A Proof of the Convexity of the Range of a Nonatomic Vector Measure Using Linear Inequalities

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ABSTRACT

This note shows how a standard result about linear inequality systems can be used to give a simple proof of the fact that the range of a nonatomic vector measure is convex, a result that is due to Liapounoff.

We denote the set of reals by R and the set of rationals by Q. Also, we let $\| \|_1$ be the l_1 norm on R^k, i.e., for every $a \in R^k$, $\|a\|_1 = \sum_{j=1}^{k} a_j$. A *measurable space* is a pair (X, Σ) where Σ is a subset of the power set $P(X)$ of X which contains the empty set and is closed under countable unions and under complements with respect to X. In particular, in this case the sets in Σ

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will be called *measurable*. A parametric family of measurable sets whose index set I is a subset of the reals, say $\{S_t : t \in I\}$, is called *increasing* if $S_{t'} \supseteq S_t$ for every $t, t' \in I$ with $t' \geqslant t$.

Throughout the remainder of this note let (X, Σ) be a given measurable space. A function $\mu : \Sigma \to R^k$ is called a *k-vector measure* on (X, Σ) if $\mu(\varnothing) = 0$ and for every countable collection of pairwise disjoint sets $S_1, S_2, \ldots$ in Σ one has $\mu(\bigcup_{i=1}^{\infty} S_i) = \sum_{i=1}^{\infty} \mu(S_i)$, where the series converges absolutely; in particular, in this case we call the integer k the *dimension* of the vector measure μ. A *scalar measure* is a vector measure with dimension 1. For a k-vector measure μ and $j \in \{1, \ldots, k\}$, we denote by μ_j the scalar measure defined for $S \in \Sigma$ by $\mu_j(S) = [\mu(S)]_j$. A vector measure μ is called *nonnegative* if $\mu(S) \geqslant 0$ for all measurable sets S, and *nonatomic* if every measurable set S with $\mu(S) \neq 0$ has a measurable subset T with $\mu(T) \neq 0$ and $\mu(T) \neq \mu(S)$.

The purpose of this note is to use a standard result about linear inequality systems to give a simple proof of the following theorem due to Liapounoff; see Liapounoff (1940), Halmos (1948) and Lindenstrauss (1966), for example.

THEOREM 1. *Let μ be a nonnegative, nonatomic vector measure. Then the set $\{\mu(S) : S \in \Sigma\}$ is convex.*

The following fact will be used in our proof. It can be established by a simple argument using Zorn's lemma. A more elementary proof that relies only on countable induction is given in the Appendix for the sake of completeness.

PROPOSITION 1. *Let μ be a nonnegative, nonatomic, scalar measure, and let S be a measurable set. Then there exists an increasing parametric family of measurable subsets of S, $\{S_t : t \in ([0, \mu(S)) \cap Q) \cup \{\mu(S)\}\}$, such that $\mu(S_t) = t$ for every $t \in ([0, \mu(S)) \cap Q) \cup \{\mu(S)\}$.*

Proof of Theorem 1. Suppose that S_0 and S_1 are measurable sets and $0 < \beta < 1$. We will show that for some measurable set T, $\mu(T) = (1 - \beta)\mu(S_0) + \beta\mu(S_1)$. We first note that it suffices to consider the case where S_0 and S_1 are disjoint, for otherwise let $S_0' \equiv S_0 \setminus (S_0 \cap S_1)$ and $S_1' \equiv S_1 \setminus (S_0 \cap S_1)$, and construct a set T' with $\mu(T') = (1 - \beta)\mu(S_0') + \beta\mu(S_1')$. Then $T \equiv T' \cup (S_0 \cap S_1)$ will satisfy $\mu(T) = (1 - \beta)\mu(S_0) + \beta\mu(S_1)$.

Let k be the dimension of μ, and let $\|\mu\|_1$ be the scalar measure defined by $\|\mu\|_1 \equiv \sum_{j=1}^{k} \mu_j$, i.e., for every measurable set U, $\|\mu\|_1(U) = \sum_{j=1}^{k} \mu_j(U) = \|\mu(U)\|_1$. Now, fix $i \in \{0, 1\}$ and let $I_i \equiv ([0, \|\mu\|_1(S_i)] \cap Q) \cup \{\|\mu\|_1(S_i)\}$. By applying Proposition 1 to $\|\mu\|_1$ and the set S_i we can construct an increasing parametric family of measurable subsets of S_i, say

$\{S_{it}: t \in I_i\}$, such that $\|\mu\|_1(S_{it}) = t$ for every $t \in I_i$. By taking set differences corresponding S_{it}'s we can define for each $p = 1, 2, \ldots$ *finite* partitions $\Pi_i^{(p)}$ of S_i into measurable sets such that $\|\mu\|_1 \leqslant 2^{-p}$ for every $U \in \Pi_i^{(p)}$; further, if $p' > p$ then $\Pi_i^{(p')}$ is a refinement of $\Pi_i^{(p)}$, i.e., all sets in $\Pi^{(p')}$ are subsets of sets in $\Pi^{(p)}$. Let $\Pi^{(p)} \equiv \Pi_0^{(p)} \cup \Pi_1^{(p)}$. In particular, $\Pi^{(p)}$ is a partition of $S_0 \cup S_1$.

Consider linear inequality systems with variables $\{x_U : U \in \Pi^{(1)}\}$ given by

$$\sum_{U \in \Pi^{(1)}} \mu(U) x_U = (1 - \beta)\mu(S_0) + \beta\mu(S_1), \tag{1}$$

$$0 \leqslant x_U \leqslant 1 \qquad \text{for all} \quad U \in \Pi^{(1)} \tag{2}$$

Let $a'^{(1)}$ be the vector in $R^{\Pi^{(1)}}$ defined by setting $a_U'^{(1)} = 1 - \beta$ for the sets $U \in \Pi^{(1)}$ that are included in S_0, and $a_U'^{(1)} = \beta$ for (the distinct class of) sets $U \in \Pi^{(1)}$ that are included in S_1. Evidently, the vector $a'^{(1)}$ satisfies (1)–(2); hence, this system is feasible. It now follows from a standard result about linear inequalities (see Chavatal, 1983, Theorem 3.4, p. 42) that there exists a solution $a^{(1)}$ of (1)–(2) such that at most k of the $a_U^{(1)}$'s are neither 0 nor 1.

For $p = 1, 2, \ldots,$ we inductively consider linear inequality systems with variables $\{x_U : U \in \Pi^{(p)}\}$ and construct special solutions $a^{(p)} \in R^{\Pi^{(p)}}$ of these systems having the property that at most k of the $a_U^{(p)}$'s are neither 0 nor 1. The first system is given by (1)–(2), and its special solution $a^{(1)}$ was constructed in the above paragraph. Next assume that for some $p \in \{2, 3, \ldots\}$, $a^{(p-1)} \in R^{\Pi^{(p-1)}}$ was constructed, and consider the pth system consisting of

$$\sum_{U \in \Pi^{(p)}} \mu(U) x_U = (1 - \beta)\mu(S_0) + \beta\mu(S_1), \tag{3}$$

$$0 \leqslant x_U \leqslant 1 \qquad \text{for all} \quad U \in \Pi^{(p)}, \tag{4}$$

$$x_U = 0 \qquad \text{for} \quad U \in \Pi^{(p)} \quad \text{for which the unique set } V \in \Pi^{(p-1)} \text{ containing } U \text{ has } a_V^{(p-1)} = 0, \tag{5}$$

$$x_U = 1 \qquad \text{for} \quad U \in \Pi^{(p)} \quad \text{for which the unique set } V \in \Pi^{(p-1)} \text{ containing } U \text{ has } a_U^{(p-1)} = 1. \tag{6}$$

Consider the vector $a'^{(p)} \in R^{\Pi^{(p)}}$ where for each set $U \in \Pi^{(p)}$ we let $a_U'^{(p)} = a^{(p-1)}V$ for the unique set $V \in R^{\Pi^{(p-1)}}$ which contains U. Evidently, $a'^{(p)}$ satisfies (3)–(6), and therefore this system is feasible. Another application of the standard result about linear inequalities shows that there exists a solution $a^{(p)}$ of (3)–(6) such that at most k of the $a_U^{(p)}$'s are neither 0 nor 1, completing our inductive construction.

For $p = 1, 2, \ldots$ let $T^{(p)} \equiv \bigcup\{U : U \in \Pi^{(p)}$ and $a_u^{(p)} = 1\}$. Then (6) assures that $T^{(1)}, T^{(2)}, \ldots,$ is an increasing sequence of sets. Further, for $p = 1, 2, \ldots,$ from Equations (3)–(6), the fact that at most k of the $a_U^{(p)}$'s are neither 0 nor 1, and the fact that $\|\mu\|_1(U) \leqslant 2^{-p}$ for every $U \in \Pi^{(p)}$ we see that

$$k\, 2^{-p} \geqslant \left\| \sum_{U \in \Pi^{(p)}} \mu(U) x_U - \mu(T^{(p)}) \right\|_1$$

$$= \|(1-\beta)\mu(S_0) + \beta\mu(S_1) - \mu(T^{(p)})\|_1. \qquad (7)$$

Let $T \equiv \bigcup_{p=1}^{\infty} T^{(p)}$. Then T is a measurable set, and (7) shows that $(1-\beta)\mu(S_0) + \beta\mu(S_1) = \lim_{p\to\infty} \mu(T^{(p)}) = \mu(T)$, completing the proof. ■

Our construction has some resemblance to the approach of Arstein (1980). But we obtain underlying extreme points from elementary arguments about linear inequalities over finite dimensional spaces, whereas he uses analytical arguments over finite dimensional spaces.

APPENDIX

The purpose of this appendix is to provide a proof of Proposition 1 that relies only on countable induction. We note that a simpler proof is available, establishing a stronger variant of the asserted result, by using Zorn's lemma.

We first establish two elementary lemmas.

LEMMA 1. *Let μ be a nonnegative, nonatomic, scalar measure, and let S be a measurable set with $\mu(S) > 0$. Then for every $\varepsilon > 0$ there exists a measurable subset T of S with $0 < \mu(T) < \varepsilon$.*

Proof. The nonatomicity of μ implies that S has a measurable subset T' with $0 \neq \mu(T)$ and $\mu(T') \neq \mu(S)$. Let T_1 be the set with smaller μ measure among T' and $S \setminus T'$. Then T_1 is a measurable subset of S with $0 <$

$\mu(T_1) \leqslant 2^{-1}\mu(S)$. By recursively iterating this procedure we can construct a sequence $T_1, T_2, \ldots$ of measurable subsets of S such that for each $k = 1, 2, \ldots$ we have $0 < \mu(T_k) \leqslant 2^{-1}\mu(T_{k-1}) \leqslant 2^{-k}\mu(S)$. The conclusion of the lemma now follows by selecting $T = T_k$ for any positive integer k with $2^{-k}\mu(S) \leqslant \varepsilon$. ■

LEMMA 2. *Let μ be a nonnegative, nonatomic, scalar measure, and let S be a measurable set with $\mu(S) \geqslant 0$. Then for each $0 \leqslant \alpha \leqslant \mu(S)$ there exists a measurable subset T of S with $\mu(T) = \alpha$.*

Proof. The conclusion of our lemma is trivial if $\alpha = 0$ or if $\alpha = \mu(S)$, by selecting $T = \varnothing$ or $T = S$, respectively. Next assume that $0 < \alpha < \mu(S)$. Let

$$\alpha_1 \equiv \sup\{\mu(U) : U \text{ is a measurable subset of } S \text{ and } \mu(U) \leqslant \alpha\}; \quad (8)$$

in particular, Lemma 1 shows that $\alpha_1 > 0$. The definition of α_1 assures that one can select a measurable subset U_1 of S satisfying

$$2^{-1}\alpha_1 \leqslant \mu(U_1) \leqslant \alpha. \quad (9)$$

We continue by inductively selecting scalars $\alpha_2, \alpha_3, \ldots$ and measurable subsets $U_2, U_3, \ldots$ of S such that

$$\alpha_k \equiv \sup\left\{\mu(U) : U \text{ is a measurable subset of } S, \right.$$

$$\left. U \cap \left(\bigcup_{j=1}^{k-1} U_j = \varnothing\right), \text{ and } \mu(U) + \sum_{j=1}^{k-1} \mu(U_j) \leqslant \alpha\right\} \quad (10)$$

and

$$U_k \cap \left(\bigcup_{j=1}^{k-1} U_j\right) = \varnothing, \qquad \mu(U_k) + \sum_{j=1}^{k-1} \mu(U_j) \leqslant \alpha, \quad \text{and}$$

$$\mu(U_k) \geqslant 2^{-1}\alpha_k. \quad (11)$$

We note that this inductive construction is possible because the selection of U_k in the kth step assures that $\sum_{j=1}^{k} \mu(U_j) \leqslant \alpha$; hence, $\mu(\varnothing) = 0$ is in the set over which the supremum in (10) in the $(k+1)$st step is taken. Let $T \equiv \bigcup_{j=1}^{\infty} U_j$. Then T is a measurable subset of S and, as the U_k's are pairwise disjoint, $\mu(T) = \sum_{j=1}^{\infty} \mu(U_j) \leqslant \alpha$.

We will next show that $\mu(T) = \alpha$. Suppose that $\mu(T) \neq \alpha$, i.e., $\varepsilon \equiv \alpha - \mu(T) > 0$. Then $\mu(S \setminus T) = \mu(S) - \mu(T) \geqslant \alpha - \mu(T) > 0$; hence, by Lemma 1, there is a measurable subset U of $S \setminus T$ with $0 < \mu(U) < \varepsilon$. Now, for each $k = 1, 2, \dots,$

$$\varnothing = U \cap T \supseteq U \cap \left(\bigcup_{j=1}^{k-1} U_j \right) \quad \text{and}$$

$$\mu(U) + \sum_{j=1}^{k-1} \mu(U_j) \leqslant \mu(U) + \mu(T) < \varepsilon + \mu(T) = \alpha; \qquad (12)$$

hence, $\mu(U)$ is an element in the set over which the supremum in (10) is taken, implying that $\alpha_k \geqslant \mu(U)$ and therefore $\mu(U_k) \geqslant 2^{-1}\alpha_k \geqslant 2^{-1}\mu(U) > 0$. Thus, we get a contradiction to the absolute convergence of $\sum_{j=1}^{\infty} \mu(U_j)$ which proves that, indeed, $\mu(T) = \alpha$. ■

Proof of Proposition 1

We start by arbitrarily ordering the rationals in the interval $[0, \mu(S))$, say $q(0), q(1), \dots,$ where $q(0) = 0$. Also, let $S_0 \neq \varnothing$ and $S_{\mu(S)} \equiv S$. We will use an inductive argument for our construction. Suppose that $S_{q(0)}, S_{q(1)}, \dots, S_{q(k)}$ have been selected such that $\{S_{q(i)} : i \in \{0, 1, \dots, k\}\} \cup \{S_{\mu(S)}\}$ is an increasing family of measurable subsets of S and $\mu(S_{q(i)}) = q(i)$ for $i \in \{0, 1, \dots, k\}$. Let $q_* \equiv \max\{q(i) : i = 0, 1, \dots, k,\ q(i) < q(k+1)\}$ [the set over which this max is taken is nonempty because it contains $q(0) = 0$], and let $q^* \equiv \min\{\{q(i) : i = 0, 1, \dots, k \text{ and } q(i) > q(k+1)\} \cup \{\mu(S)\}\}$. Then $q_* < q(k+1) < q^*$ and $\mu(S_{q^*} \setminus S_{q_*}) = q^* - q_*$. Thus, $0 < q(k+1) - q_* < q^* - q_*$, and Lemma 2 can be applied to select a measurable subset U of $S_{q^*} \setminus S_{q_*}$ such that $\mu(U) = q(k+1) - q_*$. Letting $S(k+1) \equiv S(q_*) \cup U$, we have that $\mu(S_{q_*} \cup U) = \mu(S_{q_*}) + \mu(U) = q_* + \mu(U) = q(k+1)$. Further, we have that $\{s_{q(i)} : i \equiv \{0, 1, \dots, k+1\}\} \cup \{S_{\mu(S)}\}$ is an increasing family of measurable sets. The above inductive construction establishes the conclusion of Proposition 1. ■

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