 10^{300} \).

This is very highly dependent on computational power, in terms of the size of the tree generated and in the size of the numbers whose factorisations were required.

Of the above proofs, concerned with lower bounds for the number of distinct prime divisors of \( n \), only the approach of Kishore might be describable as algorithmic. However, it was not presented as such and, because of its more ambitious main aim, it does not appear that it could easily be applied to improve the known lower bound. In [28], Pomerance gave an algorithm for finding an upper bound for all odd perfect numbers with \( t \) distinct prime factors. He showed that all such numbers must be less than

\[
(4t)^{(4t)^{e^2}}.
\]

This bound has subsequently been improved by Heath-Brown [17] to \( 4^{4t} \), and by Cook [8] to \( D^{4t} \), where \( D = 195^{1/7} = 2.123 \ldots \).

The algorithm we are about to describe will, as we will heuristically show, be stretched to show that \( t \geq 9 \), but should comfortably prove that \( t \geq 8 \). Again, of course, considerable computing power will be required.

2. Lemmata

In the following, we always use \( n \) to denote an odd perfect number, assuming one exists.

Euler showed that \( n \) must have the form \( \pi^a m^2 \), where \( \pi \) is prime, \( \pi \equiv a \equiv 1 \pmod{4} \) and \( \pi \nmid m \). In this, we refer to \( \pi \) as the special prime, and the symbol \( \pi \) will always have this significance. A maximal prime power divisor of any number will be called a component of that number. The components of \( n \) must be of the form \( p^a \), with \( a \) even or \( p \equiv a \equiv 1 \pmod{4} \); any such prime power will be called Eulerian.

We define arithmetic functions \( h \) and \( H \) by

\[
h(1) = H(1) = 1 \quad \text{and, if } m > 1,
\]

\[
h(m) = \prod_p \frac{p^{a+1} - 1}{p^a(p-1)}, \quad H(m) = \prod_p \frac{p}{p-1},
\]

where \( \prod_p p^a \) is the prime factor decomposition of \( m \). Notice that \( h(m) = \sigma(m)/m \), \( h \) and \( H \) are multiplicative, \( H(p^a) = \lim_{a \to \infty} h(p^a) \) \( (p \text{ prime}) \), \( h(\overline{m}) \leq h(m) \) if \( \overline{m} \mid m \), and \( h(m) < H(m) \) for \( m > 1 \). Also, for any prime powers \( p^a \) and \( q^b \), where \( 2 < p < q \),

\[
H(q^b) = \frac{q}{q-1} < \frac{p+1}{p} \leq h(p^a).
\]
Since $n$ is perfect, we have $h(n) = 2$. Any number $m$ for which $h(m) < 2$ (respectively, $h(m) > 2$) is called deficient (respectively, abundant). It is easy to see that any proper divisor of a perfect number must be deficient; see Lemma 2.1, below. An abundant number all of whose proper divisors are deficient is said to be primitive.

In addition to the Eulerian form above, we shall often consider $n$ to have the prime factor decomposition

$$n = \prod_{i=1}^{\mu} p_i^{a_i} \cdot \prod_{i=1}^{v} q_i^{b_i} \cdot \prod_{i=1}^{w} r_i^{c_i} = \lambda \cdot \mu \cdot v,$$

which we interpret as follows: each $p_i^{a_i}$ is a known component of $n$, each $q_i^{b_i}$ is a known prime factor of $n$ but the exponent $b_i$ is unknown, and each prime factor $r_i$ of $n$ and the exponent $c_i$ are unknown. By “known”, we mean explicitly postulated or the consequence of such an assumption. Any of $u$, $v$, $w$ may be zero, in which case we set $\lambda$, $\mu$, $\nu$, respectively, equal to 1. Writing primes as $p$, $q$ or $r$, with or without subscripts, will generally imply that they are divisors of $\lambda$, $\mu$ or $\nu$, respectively.

In the following two lemmas, $\bar{\mu}$ denotes a proper divisor of $\mu$ (except that $\bar{\mu} = 1$ if $\mu = 1$), to be specified precisely later.

**Lemma 2.1.** For any odd perfect number $n = \lambda \mu v$, as above, we have

$$h(\lambda \bar{\mu}) \leq 2 \leq h(\lambda \nu)H(\mu).$$

(2.1)

Both inequalities are strict if $v > 0$; the left-hand inequality is strict if $w > 0$.

**Proof.** We need only note that $h(\bar{\mu}) \leq h(\mu)$ and $h(\nu) \geq 1$, with strict inequality if $v > 0$ or $w > 0$, respectively, $h(\mu) < H(\mu)$ if $v > 0$, and $2 = h(\lambda \mu v) = h(\lambda)h(\mu)h(\nu)$.  

**Lemma 2.2.** Suppose $w \geq 1$, and assume $r_1 < r_2 < \cdots < r_w$. Then

$$h(\lambda \bar{\mu} r_1^{i-1}) < 2 = h(n) = h(\lambda)h(\mu)v = h(\lambda)h(\bar{\mu})\left(1 + \frac{h(\bar{\mu}^{i-1})}{r_1}\right) = h(\lambda \bar{\mu}) + \frac{1}{r_1}h(\lambda \bar{\mu} r_1^{i-1}).$$

(2.2)

with strict inequality if $v \geq 1$ or $w \geq 2$. Further, if $h(\lambda)H(\mu) < 2$, then

$$r_1 < \frac{2 + h(\lambda)H(\mu)(w-1)}{2 - h(\lambda)H(\mu)}.$$  

(2.3)

**Proof.** Since $h(\nu) \geq h(r_1^{i-1}) = 1 + h(r_1^{c-1})/r_1$, we have

$$2 = h(n) = h(\lambda)h(\mu)v \geq h(\lambda)h(\bar{\mu})\left(1 + \frac{h(\bar{\mu}^{i-1})}{r_1}\right) = h(\lambda \bar{\mu}) + \frac{1}{r_1}h(\lambda \bar{\mu} r_1^{i-1}).$$

This may be rearranged to give (2.2). The remark concerning strict inequality is clear. We now derive (2.3). For $i = 2, \ldots , w$, we have $r_i > r_{i-1} + 1 > r_{i-2} + 2 > \cdots > r_1 + i - 1$, so

$$\frac{r_i}{r_i - 1} < \frac{r_1 + i - 1}{r_1 + i - 2}.$$
Then
\[ h(\nu) < H(\nu) = \prod_{i=1}^{w} \frac{r_i}{r_i - 1} \leq \prod_{i=1}^{w} \frac{r_1 + i - 1}{r_1 + i - 2} = \frac{r_1 + w - 1}{r_1 - 1} = 1 + \frac{w}{r_1 - 1}. \]
Therefore,
\[ 2 = h(n) = h(\lambda)h(\mu)h(\nu) < h(\lambda)H(\mu) \left( 1 + \frac{w}{r_1 - 1} \right), \]
and the result follows. \( \Box \)

In practice, we mostly must assume \( c_1 \geq 1 \) and replace (2.2) by
\[ \frac{h(\lambda, \bar{\mu})}{2 - h(\lambda, \bar{\mu})} \leq r_1, \quad \text{or equivalently} \quad \frac{2}{2 - h(\lambda, \bar{\mu})} - 1 \leq r_1. \] (2.4)

On some occasions, including, in particular, when the special prime is known to divide \( \lambda \), we can assume \( c_1 \geq 2 \), in which case
\[ \frac{h(\lambda, \bar{\mu})}{2 - h(\lambda, \bar{\mu})} \left( 1 + \frac{1}{r_1} \right) \leq r_1. \] (2.5)

Then
\[ \frac{h(\lambda, \bar{\mu})}{2 - h(\lambda, \bar{\mu})} \leq \frac{r_1^2}{r_1 + 1} = r_1 - 1 + \frac{1}{r_1 + 1}, \]
so that
\[ \frac{2}{2 - h(\lambda, \bar{\mu})} - \frac{1}{r_1 + 1} \leq r_1, \]
and we may estimate \( r_1 \) on the left using (2.4). That gives us
\[ r_1 \geq \frac{2}{2 - h(\lambda, \bar{\mu})} - \frac{2 - h(\lambda, \bar{\mu})}{2}. \] (2.6)

The latter results are essentially those of Jerrard and Temperley [21] (more easily derived here; see also Hagis [14]). The inequality in (2.3) is a slight improvement, in essence, of a result of Pomerance [26, 1.4].

As communicated privately by Peter Hagis, we obtain an improvement of the estimate in (2.6) by solving the quadratic inequality implied in (2.5):
\[ r_1 \geq \frac{C + \sqrt{C^2 + 4C}}{2}, \quad \text{where} \quad C = \frac{h(\lambda, \bar{\mu})}{2 - h(\lambda, \bar{\mu})}. \]
However, (2.6) requires only rational arithmetic with no complications concerning computational precision, and is preferred by us.

**Corollary 2.3.** Suppose \( w \geq 2 \), and assume \( r_2 < r_3 < \cdots < r_w \). If \( h(\lambda)H(\mu r_1) < 2 \), then
\[ r_2 < \frac{2 + h(\lambda)H(\mu r)(w - 2)}{2 - h(\lambda)H(\mu r)}. \]
where \( r \) is the smallest prime \( r_1 \), not dividing \( \lambda \) or \( \mu \), which satisfies (2.4), or, if possible, (2.6).

**Proof.** By Lemma 2.2,

\[
r_2 < \frac{2 + h(\lambda r_1^{e_1})H(\mu)(w - 2)}{2 - h(\lambda r_1^{e_1})H(\mu)}.
\]

But \( r_1 \geq r \) and \( h(r_1^{e_1}) < H(r_1^{e_1}) = H(r_1) \leq H(r) \), and the result follows. \( \square \)

The first of the following extremely useful results is due to Hagis and Cohen [16]; the others are due to Iannucci [19,20]. We remark that the possibility of circularity does not arise: their proofs make no use of the number of prime factors that an odd perfect number might have.

**Lemma 2.4.** The largest prime factor of an odd perfect number exceeds \( 10^6 \).

**Lemma 2.5.** The second largest prime factor of an odd perfect number exceeds \( 10^4 \).

**Lemma 2.6.** The third largest prime factor of an odd perfect number exceeds 100.

3. The algorithm

We assume that \( n \) is an odd perfect number with \( t \) distinct prime factors. Appendix A consists of an extract from Appendix 1 in [5], which shows the application of the algorithm to be described below to a proof that \( t \geq 6 \). Appendix B contains a complete proof of this result, illustrating other features of the algorithm. (In these only, the prime factors of \( n \) are numbered as \( p_1, p_2, \ldots, p_t \), distinct from the use elsewhere of this notation.) It may be worthwhile for the reader to consider the appendixes in conjunction with the following description of the algorithm. In brief, the algorithm may be described as a progressive sieve, or “coin-sorter”, in which the sieve gets finer and finer, so that eventually nothing gets through.

The essence of the algorithm is the factor chain common to many problems concerned with odd perfect numbers. We shall use the terminology of graph theory to describe the branching process. Since \( 3 \mid n \) if \( t \leq 10 \), for our present purposes the even powers of 3 are the roots of the trees. If \( 3^2 \) is an exact divisor of \( n \), then, since \( \sigma(n) = 2n \), \( \sigma(3^2) = 13 \) is a divisor of \( n \), and so the children of the root \( 3^2 \) are labelled with different powers of 13. The first of these is \( 13^1 \), meaning that we assume 13 is an exact divisor of \( n \) (and hence that 13 is the special prime), the second \( 13^2 \), then \( 13^3, 13^5, \ldots \). Each of these possibilities leads to further factorisations and further subtrees. Having terminated all these, by methods to be described, we then assume that \( 3^4 \) is an exact divisor, beginning the second tree, and we continue in this manner. Only Eulerian prime powers are considered, and notice is taken of whether the special prime has been specified earlier in any path.

In the appendixes, successive indentations indicate the nodes.
We distinguish between initial components, which label the nodes and comprise initial primes and initial exponents, and consequent primes, which arise within a tree through factorisation. It is necessary to maintain a count of the total number of distinct initial and consequent primes as they arise within a path, and we let \( k \) be this number.

Often, more than one new prime will arise from a single factorisation. All are included in the count, within \( k \), and, whenever further branching is required, the smallest available consequent prime is used as the new initial prime.

To show that \( t \geq \omega \), say, we build on earlier results which have presumably shown that \( t \geq \omega - 1 \), and we suppose that \( t = \omega - 1 \). If, within any path, we have \( k > \omega - 1 \), then we clearly have a contradiction, and that path is terminated. This is one of a number of possible contradictions that may arise and which terminate a path. Our result will be proved when every path in every tree has been terminated with a contradiction (unless an odd perfect number has been found). The different possible contradictions are indicated with upper case letters.

In the contradiction just mentioned, we have too Many distinct prime factors of \( n \): this is Contradiction M. The same notation is used to indicate that there are too Many occurrences of a single prime; that is, within a path an initial prime has occurred as a consequent prime more times than the initial exponent. (So counts must also be maintained within each path of the occurrences of each initial prime as a consequent prime.) It is necessary to consider all Eulerian exponents \( a \), not just those for which \( a + 1 \) is prime (contrary to the approach adopted in [1] and [2]).

If \( k = \omega - 3 \) but none of these \( k \) primes exceeds 100, then Lemma 2.6 must be (about to be) violated: this is Contradiction P3. If \( k = \omega - 2 \) and none of these primes exceeds 10^4, then Lemma 2.5 is violated: Contradiction P2. Or if, in this case, one exceeds 10^4 but no other exceeds 100, then this is another version of Contradiction P3. If \( k = \omega - 1 \) and none of these primes exceeds 10^6, then Lemma 2.4 is violated: Contradiction P1. In this case, there are the following further possibilities: one prime exceeds 10^6 but no other exceeds 10^4, or one exceeds 10^6, another exceeds 10^4, but no other exceeds 100. These are other versions of contradictions P2 and P3, respectively. (Contradictions P1, P2 and P3 have not been employed in Appendix A. For the illustrative purpose of that appendix, these powerful results mask many other features of the algorithm. This is demonstrated in Appendix B.) These, and some of the other forms of contradiction below, require only counts or comparisons, and no calculations.

At the outset, a number \( B \) is chosen, and then the number of subtrees with a given initial prime \( p \) is bounded by taking as initial components Eulerian powers \( p^a \) with \( p^{a+1} \leq B \). If possible, these trees are continued by factorising \( \sigma(p^a) \). When \( a \) becomes so large that \( p^{a+1} > B \), which may occur with \( a = 0 \), then we write \( q^b \) for \( p^a \) and we have one more subtree with this initial prime; it is distinguished by writing its initial component as \( q^\infty \). This tree must be continued differently. In the first place, the smallest available consequent prime, which is not already an initial prime, is used to begin a new subtree. If no such primes are available, then Lemma 2.2 is used, as described below.

The product of the \( u \) initial components \( p^a \) within a path is the number \( \lambda \). Those initial primes \( q \) with exponents \( \infty \), and all consequent primes which are not initial primes, are the \( v \) prime factors of \( \mu \). If \( k < \omega - 1 \) then there are \( w = \omega - k - 1 \) remaining prime factors of \( n \), still to be found or postulated. These are the prime factors \( r \) of \( v \). The numbers \( u \),
\( v, w \) are not fixed: they vary as the path develops, for example, by taking a consequent prime as another initial prime.

If factorisation can no longer be used to provide further prime factors of \( n \), so, in particular, there are no consequent primes which are not initial primes, then the inequalities of Lemma 2.2 are used. In that lemma, \( \tilde{\mu} \) is taken to be the product of powers \( q^\beta \), where \( q \mid \mu \) and \( \beta \) is given as follows. Let \( b_0 = \min \{ b : q^{b+1} > B \} \). If \( b_0 = 0 \), then we proceed in a manner to be described later. Otherwise, let

\[
\beta = \begin{cases} 
  b_0, & \text{if } b_0 \text{ is even} \ (b_0 > 0), \\
  b_0 + 1, & \text{if } b_0 \text{ is odd},
\end{cases}
\]

with one possible exception. If \( \pi \not\mid \lambda \), and the set \( Q_1 = \{ q : q \equiv b_0 \equiv 1 \pmod{4} \} \) is non-empty, then take \( \beta = b_0 \) for \( q = \min Q_1 \). Values of \( h(p^a) \) and \( h(q^b) \) must be maintained, along with their product. This is the value of \( h(\lambda \tilde{\mu}) \) to be used in Lemma 2.2.

Lemma 2.2 is used to provide an interval, the primes within which are considered in turn as possible divisors of \( v \). If there are no primes within the interval that have not been otherwise considered, then this is Contradiction N. New primes within the interval are taken in increasing order, giving still further factors of \( n \) either through factorisation or through further applications of Lemma 2.2. There will be occasions when no new primes arise through factorisation, all being used earlier in the same path. Then again Lemma 2.2 is used to provide further possible prime factors of \( n \) (or, if \( k = \omega - 1 \), we may have found an odd perfect number). This lemma specifically supplies the smallest possible candidate for the remaining primes; a still Smaller prime subsequently arising through factorisation gives us Contradiction S.

We also denote by \( q \) any consequent prime which is not an initial prime, and, for such primes, we let \( Q_2 = \{ q : q \equiv 1 \pmod{4} \} \). Then, for such primes, we let \( \beta = 2 \) with the possible exception that, considering all primes \( q \), we let \( \beta = b_0 \) or 1, as relevant, for \( q = \min(Q_1 \cup Q_2) \), if this set is nonempty. Again, the value of \( h(\lambda \tilde{\mu}) \), defined as before, must be maintained. If this value exceeds 2, we have an Abundant divisor of \( n \), and the path is terminated: Contradiction A. This may well occur with \( k < \omega - 1 \). Values of \( H(q^b) \) must also be maintained. These, multiplied with the values of \( h(p^a) \), give values of \( h(\lambda)H(\mu) \).

If this is less than 2 and \( k = \omega - 1 \) then, for all possible values of the exponents \( b \), the postulated number \( n \) is Deficient: Contradiction D.

Contradictions A and D are in fact contradictions of Lemma 2.1. If, on the other hand, we have a postulated set of prime powers \( p^a \) and \( q^b \), for which \( h(\lambda \tilde{\mu}) \leq 2 \leq h(\lambda)H(\mu) \), then (2.1) is satisfied and we have candidates for an odd perfect number. If \( v = w = 0 \), so that we are talking only of known powers \( p^a \), then their product is an odd perfect number. Our sieving principle arises when \( v > 0 \).

In every such case where we have a set of prime powers satisfying (2.1), with \( v > 0 \), we increase the value of \( B \) and investigate that set more closely. With the larger value of \( B \), some prime powers shift from \( \mu \) to \( \lambda \), and allow further factorisation, often resulting quickly in Contradiction M or S. The value of \( h(\tilde{\mu}) \) increases, so the interval given by Lemma 2.2 shortens, and hopefully the case which led to our increasing \( B \) is no longer exceptional, or Contradiction A or D may be enforced. In that case, we revert to the earlier value of \( B \) and continue from where we were. Alternatively, it may be necessary to
increase $B$ still further, and later perhaps further again. When $w = 0$, since $h(\bar{\mu}) \rightarrow H(\mu)$ as $B \rightarrow \infty$, such cases must eventually be dispensed with, one way or the other.

We will show now that if $b_0 = 0$, the case omitted above, then $B$ may immediately be increased, because, most often, the situation of the preceding paragraph will prevail. Suppose we have a node labelled $q^\infty$, where $q > B$. If $q \mid \sigma(p^a)$, then $p^{a+1} > \sigma(p^a) \geq q > B$, so this node could not arise from its parent node by factorisation. Therefore, we must have $q = r_1$ following an application of Lemma 2.2. Assume that, in that application of Lemma 2.2, we had $w = 1$. Rearranging (2.4) and (2.3), we have, respectively,

\[ h(\lambda \bar{\mu}_1) \left(1 + \frac{1}{r_1}\right) \leq 2 \quad \text{and} \quad 2 < h(\lambda)H(\mu) \frac{r_1}{r_1 - 1}. \tag{3.1} \]

These show that we now have a number, namely $\lambda \mu'$ with $\mu' = \mu r_1^{c_1}$, satisfying (2.1), with $v > 0$, and these are the conditions for increasing $B$. (If we are entitled to assume an exponent 2 for $r_1$, so that (2.6) may be used instead of (2.4), then we obtain a correspondingly adjusted form for the left-hand inequality in (3.1).) Suppose now that, following the earlier application of Lemma 2.2, we still have $w \geq 1$. Possibly, $h(\lambda)H(\mu r_1) = h(\lambda)H(\mu') \geq 2$, so that we may argue as above. Otherwise, we may use Lemma 2.2 to find bounds for $r_2$ (as done in Corollary 2.3, in part). Since $r_2 > B$, we may then use the preceding argument, and this idea may be repeated as necessary. Thus, $B$ must eventually be increased, and there is no harm in doing so immediately.

We summarise the various contradictions:

A. There is an Abundant divisor.
B. The number is Deficient.
C. There are too Many prime factors, or a single prime has occurred too Many times.
D. There is No new prime within the given interval.
P1. There is no prime factor exceeding $10^6$.
P2. There is at most one prime factor exceeding $10^4$.
P3. There are at most two prime factors exceeding 100.
Π. None of the primes can be the special prime.
S. There is a prime Smaller than the purportedly smallest remaining prime.

One of these, Contradiction Π, was not discussed previously. It has not been used in either appendix. Within any path with $k = \omega - 1$, if $\pi$ is not implicit in an initial component and if there is no prime $q \equiv 1 \pmod{4}$, then Contradiction Π may be invoked.

There is one other point regarding Contradiction A. It is possible to use the tables of primitive abundant numbers given by Dickson [9], and corrected by Ferrier [10] and Herzog [18], to create a look-up file. Within any path, if the product $\lambda \bar{\mu}$ is a multiple of one of the numbers in the file, then $n$ is abundant: Contradiction A. There are about 500 numbers in Dickson’s tables, but these may be adjusted “upwards” to have Eulerian components, in which case many would coincide (though they may no longer be primitive). All of these contain three or four components only. The microfiche supplement to Kishore [23] contains those primitive abundant numbers $N$ with five components and satisfying $h(N) < 2 + 2/10^{10}$, and could perhaps be used similarly. Other such numbers could be appended to the list as they are found, so that subsequent runs of the program would be speeded up.
4. Further comments

To show that \( t \geq 6 \), as was done in Appendix 1 in [5], it seemed after some experimenting that taking \( B = 10^6 \) and multiplying this as necessary by 10s (to \( 10^9 \)) was the “natural” way to proceed. The present Appendix A contains an extract from that proof. There are annotations in the appendix, and in Appendix B, designed to further explain the algorithm and to give other points of interest.

The algorithm has been almost fully automated to show that \( t \geq 7 \), with completion not required if the salient features only are desired. It was apparent that the bound \( B \) needed to be extended to \( 10^{21} \) (incrementing the exponent on 10 from 6 in steps of 3).

Let us consider now the application of the algorithm to showing that \( t \geq 9 \). This requires showing that there is no odd perfect number with exactly eight distinct prime factors. It seems likely that the worst case, the hardest to dispense with, will be the path involving \( 3^\infty \), \( 5^\infty \), \( 17^\infty \), \( 257^\infty \), ..., for the reason that \( \prod_{i=0}^{\infty} F_i/(F_i - 1) = 2 \), where \( F_i = 2^{2^i} + 1 \) is the \( i \)th Fermat number. Since \( F_7 < 3.5 \times 10^{38} \), numbers larger than this should not be encountered, and this is in our favour. Also working for us is the fact that \( F_5 \), \( F_6 \) and \( F_7 \) are composite. As suggested by Appendix 1 in [5], however, and easy enough to see in general, there will arise an inequality of the approximate form \( F_6 < p_7 < 2F_6 \). This interval includes about \( \frac{1}{2}F_6 \approx 10^{19} \) odd numbers. Only the primes in this interval are required, so that the number of possibilities may be reduced somewhat by an incremental wheel. Here and elsewhere, it would not be necessary to check each number for primality. Certainly, probabilistic tests would be sufficient, with subsequent testing if necessary. There is even some interest in specifically not testing for primality in the first place, according to Leech’s comment in Guy [12] on “spoof” odd perfect numbers.

Since we must have \( n > 10^{300} \), if \( t = 8 \) then there is a component of \( n \) exceeding \( 10^{300}/8 = 10^{37.5} \). This suggests that \( B \) may have to be taken to about this size, but this is a very rough argument.

Going back a bit, it is the inequality \( F_6 < p_7 < 2F_6 \) which may well render the algorithm impractical for showing that \( t \geq 9 \) (and certainly it was the analogous case for the implementation to show \( t \geq 7 \) where \( B \) needed to be raised to around \( 10^{21} \)). By the same token, the comments above should show that proving \( t \geq 8 \) in this way is within reach. That itself would be a useful exercise in demonstrating how the problem may be mechanised, as opposed to the very deep and intricate arguments of [13] and [27], for example. Furthermore, the algorithm, when assuming \( t = 8 \), could certainly be used to whittle down the possibilities and show perhaps that such an odd perfect number must be divisible by 3, 5, 17 and 257. That would be a very worthwhile result, and would allow specific investigations using a number of known results (in [13] and [27], again) designed for handling prime divisors which are Fermat primes.

We end with some further observations concerning Lemma 2.2.

In [24], Kishore proved that, if \( p_1 < p_2 < \cdots < p_6 \) are the six smallest prime factors of an odd perfect number \( n = \prod_{i=1}^{6} p_i^{a_i} \), then

\[
p_i < 2^{2^{i-1}}(t - i + 1),
\]
for \( i = 2, \ldots, 6 \). This required his lemma, that
\[
\frac{2}{2^i - 1} \geq p_i < \frac{2 + \frac{1}{2}(t - 6)}{2 - \frac{1}{2}} = xt - 6x + 2y.
\]

Then
\[
p_i < (2^t - 1)(t - i + 1),
\]
for \( i = 2, \ldots, 6 \).

There are various ways to seek improvements to Lemma 2.2, although we do not comment on whether the increased computational complexity of the results would be worth the few cases that might be saved in the application of the algorithm.

We observe first that we may show that
\[
(4.1)
\]
which is an improvement on (2.3) when \( w \geq 3 \). For this, we note that \( r_i \geq r_{i-1} + 2 \geq r_{i-2} + 4 \geq \cdots \geq r_1 + 2t - 2 \), for \( i = 1, \ldots, w \). Then
\[
(4.1)
\]
Now (4.1) follows, in the same fashion as in the derivation of (2.3).

This leads to a corresponding, if unattractive, further improvement of Kishore’s result, above. For \( i = 2 \), it states that
\[
p_2 < \frac{18t - 4}{7}.
\]

More could be said in this area, using the results of Cohen [4] and Cohen and Hendy [6].

Note added in proof

Since this paper was accepted for publication, the results in Hagis and Cohen [16] has been improved by P.M. Jenkins (Math. Comp., to appear), as indicated in his title: “Odd perfect numbers have a prime factor exceeding 10^7”.
Appendix A

Extract from a proof that an odd perfect number has at least six distinct prime factors. The subtrees based on $3^2$, $3^4$, $3^6$ and $3^8$ have been dealt with, and we show here the subtree based on $3^{10}$, and the beginning of the subsequent subtree. The number on the left is the current value of $B$.

The complete proof requires a printout of around 20 pages. For illustrative purposes, this proof does not make use of Contradictions P1, P2 and P3. Compare this with the proof in Appendix B.

\[ 10^6 \quad 3^{10} \Rightarrow 23, 3851 \]
\[ 23^2 \Rightarrow 7, 79 \]
\[ 23^\infty \]
\[ 3851^\infty : 3.6 < p_4 < 8.3 \]
\[ 5^1 \Rightarrow 3 : 15.9 < p_5 < 16,992 \]
\[ 5^2 \Rightarrow 31 \]
\[ 5^4 \Rightarrow 11, 71 \]
\[ 5^5 \Rightarrow 32, 7, 31 \]
\[ 5^6 \Rightarrow 19531 \]
\[ 5^\infty : 49.9 < p_5 < 50.93 \]
\[ 7^2 \Rightarrow 3, 19 \]
\[ 7^4 \Rightarrow 2801 \]
\[ 7^6 \Rightarrow 29, 4733 \]
\[ 7^\infty : 10.7 < p_5 < 11.8 \]
\[ 11^2 \]
\[ 3^\infty : 2.9 < p_2 < 13 \]
\[ 5^1 \Rightarrow 3 : 9.9 < p_5 < 28 \]
\[ 11^2 \Rightarrow 7, 19 \]
\[ 11^4 \Rightarrow 5, 3221 \]
\[ 3221^\infty : 103.09 < p_5 < 103.2 \]
\[ 11^\infty : 99.9 < p_4 < 199 \]

\[ 1 \] We use $\Rightarrow$ when the number or numbers on the right follow by factorisation (here, $\sigma(3^{10}) = 23 \cdot 3851$), and we use a colon otherwise.
\[ 2 \] The number $3^{10} \cdot 7^a \cdot 23^b \cdot 3851^c$ is Deficient, for all $a, b, c \geq 2$. Contradiction $\Pi$ could also have been used here.
\[ 3 \] Since $23^5 > B = 10^6$, we write $23^\infty$ to indicate $23^b$, for any $b > 4$. A consequent prime, 3851, is available as a new initial prime.
\[ 4 \] No further consequent primes are available as initial primes, so Lemma 2.2, with $w = 2$, and (2.4) are used to obtain inequalities of the form $L < p_4 < R$.
\[ 5 \] We start with $p_4 = 5$, so we take $\pi = 5$. Lemma 2.2 is used again, this time with (2.6), to obtain inequalities of the form $L < p_5 < R$. There are No primes in the interval.
\[ 6 \] The number $3^{10} \cdot 5^2 \cdot 23^4 \cdot 31^2 \cdot 3851^2$ is Abundant.
\[ 7 \] There are too Many, namely six, distinct prime factors.
Inequalities (2.1) are satisfied with $B = 10^6$, so a “finer” sieve, with $B = 10^7$, is tried to decide this case. Possibly (in principle), we have found the five prime factors of an odd perfect number, but their exponents are not yet all determined. Movements to a finer sieve or back to the original sieve are highlighted with arrows and dots.

These are the specific components or primes being investigated. Other primes in the interval for $p_5$ may be similarly and simultaneously treated.

Since $10^6 < 3 \cdot 7 < 10^7$, $3^2$ must now be investigated. Since 797161 is not one of the above primes, including any in the interval for $p_5$, we have too Many prime factors.

With the finer sieve, this case is now void. We revert to the earlier situation.

The factor 7 is Smaller than 11.

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8 Inequalities (2.1) are satisfied with $B = 10^6$, so a “finer” sieve, with $B = 10^7$, is tried to decide this case. Possibly (in principle), we have found the five prime factors of an odd perfect number, but their exponents are not yet all determined. Movements to a finer sieve or back to the original sieve are highlighted with arrows and dots.

9 These are the specific components or primes being investigated. Other primes in the interval for $p_5$ may be similarly and simultaneously treated.

10 Since $10^6 < 3 \cdot 7 < 10^7$, $3^2$ must now be investigated. Since 797161 is not one of the above primes, including any in the interval for $p_5$, we have too Many prime factors.

11 With the finer sieve, this case is now void. We revert to the earlier situation.

12 The factor 7 is Smaller than 11.
Appendix B

Complete proof that an odd perfect number has at least six distinct prime factors. The only contradictions used here are P1, P2 and P3, and $B = 10^6$ is sufficient throughout.

$10^6 \Rightarrow 3^2 \Rightarrow 13$
$13^1 \Rightarrow 7 \text{ P3}$
$13^2 \Rightarrow 3, 61 \text{ P3}$
$13^3 \Rightarrow 30941$
$30941^\infty : p_4 < 8.2 \text{ P3}^{13}$
$13^\infty : p_3 < 11.8 \text{ P3}$

$^{13}$ The upper bound in Lemma 2.2 should be calculated first; there may be no need to calculate the lower bound.
3^4 \Rightarrow 11^2
11^2 \Rightarrow 7, 19 \ p_2
11^4 \Rightarrow 5, 3221 \ p_2
11^\infty \colon p_3 < 14.9 \ p_3
3^6 \Rightarrow 1093
1093^\infty \colon 3.005 < p_3 < 10.1, p_4 < 31.3 \ p_2 \ \footnote{An example of the use of Corollary 2.3.}
3^8 \Rightarrow 13, 757
13^1 \Rightarrow 7 \ p_2
13^2 \Rightarrow 3, 61 \ p_2
13^3 \Rightarrow 30941
\begin{align*}
757^1 & \Rightarrow 579 \ p_1 \\
757^\infty & \\
30941^\infty & \colon p_5 < 5.4 \ p_1
\end{align*}
13^\infty
\begin{align*}
757^1 & \Rightarrow 579 \ p_2 \\
757^\infty & \colon p_3 < 9.8 \ p_2
\end{align*}
3^{10} \Rightarrow 23, 3851
23^2 \Rightarrow 7, 79 \ p_1
23^\infty
\begin{align*}
3851^\infty & \colon p_4 < 8.3 \ p_2 \\
3^\infty & \colon 2.9 < p_2 < 13, p_3 < 46 \ p_3
\end{align*}

References

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