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Global existence and nonexistence of solution for Cauchy problem of multidimensional double dispersion equations

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A B S T R A C T

In this paper we consider the Cauchy problem of multidimensional generalized double dispersion equations $u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u = \Delta f(u)$, where $f(u) = a|u|^p$. By potential well method we prove the existence and nonexistence of global weak solution without establishing the local existence theory. And we derive some sharp conditions for global existence and lack of global existence solution.

1. Introduction

In this paper we study the Cauchy problem of multidimensional generalized double dispersion equations

$$u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u = \Delta f(u), \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $f(u)$ satisfies

$$\text{(H)} \quad \begin{cases} f(u) = a|u|^p, \quad a > 0, \quad \frac{n+2}{n} \leqslant p < \frac{n+2}{n-2} \text{ for } n \geqslant 3; \\ \qquad\qquad\qquad\qquad 1 \leqslant p < \infty \text{ for } n = 1, 2. \end{cases}$$

Considering the possibility of energy exchange through lateral surfaces of the waveguide in the physical study of nonlinear wave propagation in waveguide, the longitudinal displacement $u(x,t)$ of the rod satisfies the following double dispersion equation (DDE) [14,1,2]

$$u_{tt} - u_{xx} = \frac{1}{4}\left(6u^2 + au_{tt} - bu_{xx}\right)_{xx} \tag{1.3}$$

and the general cubic DDE (CDDE)

$$u_{tt} - u_{xx} = \frac{1}{4}\left(cu^3 + 6u^2 + au_{tt} - bu_{xx} + du_t\right)_{xx}, \tag{1.4}$$

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where a, b and c are positive constants. In [3,15] Chen and Wang studied the initial–boundary value problem and the Cauchy problem of the following generalized double dispersion equation which includes above Eq. (1.4) as special cases

$$u_{tt} - u_{xx} - au_{xxtt} + bu_{xxxx} - du_{xxt} = f(u)_{xx}, \tag{1.5}$$

where $a > 0$, $b > 0$ and d are constants. For the case $f'(s) \geqslant C$ (bounded below) they proved the existence of global solutions. And they also shown the nonexistence of the global solution under some other conditions to deal with the global well-posedness of (1.4). Recently in [11,12] for the nonlinear term $f(u)$ satisfying more general conditions than both convex function and $f(u) = |u|^p$, the Cauchy problem and the initial–boundary value problem for a class of generalized double dispersion equations

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} = f(u)_{xx},$$

were studied respectively. For both of above problems the authors obtained the invariant sets and sharp conditions of global existence of solution by introducing a family of potential wells. However, for the multidimensional cases it is still open to give the local and global well-posedness for the Cauchy problem (1.1), (1.2). Most recently, in [13] the authors considered the Cauchy problem of the multidimensional nonlinear evolution equation

$$u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u - k\Delta u_t = \Delta f(u), \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.6}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \tag{1.7}$$

where k is an arbitrary real constant. They first gave the existence of local solution. Then by some estimates of local solution they attempted to obtained the existence of global solution. Also they showed the lack of global existence. Note that in [13] in order to obtain the global existence of solution for problem (1.6), (1.7) the authors requested

$$\text{(A)} \quad \begin{cases} \text{(i)} \quad F(u) \geqslant 0, \quad \forall u \in \mathbb{R}; \quad \text{or} \\ \text{(ii)} \quad f'(u) \text{ is bounded below}, \quad \forall u \in \mathbb{R}. \end{cases}$$

So the global existence of solution for Cauchy problem of Eq. (1.4) and its multidimensional generalization were solved. However for Eq. (1.3) we have $f(u) = \frac{3}{2}u^2$, $F(u) = \frac{1}{2}u^3$ and $f'(u) = 3u$, which do not satisfy (A). In general, for $f(u) = a|u|^p$, $a > 0$, $p > 1$ we have $F(u) = \frac{a}{p+1}|u|^p u$ and $f'(u) = ap|u|^{p-2}u$, which do not satisfy (A) too. Therefore the results of [13] are not applicable for Eqs. (1.3) and (1.1) with $f(u) = a|u|^p$.

In this paper we study the Cauchy problem (1.1), (1.2), where $f(u)$ satisfies (H). We aim to give the sufficient and necessary conditions for the global existence of solution for problem (1.1), (1.2). In order to do this, we employ the variational methods and the existence of the invariant sets of solutions [4–10]. Throughout this paper we denote $L^p(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$ by L^p and H^s respectively, with the norm $\|\cdot\|_p = \|\cdot\|_{L^p(\mathbb{R}^n)}$, $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^n)}$ and the inner product $(u,v) = \int_{\mathbb{R}^n} uv\,dx$. We also define the space

$$H = \bigl\{u \in H^1 \bigm| (-\Delta)^{-\frac{1}{2}} u \in L^2\bigr\},$$

with the norm

$$\|u\|_H^2 = \|u\|_{H^1}^2 + \bigl\|(-\Delta)^{-\frac{1}{2}} u\bigr\|_{L^2}^2$$

and the space

$$L = \bigl\{u \in L^2 \bigm| (-\Delta)^{-\frac{1}{2}} u \in L^2\bigr\},$$

with the norm

$$\|u\|_L^2 = \|u\|_{L^2}^2 + \bigl\|(-\Delta)^{-\frac{1}{2}} u\bigr\|_{L^2}^2,$$

where $(-\Delta)^{-\alpha} v = \mathscr{F}^{-1}(|\xi|^{-2\alpha}\mathscr{F} v)$, $\mathscr{F}$ and $\mathscr{F}^{-1}$ are the Fourier transformation and the inverse Fourier transformation respectively.

Lemma 1.1. *H is dense in L.*

Proof. This lemma follows from the fact that H^1 is dense in L^2. □

We like to give the following conclusions, which will be used in the future discussions.

Proposition 1.2. *(See [13].) Assume that $s > \frac{n}{2}$, $u_0 \in H^s$, $u_1 \in H^{s-1}$, $f \in C^{[s]+1}(R)$. Then problem (1.1), (1.2) admits a unique local solution $u(t) \in C([0,T), H^s) \cap C^1([0,T), H^{s-1})$, where T is the maximal existence time of $u(t)$. Moreover if*

$$\sup_{t\in[0,T)} \left(\|u(t)\|_{H^s} + \|u_t(t)\|_{H^{s-1}}\right) < \infty,$$

then $T = \infty$.

From Proposition 1.2 we see that if we take $s = 1$, $u_0 \in H^1$, $u_1 \in L^2$ in Proposition 1.2, since $n < 2s = 2$ we can only know the local existence and global existence of solution for $n = 1$. In other words, for the case $s = 1$, $u_0 \in H^1$, $u_1 \in L^2$, it is impossible for us to derive any local and global well-posedness results for problem (1.1), (1.2) for $n > 1$. One maybe thinks this is just a special example. But indeed this restriction goes along with s perpetually. One can easily check that if $s = 2$, we can only discuss the case $n \leqslant 3$ if we like to lean on Proposition 1.2. So this local well-posedness theory does limit us to find more general conclusions. In the present paper, we aim to find another way to get more general results about the well-posedness problem for (1.1), (1.2). And certainly Proposition 1.2 is abandoned in this paper.

Definition 1.3. We call $u(x,t)$ a weak solution of problem (1.1), (1.2) on $\mathbb{R}^n \times [0,T)$, if $u \in L^\infty(0,T;H^1)$, $u_t \in L^\infty(0,T;L)$ satisfying

(i)
$$\left((-\Delta)^{-\frac{1}{2}} u_t, (-\Delta)^{-\frac{1}{2}} v\right) + (u_t, v) + \int_0^t \left((u,v) + (\nabla u, \nabla v) + \left(f(u), v\right)\right) \mathrm{d}\tau$$
$$= \left((-\Delta)^{-\frac{1}{2}} u_1, (-\Delta)^{-\frac{1}{2}} v\right) + (u_1, v), \quad \forall v \in H,\ \forall t \in [0,T). \tag{1.8}$$

(ii) There holds $u(x,0) = u_0(x)$ in H^1; and

$$u_t(x,0) = u_1(x) \quad \text{in } L. \tag{1.9}$$

(iii)
$$E(t) \leqslant E(0) \quad \text{for all } t \in [0,T), \tag{1.10}$$

where

$$E(t) = \frac{1}{2}\left(\left\|(-\Delta)^{-\frac{1}{2}} u_t\right\|^2 + \|u_t\|^2 + \|u\|_{H^1}^2\right) + \int_{\mathbb{R}^n} F(u)\,\mathrm{d}x,$$

$$F(u) = \int_0^u f(s)\,\mathrm{d}s.$$

2. Preliminary results

In this section we will give some preliminary results in order to state the main results of this paper. First for problem (1.1), (1.2) we introduce the following functionals

$$J(u) = \frac{1}{2}\left(\|u\|^2 + \|\nabla u\|^2\right) + \int_{\mathbb{R}^n} F(u)\,\mathrm{d}x = \frac{1}{2}\|u\|_{H^1}^2 + \int_{\mathbb{R}^n} F(u)\,\mathrm{d}x,$$

$$I(u) = \|u\|_{H^1}^2 + \int_{\mathbb{R}^n} u f(u)\,\mathrm{d}x,$$

$$d = \inf_{u\in\mathcal{N}} J(u),$$

$$\mathcal{N} = \left\{u \in H^1 \;\middle|\; I(u) = 0,\ \|u\|_{H^1} \neq 0\right\},$$

where $f(u)$ satisfies the assumption

$$(\mathrm{H}_0) \quad \begin{cases} f(u) = a|u|^p, \quad a > 0, \quad 1 < p \leqslant \frac{n+2}{n-2} \text{ for } n \geqslant 3; \\ \qquad\qquad\qquad\qquad\qquad 1 < p < \infty \text{ for } n = 1, 2. \end{cases}$$

Clearly if $f(u)$ satisfies (H_0), the above functionals can be well-defined on $H^1(\mathbb{R}^n)$.

Lemma 2.1. *Let $f(u)$ satisfy (H_0), $u \in H^1$ and*

$$\varphi(\lambda) = -\frac{1}{\lambda}\int_{\mathbb{R}^n} uf(\lambda u)\,\mathrm{d}x.$$

Assume that $\int_{\mathbb{R}^n} uf(u)\,\mathrm{d}x < 0$. *Then*

(i) $\varphi(\lambda)$ *is increasing on* $0 < \lambda < \infty$.
(ii) $\lim_{\lambda\to 0}\varphi(\lambda) = 0$, $\lim_{\lambda\to+\infty}\varphi(\lambda) = +\infty$.

Proof. This lemma follows from

$$\varphi(\lambda) = -\frac{1}{\lambda}\int_{\mathbb{R}^n} uf(\lambda u)\,\mathrm{d}x = -\lambda^{p-1}\int_{\mathbb{R}^n} uf(u)\,\mathrm{d}x. \qquad \square$$

Lemma 2.2. *Let* $f(u)$ *satisfy* (H_0), $u \in H^1$. *Then*

(i) $\lim_{\lambda\to 0} J(\lambda u) = 0$.
(ii) $I(\lambda u) = \lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u)$, $\forall\lambda > 0$.

Furthermore if $\int_{\mathbb{R}^n} uf(u)\,\mathrm{d}x < 0$, *then*

(iii) $\lim_{\lambda\to+\infty} J(\lambda u) = -\infty$.
(iv) *In the interval* $0 < \lambda < \infty$ *there exists a unique* $\lambda^* = \lambda^*(u)$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u)\bigg|_{\lambda=\lambda^*} = 0.$$

(v) $J(\lambda u)$ *is increasing on* $0 < \lambda \leqslant \lambda^*$, *decreasing on* $\lambda^* \leqslant \lambda < \infty$ *and takes the maximum at* $\lambda = \lambda^*$.
(vi) $I(\lambda u) > 0$ *for* $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ *for* $\lambda^* < \lambda < \infty$ *and* $I(\lambda^* u) = 0$.

Proof. Parts (i)–(iii) are obvious.

Note that $\int_{\mathbb{R}^n} uf(u)\,\mathrm{d}x \neq 0$ implies $\|u\|_{H^1} \neq 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u) = \lambda\big(\|u\|_{H_1}^2 - \varphi(\lambda)\big), \tag{2.1}$$

which together with Lemma 2.1 gives parts (iv) and (v).

Part (vi) follows from part (ii) and (2.1). $\square$

Lemma 2.3. *Let* $f(u)$ *satisfy* (H_0), $u \in H^1$. *Then*

(i) *If* $0 < \|u\|_{H^1} < r_0$, *then* $I(u) > 0$;
(ii) *If* $I(u) < 0$, *then* $\|u\|_{H^1} > r_0$;
(iii) *If* $I(u) = 0$ *and* $\|u\|_{H^1} \neq 0$, *i.e.* $u \in \mathcal{N}$, *then* $\|u\|_{H^1} \geqslant r_0$, *where*

$$r_0 = \left(\frac{1}{aC_*^{p+1}}\right)^{\frac{1}{p-1}}, \quad C_* = \sup_{u\in H^1,\,u\neq 0}\frac{\|u\|_{p+1}}{\|u\|_{H^1}}.$$

Proof. (i) If $0 < \|u\|_{H^1} < r_0$, then $I(u) > 0$ follows from

$$\int_{\mathbb{R}^n} uf(u)\,\mathrm{d}x \leqslant \int_{\mathbb{R}^n}\big|uf(u)\big|\,\mathrm{d}x = a\|u\|_{p+1}^{p+1} \leqslant aC_*^{p+1}\|u\|_{H^1}^{p+1}$$
$$= aC_*^{p+1}\|u\|_{H^1}^{p-1}\|u\|_{H^1}^2 < \|u\|_{H^1}^2.$$

(ii) If $I(u) < 0$, then $\|u\|_{H^1} > r_0$ follows from

$$\|u\|_{H^1}^2 < -\int_{\mathbb{R}^n} uf(u)\,\mathrm{d}x \leqslant aC_*^{p+1}\|u\|_{H^1}^{p-1}\|u\|_{H^1}^2.$$

(iii) If $I(u) = 0$ and $\|u\|_{H^1} \neq 0$, then we have

$$\|u\|_{H^1}^2 = -\int_{\mathbb{R}^n} uf(u)\,dx \leqslant aC_*^{p+1}\|u\|_{H^1}^{p-1}\|u\|_{H^1}^2,$$

which together with $\|u\|_{H^1} \neq 0$ gives $\|u\|_{H^1} \geqslant r_0$. □

Lemma 2.4. *Let $f(u)$ satisfy* (H_0). *Then*

(i)
$$d \geqslant d_0 = \frac{p-1}{2(p+1)}\left(\frac{1}{aC_*^{p+1}}\right)^{\frac{2}{p-1}}. \tag{2.2}$$

(ii) *If $u \in H^1$ and $I(u) < 0$, then*

$$I(u) < (p+1)\big(J(u) - d\big). \tag{2.3}$$

Proof. (i) For any $u \in \mathcal{N}$, by Lemma 2.3 we have $\|u\|_{H^1} \geqslant r_0$ and

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|_{H^1}^2 + \int_{\mathbb{R}^n} F(u)\,dx = \frac{1}{2}\|u\|_{H^1}^2 + \frac{1}{p+1}\int_{\mathbb{R}^n} uf(u)\,dx \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|_{H^1}^2 + \frac{1}{p+1}I(u) = \frac{p-1}{2(p+1)}\|u\|_{H^1}^2 \geqslant \frac{p-1}{2(p+1)}r_0^2, \end{aligned}$$

which gives (2.2).

(ii) Let $u \in H^1$ and $I(u) < 0$, then from Lemma 2.2 it follows that there exists a λ^* such that $0 < \lambda^* < 1$ and $I(\lambda^* u) = 0$. From the definition of d we have

$$\begin{aligned} d \leqslant J(\lambda^* u) &= \frac{1}{2}\|\lambda^* u\|_{H^1}^2 + \int_{\mathbb{R}^n} F(\lambda^* u)\,dx \\ &= \frac{1}{2}\|\lambda^* u\|_{H^1}^2 + \frac{1}{p+1}\int_{\mathbb{R}^n} \lambda^* u f(\lambda^* u)\,dx \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|\lambda^* u\|_{H^1}^2 + \frac{1}{p+1}I(\lambda^* u) \\ &= \frac{p-1}{2(p+1)}\|\lambda^* u\|_{H^1}^2 = \lambda^{*2}\frac{p-1}{2(p+1)}\|u\|_{H^1}^2 \\ &< \frac{p-1}{2(p+1)}\|u\|_{H^1}^2. \end{aligned}$$

From this and

$$J(u) = \frac{p-1}{2(p+1)}\|u\|_{H^1}^2 + \frac{1}{p+1}I(u)$$

we get

$$d < \frac{p-1}{2(p+1)}\|u\|_{H^1}^2 = J(u) - \frac{1}{p+1}I(u),$$

which gives (2.3). □

Now we define two subsets of $H^1(\mathbb{R}^n)$ which will be proved to be invariant under the flow generated by problem (1.1), (1.2). Set

$$\begin{aligned} W &= \big\{u \in H^1 \;\big|\; I(u) > 0,\ J(u) < d\big\} \cup \{0\}; \\ V &= \big\{u \in H^1 \;\big|\; I(u) < 0,\ J(u) < d\big\}, \end{aligned}$$

and

$$\begin{aligned} W' &= \big\{u \in H^1 \;\big|\; I(u) > 0\big\} \cup \{0\}; \\ V' &= \big\{u \in H^1 \;\big|\; I(u) < 0\big\}. \end{aligned}$$

3. Invariant sets

This section will show that the subsets W and V of $H^1(\mathbb{R}^n)$ are invariant under the flow of (1.1), (1.2).

Theorem 3.1. *Let $f(u)$ satisfy* (H_0), $u_0 \in H^1$, $u_1 \in L$. *Assume that $E(0) < d$. Then both sets W' and V' are invariant under the flow of* (1.1), (1.2).

Proof. We only prove the invariance of W', the proof for the invariance of V' is similar. Let $u(t)$ be any weak solution of problem (1.1), (1.2) with $u_0 \in W'$, T be the maximal existence time of $u(t)$. Next we prove that $u(t) \in W'$ for $0 < t < T$. Arguing by contradiction we assume there is a $\bar{t} \in (0, T)$ such that $u(\bar{t}) \notin W'$. According to the continuity of $I(u(t))$ with respect to t, there is a $t_0 \in (0, T)$ such that $u(t_0) \in \partial W'$. From the definition of W' and (i) of Lemma 2.3 we have $B_{r_0} \subset W'$, $B_{r_0} = \{u \in H^1 \mid \|u\|_{H^1} < r_0\}$. Hence we know $0 \notin \partial W'$. So $u(t_0) \in \partial W$ reads $I(u(t_0)) = 0$ with $\|u(t_0)\|_{H^1} \neq 0$. The definition of d tells $J(u(t_0)) \geqslant d$, which contradicts

$$\frac{1}{2}\left(\left\|(-\Delta)^{-\frac{1}{2}} u_t\right\|^2 + \|u_t\|^2\right) + J(u) \leqslant E(0) < d, \quad 0 \leqslant t < T. \tag{3.1}$$

So the prove can be completed. □

And the following corollary can be concluded from the above Theorem 3.1.

Corollary 3.2. *Let $f(u)$ satisfy* (H_0), $u_0 \in H^1$, $u_1 \in L$. *Assume that $E(0) < d$. Then*

(i) *All weak solutions of problem* (1.1), (1.2) *belong to W provided $I(u_0) > 0$ or $\|u_0\|_{H^1} = 0$.*
(ii) *All weak solutions of problem* (1.1), (1.2) *belong to V provided $I(u_0) < 0$.*

Next we consider the case $E(0) \leqslant 0$, which is a special case of the energy restriction $E(0) < d$.

Corollary 3.3. *Let $f(u)$ satisfy* (H_0), $u_0 \in H^1$, $u_1 \in L$. *Assume that $E(0) < 0$ or $E(0) = 0$, $\|u_0\|_{H^1} \neq 0$. Then all weak solutions of problem* (1.1), (1.2) *belong to V.*

Proof. Let $u(t)$ be any weak solution of problem (1.1), (1.2) with $E(0) < 0$ or $E(0) = 0$, $\|u_0\|_{H^1} \neq 0$, T be the maximal existence time of $u(t)$. From

$$\frac{1}{2}\left(\left\|(-\Delta)^{-\frac{1}{2}} u_1\right\|^2 + \|u_1\|^2\right) + \frac{p-1}{2(p+1)}\|u_0\|_{H^1}^2 + \frac{1}{p+1} I(u_0) = E(0),$$

we see that if $E(0) < 0$ or $E(0) = 0$ with $\|u_0\|_{H^1} \neq 0$, then $I(u_0) < 0$. Hence from Corollary 3.2 we get $u(t) \in V$ for $0 \leqslant t < T$. □

4. Existence and nonexistence of global solution

In this section we study the existence and nonexistence of global solution for problem (1.1), (1.2). And we give some sharp conditions for global well-posedness. These results are independent of the local existence theory, so they are not restricted by the conditions for the local solution.

Let $u_0 \in H$, $u_1 \in L$, $\{w_j\}_{j=1}^{\infty}$ be a basis function system in H. Construct the approximate solutions of problem (1.1), (1.2)

$$u_m(x,t) = \sum_{j=1}^{m} g_{jm}(t) w_j(x), \quad m = 1, 2, \ldots \tag{4.1}$$

satisfying

$$\left((-\Delta)^{-\frac{1}{2}} u_{mtt}, (-\Delta)^{-\frac{1}{2}} w_s\right) + (u_{mtt}, w_s) + (u_m, w_s) + (\nabla u_m, \nabla w_s) + \left(f(u_m), w_s\right) = 0,$$
$$s = 1, 2, \ldots, m, \tag{4.2}$$

$$u_m(x,0) = \sum_{j=1}^{m} a_{jm} w_j(x) \to u_0(x) \quad \text{in } H, \tag{4.3}$$

$$u_{mt}(x,0) = \sum_{j=1}^{m} b_{jm} w_j(x) \to u_1(x) \quad \text{in } L. \tag{4.4}$$

Multiplying (4.2) by $g'_{sm}(t)$ and summing for s we get

$$\frac{\mathrm{d}}{\mathrm{d}t}E_m(t)=0,$$

and

$$E_m(t)=E_m(0), \tag{4.5}$$

where

$$E_m(t)=\frac{1}{2}\left(\left\|(-\Delta)^{-\frac{1}{2}}u_{mt}\right\|^2+\|u_{mt}\|^2+\|u_m\|^2+\|\nabla u_m\|^2\right)+\int_{\mathbb{R}^n}F(u_m)\,\mathrm{d}x,$$

$$F(u)=\int_0^u f(s)\,\mathrm{d}s. \tag{4.6}$$

Lemma 4.1. *Let $f(u)$ satisfy* (H)*, $u_0\in H$, $u_1\in L$. Then $F(u_0)\in L^1$. And for the approximate solutions u_m defined by* (4.1)–(4.4) *there holds $E_m(0)\to E(0)$ as $m\to\infty$, where*

$$E(0)=\frac{1}{2}\left(\left\|(-\Delta)^{-\frac{1}{2}}u_1\right\|^2+\|u_1\|^2+\|u_0\|^2+\|\nabla u_0\|^2\right)+\int_{\mathbb{R}^n}F(u_0)\,\mathrm{d}x.$$

Proof. First from (H) we have

$$\left|F(u)\right|\leqslant\frac{a}{p+1}|u|^{p+1},\quad \forall u\in\mathbb{R},$$

where $2(1+\frac{1}{n})\leqslant p+1<\frac{2n}{n-2}$ for $n\geqslant 3$, $2<p+1<\infty$ for $n=1,2$. From this and $u_0\in H^1$ we get $F(u_0)\in L^1$.

From (4.3), (4.4) we get that as $m\to\infty$

$$\left\|(-\Delta)^{-\frac{1}{2}}u_{mt}(0)\right\|^2+\left\|u_{mt}(0)\right\|^2+\left\|u_m(0)\right\|^2+\left\|\nabla u_m(0)\right\|^2\to\left\|(-\Delta)^{-\frac{1}{2}}u_1\right\|^2+\|u_1\|^2+\|u_0\|^2+\|\nabla u_0\|^2.$$

Next we prove

$$\int_{\mathbb{R}^n}F\big(u_m(0)\big)\,\mathrm{d}x\to\int_{\mathbb{R}^n}F(u_0)\,\mathrm{d}x\quad\text{as } m\to\infty.$$

In fact we have

$$\left|\int_{\mathbb{R}^n}F\big(u_m(0)\big)\,\mathrm{d}x-\int_{\mathbb{R}^n}F(u_0)\,\mathrm{d}x\right|\leqslant\int_{\mathbb{R}^n}\left|f(\varphi_m)\right|\left|u_m(0)-u_0\right|\mathrm{d}x\leqslant\left\|f(\varphi_m)\right\|_r\left\|u_m(0)-u_0\right\|_q,\quad 1<q,r<\infty,$$
$$\frac{1}{q}+\frac{1}{r}=1,$$

where $\varphi_m=u_0+\theta(u_m(0)-u_0)$, $0<\theta<1$.

(i) If $n\geqslant 3$. Choose $q=\frac{2n}{n-2}$, $r=\frac{2n}{n+2}$. We have

$$\left\|u_m(0)-u_0\right\|_q\leqslant C\left\|u_m(0)-u_0\right\|_{H^1}\to 0\quad\text{as } m\to\infty,$$

$$\left\|f(\varphi_m)\right\|_r^r=\int_{\mathbb{R}^n}\big(a|\varphi_m|^p\big)^r\,\mathrm{d}x=A\|\varphi_m\|_{pr}^{pr}.$$

From (H) we have $2\leqslant pr\leqslant\frac{2n}{n-2}$, hence $\|f(\varphi_m)\|_r\leqslant C$.

(ii) If $n=1,2$. Choose $q=r=2$, then we have

$$\left\|u_m(0)-u_0\right\|_q\leqslant\left\|u_m(0)-u_0\right\|\to 0\quad\text{as } m\to\infty,$$

$$\left\|f(\varphi_m)\right\|_r^r=\left\|f(\varphi_m)\right\|^2\leqslant A\|\varphi_m\|_{2p}^{2p}.$$

Since $2<2p<\infty$, we get $\|f(\varphi_m)\|_r<C$.

Thus for above two cases we always have

$$\int_{\mathbb{R}^n} F\big(u_m(0)\big)\,\mathrm{d}x \to \int_{\mathbb{R}^n} F(u_0)\,\mathrm{d}x \quad \text{as } m\to\infty$$

and $E_m(0)\to E(0)$ as $m\to\infty$. □

Corollary 4.2. *Let $f(u)$, u_0 and u_1 satisfy the conditions of Lemma* 4.1. *Assume that $E(0)<d$. Then $E_m(0)<d$ for sufficiently large m.*

Lemma 4.3. *Let $f(u)$ satisfy* (H), *$u_0\in H$, $u_1\in L$, $E(0)<d$. Assume that $I(u_0)>0$ or $\|u_0\|_{H^1}=0$, i.e. $u_0\in W'$. Then for the approximate solutions u_m defined by* (4.1)–(4.4) *there holds $u_m\in W'$ for $0\leqslant t<\infty$ and sufficiently large m.*

Proof. Arguing by contradiction, we assume that there exists a $\bar t>0$ such that $u_m(\bar t)\notin W'$ for some sufficiently large m. Then by the continuity of $I(u_m)$ with respect to t it follows that there exists a $t_0>0$ such that $u_m(t_0)\in\partial W'$. On the other hand, from the definition of W' we have $0\notin\partial W'$. Hence $I(u_m(t_0))=0$ and $\|u_m(t_0)\|_{H^1}\neq 0$ for some sufficiently large m. From the definition of d we get $J(u_m(t_0))\geqslant d$, which contradicts (by (4.5))

$$E_m(t)=\frac{1}{2}\Big(\big\|(-\Delta)^{-\frac12}u_{mt}\big\|^2+\|u_{mt}\|^2\Big)+J(u_m)=E_m(0)<d,\quad 0\leqslant t<\infty \tag{4.7}$$

for sufficiently large m. □

Corollary 4.4. *Under the assumption of Lemma* 4.3 *we have*

$$\|u_m\|_{H^1}^2\leqslant\frac{2(p+1)}{p-1}d,\qquad \big\|(-\Delta)^{-\frac12}u_{mt}\big\|^2+\|u_{mt}\|^2<2d,\quad 0\leqslant t<\infty, \tag{4.8}$$

for sufficiently large m.

Proof. From (4.7) we get that for sufficiently large m there holds

$$\frac12\Big(\big\|(-\Delta)^{-\frac12}u_{mt}\big\|^2+\|u_{mt}\|^2\Big)+\frac{p-1}{2(p+1)}\|u_m\|_{H^1}^2+\frac{1}{p+1}I(u_m)=E_m(0)<d,\quad 0\leqslant t<\infty,$$

which together with $u_m(t)\in W'$ gives (4.8). □

Theorem 4.5 *(Global existence). Let $f(u)$ satisfy* (H), *$u_0\in H$, $u_1\in L$. Assume that $E(0)<d$, $I(u_0)>0$ or $\|u_0\|_{H^1}=0$. Then problem* (1.1), (1.2) *admits a global weak solution $u(t)\in L^\infty(0,\infty;H^1)$ with $u_t(t)\in L^\infty(0,\infty;L)$ and $u(t)\in W$ for $0\leqslant t<\infty$.*

Proof. For problem (1.1)–(1.2), construct the approximate solutions $u_m(x,t)$ by (4.1)–(4.4). From Corollary 4.4 it follows that $\{u_m\}$ in $L^\infty(0,\infty;H^1)$; $\{u_{mt}\}$ in $L^\infty(0,\infty;L)$ are bounded respectively. Moreover by an argument similar to that in the proof of Lemma 4.1 we can get $\{f(u_m)\}$ are bounded in $L^\infty(0,\infty;L^r)$, where r is defined in the proof of Lemma 4.1. Hence there exists a u and a subsequence $\{u_\nu\}$ of $\{u_m\}$ such that as $\nu\to\infty$

$u_\nu\to u$ in $L^\infty\big(0,\infty;H^1\big)$ weakly star and a.e. in $Q=\mathbb{R}^n\times[0,\infty)$;

$u_{\nu t}\to u_t$ in $L^\infty(0,\infty;L)$ weakly star;

$f(u_\nu)\to\chi=f(u)$ in $L^\infty\big(0,\infty;L^r\big)$ weakly star.

Integrating (4.2) with respect to t from 0 to t we get

$$\begin{aligned}&\Big((-\Delta)^{-\frac12}u_{mt},(-\Delta)^{-\frac12}w_s\Big)+(u_{mt},w_s)+\int_0^t\Big((u_m,w_s)+(\nabla u_m,\nabla w_s)+\big(f(u_m),w_s\big)\Big)\,\mathrm{d}\tau\\&\quad=\Big((-\Delta)^{-\frac12}u_{mt}(0),(-\Delta)^{-\frac12}w_s\Big)+\big(u_{mt}(0),w_s\big).\end{aligned} \tag{4.9}$$

Let $m=\nu\to\infty$ in (4.9) we obtain

$$\begin{aligned}&\Big((-\Delta)^{-\frac12}u_t,(-\Delta)^{-\frac12}w_s\Big)+(u_t,w_s)+\int_0^t\Big((u,w_s)+(\nabla u,\nabla w_s)+\big(f(u),w_s\big)\Big)\,\mathrm{d}\tau\\&\quad=\Big((-\Delta)^{-\frac12}u_1,(-\Delta)^{-\frac12}w_s\Big)+(u_1,w_s),\quad\forall s\end{aligned}$$

and

$$\big((-\Delta)^{-\frac{1}{2}}u_t,(-\Delta)^{-\frac{1}{2}}v\big)+(u_t,v)+\int_0^t\big((u,v)+(\nabla u,\nabla v)+\big(f(u),v\big)\big)\,d\tau$$
$$=\big((-\Delta)^{-\frac{1}{2}}u_1,(-\Delta)^{-\frac{1}{2}}v\big)+(u_1,v),\quad \forall v\in H,\ \forall t\in[0,\infty).$$

On the other hand, from (4.3) and (4.4) we have $u(x,0)=u_0(x)$ in H^1 and $u_t(x,0)=u_1(x)$ in L.

Next we prove that above u satisfies (1.10).

Note that the embedding $H^1\hookrightarrow L^{p+1}$ is compact under the condition $2(1+\frac{1}{n})\leqslant p+1<\frac{2n}{n-2}$ for $n\geqslant 3$; $2<p+1<\infty$ for $n=1,2$. Thus from $\{u_m\}$ is bounded in $L^\infty(0,\infty;H^1)$ it follows that there exists a subsequence $\{u_\nu\}$ of $\{u_m\}$ such that as $\nu\to\infty$

$$u_\nu\to u\quad \text{in } L^{p+1} \text{ strongly for each } t>0.$$

Hence

$$\left|\int_{\mathbb{R}^n}F(u_\nu)\,dx-\int_{\mathbb{R}^n}F(u)\,dx\right|\leqslant\int_{\mathbb{R}^n}\big|f(v_\nu)\big||u_\nu-u|\,dx\leqslant\big\|f(v_\nu)\big\|_{\bar r}\|u_\nu-u\|_{\bar q},$$

where $\bar q=p+1$, $\bar r=\frac{p+1}{p}$, $u_\nu=u+\theta(u_\nu-u)$, $0<\theta<1$. From

$$\|u_\nu-u\|_{\bar q}\to 0\quad \text{as } \nu\to\infty,$$

and

$$\big\|f(v_\nu)\big\|_{\bar r}^{\bar r}=\int_{\mathbb{R}^n}\big(a|v_\nu|^p\big)^{\bar r}\,dx=a^{\frac{p+1}{p}}\|v_\nu\|_{p+1}^{p+1}\leqslant C$$

we get

$$\int_{\mathbb{R}^n}F(u_\nu)\,dx\to\int_{\mathbb{R}^n}F(u)\,dx\quad \text{as } \nu\to\infty.$$

Hence

$$\begin{aligned}\frac{1}{2}\big(\big\|(-\Delta)^{-\frac{1}{2}}u_t\big\|^2+\|u_t\|^2+\|u\|_{H^1}^2\big)&\leqslant\frac{1}{2}\Big(\liminf_{\nu\to\infty}\big\|(-\Delta)^{-\frac{1}{2}}u_{\nu t}\big\|^2+\liminf_{\nu\to\infty}\|u_{\nu t}\|^2+\liminf_{\nu\to\infty}\|u_\nu\|_{H^1}^2\Big)\\&\leqslant\frac{1}{2}\liminf_{\nu\to\infty}\big(\big\|(-\Delta)^{-\frac{1}{2}}u_{\nu t}\big\|^2+\|u_{\nu t}\|^2+\|u_\nu\|_{H^1}^2\big)\quad\big(\text{by (4.5)}\big)\\&=\liminf_{\nu\to\infty}\Big(E_\nu(0)-\int_{\mathbb{R}^n}F(u_\nu)\,dx\Big)=\lim_{\nu\to\infty}\Big(E_\nu(0)-\int_{\mathbb{R}^n}F(u_\nu)\,dx\Big)\\&=E(0)-\int_{\mathbb{R}^n}F(u)\,dx\end{aligned}$$

which gives $E(t)\leqslant E(0)$ for $0\leqslant t<\infty$. Therefore above $u(x)$ is a global weak solution of problem (1.1), (1.2). Finally from Corollary 3.2 we get $u(t)\in W$ for $0\leqslant t<\infty$. □

Corollary 4.6. *Under the conditions of Theorem* 4.5*, for the global weak solution of problem* (1.1), (1.2) *given in Theorem* 4.5 *we further have*

$$u(t)\in L^\infty(0,T;H),\quad \forall T>0.$$

Proof. From

$$(-\Delta)^{-\frac{1}{2}}u=\int_0^t(-\Delta)^{-\frac{1}{2}}u_\tau\,d\tau+(-\Delta)^{-\frac{1}{2}}u_0,\quad 0\leqslant t<\infty$$

we get

$$\begin{aligned}\left\|(-\Delta)^{-\frac{1}{2}}u\right\| &\leqslant \int_0^t \left\|(-\Delta)^{-\frac{1}{2}}u_\tau\right\| \mathrm{d}\tau + \left\|(-\Delta)^{-\frac{1}{2}}u_0\right\| \\ &\leqslant T \max_{0\leqslant t\leqslant T}\left(\left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|\right) + \left\|(-\Delta)^{-\frac{1}{2}}u_0\right\|, \quad 0 \leqslant t \leqslant T,\end{aligned}$$

which gives

$$(-\Delta)^{-\frac{1}{2}}u \in L^\infty\left(0, T; L^2\right), \quad \forall T > 0$$

and

$$u(t) \in L^\infty(0, T; H), \quad \forall T > 0. \qquad \square$$

Corollary 4.7. *If in Theorem* 4.5 *the assumption "$E(0) < d$, $I(u_0) > 0$ or $\|u_0\|_{H^1} = 0$" is replaced by "$0 < E(0) < d$, $\|u_0\|_{H^1} < r_0$", where r_0 is defined in Lemma* 2.3. *Then the conclusion of Theorem* 4.5 *still holds and*

$$\|u\|_{H^1}^2 \leqslant \frac{2(p+1)}{p-1}E(0), \qquad \left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|^2 + \|u_t\|^2 \leqslant 2E(0), \quad 0 \leqslant t < \infty.$$

Proof. Note that $\|u_0\|_{H^1} < r_0$ implies $0 < \|u_0\|_{H^1} < r_0$ or $\|u_0\|_{H^1} = 0$. If $0 < \|u_0\|_{H^1} < r_0$, Lemma 2.3 immediately gives $I(u_0) > 0$. Then the proof is completed by Theorem 4.5 and

$$\frac{1}{2}\left(\left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|^2 + \|u_t\|^2\right) + \frac{p-1}{2(p+1)}\|u\|_{H^1}^2 + \frac{1}{p+1}I(u) = E(t) \leqslant E(0) < d, \quad 0 \leqslant t < \infty. \qquad \square$$

Next we prove the global nonexistence of weak solution for problem (1.1), (1.2).

Theorem 4.8 *(Global nonexistence in V). Let $f(u)$ satisfy* (H), *$u_0 \in H$, $u_1 \in L$. Assume that $E(0) < d$, $I(u_0) < 0$. Then problem* (1.1), (1.2) *does not admits any global weak solution.*

Proof. Let $u(t) \in L^\infty(0, \infty; H^1)$ with $u_t(t) \in L^\infty(0, \infty; L)$ be any weak solution of problem (1.1), (1.2), T be the maximal existence time of $u(t)$. Now we need to show $T < \infty$. Arguing by contradiction, we suppose that $T = +\infty$. Let

$$\phi(t) = \left\|(-\Delta)^{-\frac{1}{2}}u\right\|^2 + \|u\|^2.$$

From the proof of Corollary 4.6 (note that in the proof of Corollary 4.6 the assumptions $u \in L^\infty(0, \infty; H^1)$, $u_t \in L^\infty(0, \infty; L)$ and $u_0 \in H$ are required only), it follows that $\phi(t)$ is well defined for $0 \leqslant t < \infty$. Then we have

$$\begin{aligned}\dot\phi(t) &= 2\left((-\Delta)^{-\frac{1}{2}}u_t, (-\Delta)^{-\frac{1}{2}}u\right) + 2(u_t, u), \\ \ddot\phi(t) &= 2\left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|^2 + 2\|u_t\|^2 + 2\left((-\Delta)^{-\frac{1}{2}}u_{tt}, (-\Delta)^{-\frac{1}{2}}u\right) + 2(u_{tt}, u) \\ &= 2\left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|^2 + 2\|u_t\|^2 + 2\left((-\Delta)^{-1}u_{tt}, u\right) + 2(u_{tt}, u) \\ &= 2\left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|^2 + 2\|u_t\|^2 - 2\left(\|u\|_{H^1}^2 + \int_{-\infty}^{\infty} uf(u)\,\mathrm{d}x\right) \\ &= 2\left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|^2 + 2\|u_t\|^2 - 2I(u). \end{aligned} \tag{4.10}$$

From Schwartz inequality we get

$$\begin{aligned}\left(\dot\phi(t)\right)^2 &= 4\left(\left((-\Delta)^{-\frac{1}{2}}u, (-\Delta)^{-\frac{1}{2}}u_t\right) + (u, u_t)\right)^2 \\ &= 4\left(\left((-\Delta)^{-\frac{1}{2}}u, (-\Delta)^{-\frac{1}{2}}u_t\right)^2 + (u, u_t)^2 + 2\left((-\Delta)^{-\frac{1}{2}}u, (-\Delta)^{-\frac{1}{2}}u_t\right)(u, u_t)\right) \\ &\leqslant 4\left(\left\|(-\Delta)^{-\frac{1}{2}}u\right\|^2\left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|^2 + \|u\|^2\|u_t\|^2 + \left\|(-\Delta)^{-\frac{1}{2}}u\right\|^2\|u_t\|^2 + \|u\|^2\left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|^2\right)\end{aligned}$$

which gives

$$\left(\dot\phi(t)\right)^2 \leqslant 4\phi(t)\left(\left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|^2 + \|u_t\|^2\right).$$

Hence

$$\phi(t)\ddot{\phi}(t) - \frac{p+3}{4}\big(\dot{\phi}(t)\big)^2 \geqslant \phi(t)\big(-(p+1)\big(\big\|(-\Delta)^{-\frac{1}{2}}u_t\big\|^2 + \|u_t\|^2\big) - 2I(u)\big).$$

From (1.10) we get

$$-(p+1)\big(\big\|(-\Delta)^{-\frac{1}{2}}u_t\big\|^2 + \|u_t\|^2\big) \geqslant 2(p+1)\big(J(u) - E(0)\big) > 2(p+1)\big(J(u) - d\big).$$

So by Theorem 3.1 and (2.3) we have

$$\phi(t)\ddot{\phi}(t) - \frac{p+3}{4}\big(\dot{\phi}(t)\big)^2 \geqslant 2\phi(t)\big((p+1)\big(J(u) - d\big) - I(u)\big) > 0$$

and

$$\big(\phi^{-\alpha}(t)\big)'' = \frac{-\alpha}{\phi(t)^{\alpha+2}}\big(\phi(t)\ddot{\phi}(t) - (\alpha+1)\big(\dot{\phi}(t)\big)^2\big) < 0, \quad \alpha = \frac{p-1}{4},\ 0 < t < \infty. \tag{4.11}$$

On the other hand, from (4.10) and (2.3) we get

$$\ddot{\phi}(t) \geqslant -2I(u) > 2(p+1)\big(d - J(u)\big) \geqslant 2(p+1)\big(d - E(0)\big) = \delta_0 > 0,$$
$$\dot{\phi}(t) \geqslant \delta_0 t + \dot{\phi}(0), \quad 0 < t < \infty.$$

Hence there exists a $t_0 \geqslant 0$ such that $\dot{\phi}(t) > \dot{\phi}(t_0) > 0$ for $t > t_0$ and

$$\phi(t) > \dot{\phi}(t_0)(t - t_0) + \phi(t_0) \geqslant \dot{\phi}(t_0)(t - t_0), \quad t_0 < t < \infty.$$

Therefore there exits a $t_1 > 0$ such that $\phi(t_1) > 0$ and $\dot{\phi}(t_1) > 0$. From this and (4.11) it follows that there exists a $T_1 > 0$ such that

$$\lim_{t \to T_1} \phi^{-\alpha}(t) = 0$$

and

$$\lim_{t \to T_1} \phi(t) = +\infty, \tag{4.12}$$

which contradicts $T = +\infty$. So we prove the nonexistence of global weak solutions. □

From above Theorem 4.8 and Corollary 3.3 we can conclude

Corollary 4.9. *Let $f(u)$ satisfy* (H), $u_0 \in H$, $u_1 \in L$. *Assume that $E(0) < 0$ or $E(0) = 0$, $\|u_0\|_{H^1} \neq 0$. Then problem* (1.1), (1.2) *does not admits any global weak solution.*

From Theorems 4.5 and 4.8 we can obtain a sharp condition for existence and nonexistence of global weak solution for problem (1.1), (1.2) as follows.

Theorem 4.10. *Let $f(u)$, u_0 and u_1 be same as those in Theorem* 4.5. *Assume that $E(0) < d$. Then when $I(u_0) > 0$ problem* (1.1), (1.2) *admits a global weak solution*; *and when $I(u_0) < 0$ the problem* (1.1), (1.2) *does not admit any global weak solution.*

The above theorem can be followed by another form as follows for problem (1.1), (1.2).

Theorem 4.11. *Let $f(u)$, u_0 and u_1 be same as those in Theorem* 4.5. *Assume that $E(0) < d_0$, where d_0 is defined in Lemma* 2.4, *i.e.*

$$d_0 = \frac{p-1}{2(p+1)}\left(\frac{1}{aC_*^{p+1}}\right)^{\frac{2}{p-1}}.$$

Then when $\|u_0\|_{H^1} < r_0$ problem (1.1), (1.2) *admits a global weak solution*; *and when $\|u_0\|_{H^1} \geqslant r_0$ problem* (1.1), (1.2) *does not admit any global weak solution, where r_0 is defined in Lemma* 2.3, *i.e.*

$$r_0 = \left(\frac{1}{aC_*^{p+1}}\right)^{\frac{1}{p-1}}, \quad C_* = \sup_{u \in H^1,\, u \neq 0} \frac{\|u\|_{p+1}}{\|u\|_{H^1}}.$$

Proof. We will complete this proof by considering case $\|u_0\|_{H^1} < r_0$ and case $\|u_0\|_{H^1} \geqslant r_0$ separately as follows.

(i) Since $\|u_0\|_{H^1} < r_0$ implies $0 < \|u_0\|_{H^1} < r_0$ or $\|u_0\|_{H^1} = 0$. If $0 < \|u_0\|_{H^1} < r_0$, from Lemma 2.3 we can derive $I(u_0) > 0$, which makes Theorem 4.5 work to make sure the weak solution exists globally.

(ii) If $\|u_0\|_{H^1} \geqslant r_0$, then by

$$\frac{1}{2}\left(\left\|(-\Delta)^{-\frac{1}{2}}u_1\right\|^2 + \|u_1\|^2\right) + \frac{p-1}{2(p+1)}\|u_0\|_{H^1}^2 + \frac{1}{p+1}I(u_0) = E(0) < d_0 = \frac{p-1}{2(p+1)}\left(\frac{1}{aC_*^{p+1}}\right)^{\frac{2}{p-1}}$$
$$= \frac{p-1}{2(p+1)}r_0^2,$$

we get $I(u_0) < 0$. Then Theorem 4.8 gives that there is no global weak solution for problem (1.1), (1.2). □

Note that $F(u_0) = \frac{a}{p+1}|u_0|^p u_0$. Hence if $u_0(x) \geqslant 0$ a.e. in $\mathbb{R}^n$, then from $E(0) < d_0$ we can obtain $\|u_0\|_{H^1}^2 \leqslant \frac{p-1}{p+1}r_0^2 < r_0^2$. Therefore we have the following corollary.

Corollary 4.12. *Let $f(u)$, u_0 and u_1 be same as those in Theorem* 4.5. *Assume that $E(0) < d_0$ and $u_0(x) \geqslant 0$ a.e. in $\mathbb{R}^n$. Then problem* (1.1), (1.2) *admits a global weak solution.*

5. Generalization and example

In this section we give a generalization and an example for the results of this paper.

Generalization 5.1. Clearly all the results of this paper also hold when we replace Eq. (1.1) by

$$u_{tt} - \Delta u - a\Delta u_{tt} + b\Delta^2 u = \Delta f(u),$$

where a and b are positive constants.

Example 5.2. Consider the multidimensional generalization of Eq. (1.3)

$$u_{tt} - \Delta u - a\Delta u_{tt} + b\Delta^2 u = \Delta f(u), \tag{5.1}$$

where a and b are positive constants, $f(u) = \frac{3}{2}u^2$. It is easy to check that this $f(u)$ satisfies (H) for $1 \leqslant n \leqslant 5$. Hence from Theorem 4.11 we can obtain the following

Theorem 5.3. *Let $f(u) = \frac{3}{2}u^2$, $1 \leqslant n \leqslant 5$, $u_0 \in H$ and $u_1 \in L$. Assume that*

$$\left\|(-\Delta)^{-\frac{1}{2}}u_1\right\|^2 + \|u_1\|^2 + \|u_0\|^2 + \|\nabla u_0\|^2 + \int_{\mathbb{R}^n} u_0^3(x)\,\mathrm{d}x < \frac{1}{3}\left(\frac{2}{3C_*^3}\right)^2. \tag{5.2}$$

Then when

$$\|u_0\|^2 + \|\nabla u_0\|^2 < \left(\frac{2}{3C_*^3}\right)^2$$

problem (5.1), (1.2) *admits a global weak solution*; *and when*

$$\|u_0\|^2 + \|\nabla u_0\|^2 \geqslant \left(\frac{2}{3C_*^3}\right)^2$$

the problem (5.1), (1.2) *does not admit any global weak solution.*

Corollary 5.4. *Let $f(u)$, n, u_0 and u_1 be the same with those in Theorem* 5.3. *Assume that $u_0(x) \geqslant 0$ in $\mathbb{R}^n$ and* (5.2) *holds. Then problem* (5.1), (1.2) *admits a global weak solution.*

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