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Behavior of Solutions of Model Equations for Incompressible Fluid Flow

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We study the behavior of solutions of model equations of inviscid incompressible fluid flow proposed by Constantin, Lax and Majda together with a viscous version studied by Schochet. A condition is found on initial data to guarantee that solution of viscous equation remains smooth when the inviscid solution blows up. We prove

1. INTRODUCTION

Here we are concerned with the following equation

$$\omega_t(x, t) = \omega(x, t)\, H(\omega)(x, t), \tag{1.1}$$

together with its viscous version

$$\omega_t(x, t) = \omega_{xx}(x, t) + \omega(x, t)\, H(\omega)(x, t), \tag{1.2}$$

where x is in R^1 and H is the Hilbert transform. Equation (1.1) was introduced by Constantin, Lax and Majda in [2] as a model of the vorticity equation of the 3-D incompressible Euler equations. The initial value problem with equation (1.1) is solved in [2], where it is also shown that there exist initial data such that the solution blows up in finite time. Schochet has obtained a class of explicit solutions for equation (1.2) which too blow up in finite time, see [6].

In this paper, we study the behavior of solutions of equation (1.2). In Section 3 we compare the life spans for solutions of equations (1.1) and (1.2). We show for initial data ω_0, if there exists $x_0 \in R$ such that

$$\omega_0(x_0) = 0, \qquad \text{and} \qquad H(\omega_0)(x_0) = \max_{x \in R} H(\omega_0)(x),$$

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then solution for (1.2) remains smooth even as the solution for equation (1.1) with the same initial data ω_0 blows up. On the other hand, it has been demonstrated by Schochet in [6] that for model equation (1.2) viscosity hastens the instability of solution for some other initial data.

In Section 4 we compare the behavior of blowing up of solutions of viscous model equation to that of solutions of semi-linear heat equation. We observe that solutions obtained by Constantin, Lax and Majda and solutions obtained by Schochet converge to definite distributions when t tends to critical time. In formulas we have

$$\lim_{t \to t_{critical}^-} \int_{-\infty}^{+\infty} \omega(x, t)\, \phi(x)\, dx = \langle G, \phi \rangle,$$

where ω is a solution of Constantin, Lax and Majda for equation (1.1), or a solution of Schochet for equation (1.2), G is a distribution depending on initial data ω_0 and ϕ is any $C_0^\infty(R)$ test function. We include a discussion of how this observation may relate to problem of continuing the solutions beyond critical time.

Finally in Section 5 we prove for initial data ω_0 such that

$$\omega_0(x) > 0 \qquad \text{for all} \quad x \in R$$

or

$$\omega_0(x) < 0 \qquad \text{for all} \quad x \in R,$$

where explicit solution is not known, global smooth solution exists for equation (1.2).

Section 2 next contains some preliminary materials.

2. A New Proof of Local Existence Result

2.1. *Preliminaries*

The equation of Constantin, Lax and Majda is a 1-dimensional model of the vorticity equation for the 3-dimensional incompressible Euler equations. The initial value problem takes the form

$$\begin{aligned} \omega_t &= \omega H(\omega) \\ \omega(x, 0) &= \omega_0(x). \end{aligned} \tag{2.1}$$

In most of our study, (2.1) will be considered together with its viscous counterpart

$$\begin{aligned} \omega_t &= \omega_{xx} + \omega H(\omega) \\ \omega(x, 0) &= \omega_0(x). \end{aligned} \tag{2.2}$$

A beautiful complexification technique is invented in [2] which provides solutions to equation (2.1). Write

$$Q(x, t) = \omega(x, t) + H(\omega)(x, t)\, i.$$

Then if both Q and $(-(i/2)\, Q^2)$ are in Hardy space $H^2(E_2^+)$, by taking Hilbert transform with equation (2.1), we have

$$H_t(\omega) = \operatorname{Im}\left\{-\frac{i}{2}\, Q^2\right\} = \frac{1}{2}\,(H^2(\omega) - \omega^2). \tag{2.3}$$

Combining (2.3) with (2.1), we arrive at another form of the initial value problem

$$\begin{aligned} Q_t(x, t) &= -\frac{i}{2}\, Q^2(x, t) \\ Q(x, 0) &= Q_0(x), \end{aligned} \tag{2.4}$$

with

$$\operatorname{Im}\{Q\} = H(\operatorname{Re}\{Q\}), \tag{2.5}$$

where $Q_0(x) = \omega_0(x) + iH(\omega_0)(x)$ belongs to $H^2(E_2^+)$ if ω_0 is in L^2. A very useful result of these steps is that we get rid of the non-locality present in (2.1), and (2.4) is simply an ODE with fixed x variable. But there is a point of caution here: (2.4) and (2.5) as equations of Q are apparently overdetermined in form, in general one can only pose the problem: find a complex valued function $Q(x, t)$ such that

$$\begin{aligned} Q_t &= -\frac{i}{2}\, Q^2 \\ Q(x, 0) &= Q_0(x). \end{aligned} \tag{2.6}$$

Equation (2.6) is much easier to deal with. Solution formula follows immediately

$$Q(x, t) = \frac{Q(x, 0)}{1 + (t/2)\, iQ(x, 0)}. \tag{2.7}$$

It is easy to check that (2.7) is good only for finite time when there is $x_0 \in R$ such that the initial data satisfies

$$\omega_0(x_0) = 0 \qquad \text{and} \qquad H(\omega_0(x_0)) > 0. \tag{2.8}$$

In fact life span of solution in this case is

$$t_{crit.} = \frac{2}{(\max_{x \in R,\ \omega(x)=0} H(\omega_0(x)))}.$$

To show that $Q(x, t)$ in (2.7) solves (2.1) , we need to prove that this solution is in $H^2(E_2^+)$ for all t before critical time to justify (2.5).

In [2], this fact follows from the local existence proof using results for Lipschitz nonlinear differential equations and the observation that $H^1(R)$ is a Banach algebra, where Hilbert transform is a continuous map. Later in this section we will provide a simple direct verification of relation (2.5) for solutions (2.7) using a simple lemma in complex variable.

Another important fact found in [2] is that, in case the initial data ω_0 is periodic (say 2π), and if we model the velocity field by

$$u(x, t) = \int_0^x \omega(\xi, t)\, d\xi,$$

then

$$\int_{-\pi}^{\pi} |\omega(\xi, t)|^p\, d\xi \to \infty \qquad \text{as} \quad t \to t_{crit.},$$

$$\int_{-\pi}^{\pi} |u(\xi, t)|^p\, d\xi < M(p) \qquad \text{as} \quad t \to t_{crit.},$$

for any $1 \leqslant p < \infty$. Our observation proved later that solutions converge in distribution sense when blowing up shares similar basis with this property, though one can not be deduced from the other. From now on, we use $Q(x, t)$ to stand for the solution given by (2.7).

Similarly, by complexifying the viscous model equation, we also get another form of (2.2): find a complex valued function Q, such that

$$\begin{aligned} Q_t &= Q_{xx} - \frac{i}{2} Q^2 \\ Q(x, 0) &= Q_0(x). \end{aligned} \tag{2.9}$$

A Family of Explicit Solutions of Schochet

A family of explicit solutions are found by Schochet in [6] for nonlinear equation (2.9). They take the following form

$$\begin{aligned} Q &= Q(x, t, z_1(0), z_2(0), \pm) \\ &= \omega(x, t) + i[H(\omega)](x, t) \\ &= -\frac{k_\pm i}{(x - z_1(t))(x - z_2(t))} - \frac{12i}{(x - z_1(t))^2} - \frac{12i}{(x - z_2(t))^2} \end{aligned} \tag{2.10}$$

with $k_\pm = 12(6 \pm \sqrt{6})$, while $z_1(t)$, $z_2(t)$ are initially on the lower half plane so that the initial data is in $H^2(E_2^+)$ and

$$\begin{aligned} z_1(t) &= \tfrac{1}{2}\left[z_1(0) + z_2(0) + ([z_1(0) - z_2(0)]^2 - \tfrac{5}{3} k_\pm t)^{1/2}\right], \\ z_2(t) &= \tfrac{1}{2}\left[z_1(0) + z_2(0) - ([z_1(0) - z_2(0)]^2 - \tfrac{5}{3} k_\pm t)^{1/2}\right]. \end{aligned}$$

Because the imaginary part of

$$([z_1(0) - z_2(0)]^2 - \tfrac{5}{3} k_\pm t)^{1/2}$$

goes to $+\infty$ as $t \to +\infty$, $z_1(t)$ must move up to cross the real line in finite time. It is at that time that the solution (2.10) blows up.

Obviously initial data of solutions in formula (2.10) consist of only a subset of all data in $H^2(E_2^+)$. From now on, we use $Q(x, t, z_1, z_2, \pm)$ to stand for the solution given by (2.10).

2.2. *A Direct Verification that $Q(x, t)$ Remains in $H^2(E_2^+)$ for $t < t_{crit.}$*

The proof is based on the following simple lemma

LEMMA 1. *Let $f(z)$ be a complex analytic function defined on the upper half plane E_2^+. Suppose $f(z) \to 0$ as $|z| \to \infty$, and there exists z_0 in E_2^+ where* $\mathrm{Re}\{f(z_0)\} = 0$ *and* $\mathrm{Im}\{f(z_0)\} > 0$. *Then there is a curve in E_2^+ connecting z_0 to some point on real line*

$$z(t)\colon [0, t_m) \to E_2^+, \qquad z(0) = z_0$$

such that

$$\mathrm{Re}\{f(z(t))\} = 0 \qquad \text{for all} \quad t \in [0, t_m)$$

and $\mathrm{Im}\{f(z(t))\}$ *is strictly increasing with respect to t. In particular*

$$\max_{z \in E_2^+,\ \mathrm{Re}\{f(z)\} = 0} (\mathrm{Im}\{f(z)\}) = \max_{x \in R,\ \mathrm{Re}\{f(x)\} = 0} (\mathrm{Im}\{f(x)\}).$$

This lemma looks like the familiar maximum principle. Indeed we can say something accordingly with respect to $f(z)$. For example, $|f(z)|$, $\operatorname{Re}\{f(z)\}$ and $\operatorname{Im}\{f(z)\}$ take on extreme value only on real line and at ∞. However our lemma is slightly different in that we have the condition $\operatorname{Re}\{f(z)\}=0$ in selecting a maximum for $\operatorname{Im}\{f(z)\}$. To prove Lemma 1, one can begin by looking at the map f, which is assumed not to be a constant, around a small neighborhood of z_0, where $f(z) \sim f(z_0)+(f^{(n)}(z_0)/n!)(z-z_0)^n$, $n \geqslant 1$ and $f^{(n)}(z_0) \neq 0$. It is then easy to see that the map we want to construct, $z(t)$, exists locally; and we can choose parameter $t=\operatorname{Im}\{f(z(t))\}-\operatorname{Im}\{f(z_0)\}$. Now $|f(z(t))|=\operatorname{Im}\{f(z(t))\}>\operatorname{Im}\{f(z_0)\}>0$. Hence, in extending the domain of definition, $Z(t)$ has to be localized within some finite region $B^+(r)=\{z \mid |z| \leqslant r, \operatorname{Im}\{z\}>0\}$, because $f(z) \to 0$ when $z \to \infty$. Now for any $\delta>0$, $B^+_\delta(r)=\{z \mid |z| \leqslant r, \operatorname{Im}\{z\} \geqslant \delta\}$ is a compact region in the interior of E^+_2, which can be covered by finite number of small open discs where f is well approximated by the local Taylor expansions as above. As a result, the domain of definition for $z(t)$ can be extended uniformly within $B^+_\delta(r)$: there exists $\varepsilon>0$, such that if $z(t)$ is defined for $0 \leqslant t \leqslant t_n$ and $z(t_n) \in B^+_\delta$, then $z(t)$ can be defined for $t \in (t_n, t_n+\varepsilon)$, where $\operatorname{Im}\{f(z(t_n+\varepsilon))\}-\operatorname{Im}\{f(t_n)\}=\varepsilon$. As $\operatorname{Im}\{f(z)\}$ is bounded for $z \in B^+_\delta$, $z(t)$ must leave the compact region after finite steps of extensions. In other words, $z(t)$ approaches the real line when t tends to the upper limit. Of course the detail behavior of $z(t)$ when t increases depends on the regularity property of f near the real line. If f is assumed to be analytic at R^1, above argument (let $\delta=0$) shows that $f(z(t))$ reaches the real line in finite steps. While if f is only assumed to be continuous at the real line, we can conclude from this construction of $z(t)$ that

$$\max_{z \in E^+_2,\ \operatorname{Re}\{f(z)\}=0} (\operatorname{Im}\{f(z)\}) = \max_{x \in R,\ \operatorname{Re}\{f(x)\}=0} (\operatorname{Im}\{f(x)\}).$$

In [2], $\omega \in H^1(R)$ and so the complex function $\omega+H(\omega)(z)$ is continuous at the real line.

We now prove that when ω_0, $H(\omega_0)$ decay at infinity, relation (2.5) holds for solution in (2.7) for any $t<t_{critical}$. Hence $Q(x, t)$ must be a solution for the vorticity model equation (2.1).

Proof. We recall the explicit solution formula

$$Q(x, t)=\frac{Q(x, 0)}{1+\frac{1}{2}\, tiQ(x, 0)}.$$

In terms of ω_0 and $H(\omega_0)$, this is

$$Q(x, t)=\frac{\omega_0(x)+iH(\omega_0)(x)}{(1-(t/2)\, H(\omega_0)(x))+(1/2)\, \omega_0(x) i}. \tag{2.11}$$

As $\omega_0 \in L^2$, the domain of definition of $\omega_0 + iH(\omega_0)$ can be analytically extended to include E_2^+. Then by (2.11), $Q(z, t)$ is defined for $z \in E_2^+$. In verifying that $Q(z, t)$ is analytic and in $H^2(E_2^+)$, we first show that there is no singularity for $Q(z, t)$ in E_2^+ for $t < t_{critical}$. Obviously

$$\left(1 - \frac{t}{2} H(\omega_0)(x)\right) + \frac{1}{2} \omega_0(x) i \neq 0 \qquad \text{for all} \quad x \in R, \quad t \in (0, t_{critical}). \qquad (2.12)$$

Because $\omega_0(x)$, $H(\omega_0)(x)$ decay at infinity, $\omega_0(z) + iH(\omega_0)(z) \to 0$ as $z \to \infty$. Using Lemma 1, we have from (2.12)

$$\left(1 - \frac{t}{2} H(\omega_0)(z)\right) + \frac{1}{2} \omega_0(z) i \neq 0 \qquad \text{for all} \quad z \in E_2^+, \quad t \in (0, t_{critical}).$$

In addition, we have $|Q(x + yi, t)| \sim O(|\omega_0(x + yi) + iH(\omega_0)(x + yi)|)$ $(y \geqslant 0)$, as $x \to \infty$. As a result $Q(z, t)$ is in $H^2(E_2^+)$ for all t before $Q(x, t)$ breaks down, which implies relation (2.5) for Q; then because $Q(x, t)$ satisfies ODE

$$Q_t(x, t) = -\frac{i}{2} Q^2(x, t),$$

we conclude that the real part is indeed a solution for (2.1).

3. Comparison of Life Spans

It is a surprising result in [6] that solution of viscous model equation, $Q(x, t, z_1, z_2, +)$, with

$$z_1(0) = x_1 - \frac{ic}{2} |x_1 - x_2|, \qquad z_2(0) = x_2 - \frac{ic}{2} |x_1 - x_2|,$$

$x_1, x_2 \in R$, $c < 0.219$, blows up sooner than the solution of the inviscid model equation with the same initial data. In general viscosity is known to have regularizing effect on solutions. Constantin shows in [5] that, at any given time, solution of the genuine Navier–Stokes equation remains smooth if inviscid Euler flow exists, providing that the viscosity is small; and we note that for the later to become singular, vorticity must blow up, see [1]. For nonlinear heat equations, Friedman and Lacey prove in [4] that viscosity increases the life spans of solutions. In this section, we identify a class of initial data for the model equations where viscosity decreases instability.

THEOREM 1. *Suppose the initial data* ω_0, $H(\omega_0) \in L^2 \cap L^\infty$, *decay at infinity, and solution* $Q(x, t)$ *for equation* (2.1) *blows up at* x_0. *If in addition*

$$H(\omega_0)(x_0) = \max_{x \in R} H(\omega_0)(x),$$

then the solution for equations (2.2) *with same initial data* ω_0 *remains smooth when solution for equations* (2.1) *blows up.*

It is readily verified that following initial data in $H^2(E_2^+)$:

$$\omega_0(x) + H(\omega_0)(x)\, i = \frac{(-1)^n\, bi}{(x - x_0 + ai)^{2n}}, \qquad \text{or} \qquad \frac{(-1)^{n+1}\, b}{(x - x_0 + ai)^{2n+1}},$$

where $x_0 \in R$, $a > 0$, $b > 0$, n is any positive integer, will satisfy conditions of Theorem 1.

Schochet points out in [6] that when we have two locations where $H\omega$ is large, one might diffuse toward the other so that the blow up of viscous solution happens sooner. The condition in Theorem 1 corresponds to a single hump situation. For the viscous solution, blow up does not necessarily occur at the point where ω vanishes. The key point in the proof below is that ω being non zero does not increase the instability of viscous solution. In the proof of Theorem 1, we make frequent uses of the comparison theorem:

THEOREM 2 (Comparison Theorem). *Suppose* $f(x, t, u)$ *is a smooth function and* $D = \Omega \times (0, T)$ *is a bounded domain in* $R^n \times R_+$, *with* Ω *simply connected,* $\partial\Omega$ *smooth. Let*

$$pu = u_t - \sum_{i, j=1}^{n} (a_{ij}(x, t)\, u_{x_i})_{x_j};$$

we assume that p *is uniformly parabolic. Let* u *and* v *each be* C^2 *functions of* x *in* $\bar{\Omega}$, C^1 *function of* t *on* $[0, T]$, *and satisfy the following three conditions*:

$$pu - f(x, t, u) \geqslant pv - f(x, t, v), \tag{i}$$

$$u(x, 0) \geqslant v(x, 0), \tag{ii}$$

$$u(x, t) > v(x, t) \qquad \text{for} \quad (x, t) \in \partial\Omega \times [0, T]. \tag{iii}$$

Then

$$u(x, t) > v(x, t) \qquad \text{for all} \quad (x, t) \in \Omega \times (0, T).$$

Remark 1. We can have a variation on condition (i) of this theorem. Suppose $u(x, t) > 0$ for all $(x, t) \in \bar{\Omega} \times [0, T]$, then the result is still true if (i) holds whenever $v \geqslant 0$, that is,

$$pu - f(x, t, u) \geqslant pv - f(x, t, v) \qquad \text{for} \quad (x, t), \quad \text{such that} \quad v(x, t) \geqslant 0. \qquad \text{(i)}'$$

Proof of Theorem 1. We begin by first comparing the life spans of the imaginary parts of solutions to (2.6) and (2.9). For brevity we use $\omega_1 + iH_1$, $\omega_2 + iH_2$ to stand for two solutions and $\omega_0 + iH_0$ for initial data. Hence

$$\text{I:} \quad \partial_t H_1(x, t) = \tfrac{1}{2}(H_1^2 - \omega_1^2),$$
$$\text{II:} \quad \partial_t H_2(x, t) = \partial_{xx} H_2 + \tfrac{1}{2}(H_2^2 - \omega_2^2),$$

with the same initial data ω_0 which satisfies

$$\text{A:} \quad \omega_0(x_0) = 0,$$
$$\text{B:} \quad H_0(x_0) > 0,$$
$$\text{C:} \quad H_0(x_0) = \max_{x \in R} H_0(x).$$

We show that $H_2(x, t)$ remains smooth even when $H_1(x, t)$ blows up. First we observe that

$$\max_{x \in R} H_1(x, t) = H_1(x_0, t),$$

which follows from solution formula (2.7). Hence $H_1(x, t)$ blows up if and only if $H_1(x_0, t)$ blows up. By A and (2.11), $\omega_1(x_0, t) \equiv 0$. Hence I gives

$$\partial_t H_1(x_0, t) = \tfrac{1}{2} H_1^2(x_0, t).$$

Define

$$H_1^*(x, t) = H_1(x_0, t) \qquad \forall x \in R.$$

Then

$$\partial_t H_1^*(x, t) = \partial_{xx} H_1^*(x, t) + \tfrac{1}{2} H_1^*(x, t)^2.$$

Now the problem of comparing $H_1(x, t)$ and $H_2(x, t)$ becomes the problem of comparing $H_1^*(x, t)$ and $H_2(x, t)$, which fits into the framework of the comparison theorem. First, as $\omega_0(x)$, $H_0(x)$ decay at infinity, by some standard methods with parabolic equations, $\omega_2(x, t)$, $H_2(x, t)$ must tend to 0 as $x \to \infty$ for fixed t before blowing up. Let $u = H_1^*$, $v = H_2$. Then condition (ii) of the comparison theorem is obviously satisfied. Condition (iii) holds if Ω, an interval here, is sufficiently large due to the decay property of H_2 in x variable. As for condition (i), now let

$$pg = g_t - g_{xx}, \qquad f(g) = \tfrac{1}{2} g^2.$$

We have

$$(pu - f(u)) - (pv - f(v)) = \tfrac{1}{2}\,\omega_2^2 \geqslant 0.$$

So we conclude

$$H_1^*(x, t) > H_2(x, t)$$

for $t > 0$ before H_1^* blows up. Next we show that H_1^* blows up sooner from the strict inequality above. By continuity there exists a positive number ε, such that

$$H_2(x, t_1 + \varepsilon) \leqslant H_1^*(x, t_1).$$

Now we use comparison theorem for $H_1^*(x, t)$ with initial data $H_1^*(x, t_1)$ and $H_2(x, t)$ with initial data $H_2(x, t_1 + \varepsilon)$. We have

$$H_1^*(x, t_1 + t) \geqslant H_2(x, t_1 + \varepsilon + t)$$

as long as $H_1^*(x, t_1 + t)$ remains finite. This implies H_1^* blows up at least ε sooner than H_2. To complete the proof of the Theorem 1, we still have to show that, if $\max_{x \in R} H_2(x, t)$ remains finite, then so do $\max_{x \in R} |\omega_2(x, t)|$ and $\min_{x \in R} H_2(x, t)$.

LEMMA 2. *Let ω, H be a solution to the following system*

$$\omega_t = \omega_{xx} + \omega H$$
$$H_t = H_{xx} + \tfrac{1}{2}(H^2 - \omega^2) \qquad \text{for} \quad x \in R, \quad t \in [0, T), \quad T > 0.$$

Suppose

$$H(x, t) \leqslant M \qquad \text{for} \quad x \in R, \quad t \in [0, T).$$

Then

$$|\omega(x, t)| \leqslant \max_{x \in R} (|\omega_0|)\, e^{MT},$$

$$H(x, t) \geqslant \min_{x \in R} (H_0(x)) - \tfrac{1}{2} \max_{x \in R} (|\omega_0(x)|^2)\, Te^{2MT}.$$

Proof. Let ω^*, ω_* be two functions of t that solve the ODE with different initial data:

$$\omega_t^* = M\omega^* \qquad\qquad \omega_{*t} = M\omega_*$$
$$\omega^*(0) = \max(0, \max_{x \in R} (\omega_0(x))), \qquad \omega_*(0) = \min(0, \min_{x \in R} (\omega_0(x))).$$

By using comparison theorem in the way that we just did, together with the variation of condition (i) in our Remark 1, we have

$$\omega_*(t) \leqslant \omega(x, t) \leqslant \omega^*(t) \qquad \text{for all} \quad x \in R, \quad t \in [0, T).$$

Integrating these two ODEs, we get

$$|\omega(x, t)| \leqslant (\max_{x \in R} |\omega_0(x)|)\, e^{MT}.$$

Apply comparison theorem once more now with equation for H, then the lower bound of H can be dominated by function $H_*(t)$ which is solution of

$$H_{*t} = -\tfrac{1}{2} (\max_{x \in R} |\omega_0(x)|)^2\, e^{2MT}$$

$$H_*(0) = \min_{x \in R} (H_0(x)).$$

Lemma 2 follows immediately.

4. Solutions Blow Up As Distributions

In many physical situations, the development of singularities in the solutions is intimately connected with the problem of continuing the solution beyond critical time in the weak sense. As a first step in this direction, we establish a few basic facts below concerning the solutions of model equations. We show that solutions given by (2.7) and (2.10) converge in distribution sense as $t \to t_{crit.}$. As a result, one may first drop the constraint imposed by relation (2.5) to simplify the problem and treat continuation of solutions as an initial value problems for differential equations, (2.6), (2.9), with distributional initial data and t starting from $t_{crit.}$. We point out here that, while this consideration raises an interesting problem on the other aspects of viscous model equation, it is straight forward to verify that such continuation does not exist for equation (2.6). We remark here that the uniform boundedness of L^p norm of model velocity observed in [2] already implies that there exists a *subsequence*, $t_n \to t_{crit.}$, such that $\omega(t_n)$ converge to a distribution. Our explicit computations below also apply to the more singular viscous solutions of Schochet.

The main underlying technical point involved here can be illustrated very briefly as follows: the complex function

$$v(x, t) = \frac{2}{(x + (t_{crit.} - t)\, i)^2}$$

will have some cancellation at $x=0$ and converge in inner product with a C_0^∞ test function of x as t tends to t_{crit}; on the other hand the familiar real function

$$v_*(x, t) = \frac{2}{x^2 + (t_{crit.} - t)^2}$$

will make inner product with test functions which are not zero at $x=0$ go to ∞, even though they both tend to

$$v_0(x) = \frac{2}{x^2}$$

as t tends to $t_{crit.}$ for almost all x.

THEOREM 3. *Suppose $Q(x, t)$ in (2.6) blows up at finite points x_j, $1 \leqslant j \leqslant n$ at $t = t_{crit}$. If in addition the initial data $Q_0 = \omega_0 + iH(\omega_0)$ satisfies*

$$Q_0(z) = Q_0(x_j) + (a_j + b_j i)(z - x_j) + O((z - x_j)^2),$$

with $a_j \neq 0$, for z in small neighborhoods around x_j in E_2^+. Then $Q(x, t)$ converges in distribution sense as $t \to t_{crit.}$.

Remark 2. In case $n=1$, write $Q_0(z) = Q_0(x_1) + (\tilde{a}_1 + \tilde{b}_1 i)(z - x_1)$, where $\tilde{a}_1 = a_1 + O(z - x_1)$, $\tilde{b}_1 = b_1 + O(z - x_1)$. Let $\mathrm{Re}\{Q_0(z)\} = 0$. We have $x - x_1 = (\tilde{b}_1/\tilde{a}_1)(y)$, where $\mathrm{Im}\{Q_0(z)\} = \mathrm{Im}\{Q_0(x_1)\} + ((\tilde{a}_1{}^2 + \tilde{b}_1{}^2)/\tilde{a}_1)(y)$. By Lemma 1, $a_1 < 0$, because $\max_{z \in E_2^+,\, \omega_0(z)=0} H(\omega_0(z))$ is achieved only on the real line.

We give a proof of Theorem 3 for the case $n=1$, $x_1 = 0$. The general situation can be proved similarly.

Proof. For any real test function ϕ,

$$\begin{aligned}
\int_{-\infty}^{+\infty} Q(x, t)\, \phi(x)\, dx &= \int_{-\infty}^{+\infty} \frac{Q(x, 0)}{1 + (t/2)\, iQ(x, 0)}\, \phi(x)\, dx \\
&= \frac{2}{it} \int_{-\infty}^{+\infty} \left(1 - \frac{1}{1 + (t/2)\, iQ(x, 0)}\right) \phi(x)\, dx \\
&= \int_{-\infty}^{+\infty} \frac{2\phi(x)}{it}\, dx - \int_{-\infty}^{+\infty} \frac{4\, \phi(x)}{it^2((2/t) + iQ(x, 0))}\, dx.
\end{aligned}$$

On the right hand side above, the first term, denoted by I, converges to

$$\int_{-\infty}^{+\infty} \frac{2\phi(x)}{it_{crit.}}\, dx$$

at critical time. Because $x=0$ is the only singular point, the denominator of the second integrand vanishes only when $t=t_{crit.}$ and only at $x=0$. By assumption

$$\frac{1}{(2/t)+iQ(x,0)} = \frac{1}{(2/t)+(H(\omega_0)(0)\, i+ax+bxi)\, i+O(x^2)}$$
$$= \frac{1}{((2/t)-H(\omega_0)(0)-bx)+iax+O(x^2)}.$$

Denote $2/t-H(\omega_0)(0)>0$ as $\varepsilon(t)$. Then

$$\frac{1}{\varepsilon(t)-bx+iax+O(x^2)}$$
$$= \frac{1}{\varepsilon(t)-bx+iax} - \frac{O(x^2)}{(\varepsilon(t)-bx+iax)(\varepsilon(t)-bx+iax+O(x^2))}.$$

Denote the first and second terms of the right hand side above by $C_1(x,t)$ and $C_2(x,t)$, their inner products with $(4\phi/it^2)$ by II and III respectively. We have

$$|C_2(x,t)| \leqslant \frac{O(x^2)}{|ax|\,(|ax|-O(x^2))},$$

which is bounded near $x=0$. Because when x is fixed, $C_2(x,t)$ has limit as $t\to t_{crit.}$ if $x\neq 0$, $\mathrm{III}=(4/it^2)\int_{-\infty}^{+\infty} C_2(x,t)\,\phi(x)\,dx$ must converge as $t\to t_{crit.}$. We can treat $C_1(x,t)$ in a more direct way:

$$\mathrm{II} = \frac{4}{it^2}\int_{-\infty}^{+\infty} \frac{1}{\varepsilon(t)-bx+iax}\,\phi(x)\,dx$$
$$= \frac{4}{it^2(ai-b)}\int_{-\infty}^{+\infty} \frac{1}{\varepsilon_1(t)+x}\,\phi(x)\,dx$$
$$= \frac{4}{it^2(ai-b)}\int_{-\infty}^{+\infty} (\ln(\varepsilon_1(t)+x))_x\,\phi(x)\,dx$$
$$= -\frac{4}{it^2(ai-b)}\int_{-\infty}^{+\infty} \ln(\varepsilon_1(t)+x)\,\phi_x(x)\,dx,$$

where

$$\varepsilon_1(t) = \frac{\varepsilon(t)}{ai - b}$$

is in the upper half of complex plane for $t < t_{crit.}$ (recall $\varepsilon(t) > 0$, $a < 0$) and tends to zero as $t \to t_{crit.}$. Observe that $\ln(|x|)$ is a weak singularity. We have , as $t \to t_{crit.}$,

$$\int_{-\infty}^{+\infty} \mathrm{Re}\{\ln(\varepsilon_1(t) + x)\}\ \phi_x(x)\, dx \to \int_{-\infty}^{+\infty} \ln(|x|)\ \phi_x(x)\, dx,$$

$$\int_{-\infty}^{+\infty} \mathrm{Im}\{\ln(\varepsilon_1(t) + x)\}\ \phi_x(x)\, dx \to \int_{-\infty}^{+\infty} \pi(1 - \alpha(x))\ \phi_x(x)\, dx,$$

where $\alpha(x)$ is the characteristic function of positive x line. Hence II converges as t tends to critical time. Theorem 3 is proved.

THEOREM 4. *Schochet's explicit solutions* (2.10) *for equations* (2.2) *converge in distribution sense when t tends to critical time.*

We do not know yet whether such statement is true with solutions of viscous model equation for general $H^2(E_2^+)$ initial data not in form of (2.10). Proof of Theorem 4 is similar to the case for Constantin–Lax–Majda solutions above. Observe that terms like $1/(x - z(t))$, $1/(x - z(t))^2$ can be written as first and second derivative of the *ln* function. We now evaluate $\int_{-\infty}^{+\infty} Q(x, t, z_1, z_2, \pm)\, \phi(x)\, dx$, which, by (2.10), is

$$\int_{-\infty}^{+\infty} \frac{-k_{\pm} i}{(x - z_1)(x - z_2)}\, \phi(x) dx + \int_{-\infty}^{+\infty} \frac{-12i}{(x - z_1)^2}\, \phi(x) dx$$
$$+ \int_{-\infty}^{+\infty} \frac{-12i}{(x - z_2)^2}\, \phi(x)\, dx.$$

Among z_1, z_2, only $z_1(t)$ will go up to hit the real line. Then

$$\int_{-\infty}^{+\infty} Q(x, t, z_1, z_2, \pm)\ \ \phi(x)\, dx$$
$$= \frac{-k_{\pm} i}{z_2 - z_1} \left(\int_{-\infty}^{+\infty} \ln(x - z_1)\ \phi_x(x)\, dx + \int_{-\infty}^{+\infty} \frac{\phi(x)}{x - z_2}\, dx \right)$$
$$+ 12i \int_{-\infty}^{+\infty} \ln(x - z_1)\ \phi_{xx}\, dx - \int_{-\infty}^{+\infty} \frac{12i}{(x - z_2)^2}\, \phi(x).$$

All integrations have limits as $t \to t_{crit.}$. This finishes our proof.

Similar to the observation in [2] on comparison between singularity of inviscid solution of equation (2.1) and a local scaler quadratic equation, $\omega_t(x, t) = \omega^2$, the behavior of solutions for the complexified semilinear heat equation (2.9) is sharply different from the real one in that blowing up of solutions occurs in a very restrained way. This can be seen from the following characterization of singularities for semilinear heat equations .

THEOREM 5. (Giga, Kohn, [5]). *Let u solve the semilinear heat equation*

$$u_t - \Delta u - |u|^{p-1} u = 0 \tag{4.1}$$

on $Q_1 = B_1 \times (-1, 0)$, *and assume that*

$$|u(x, t)|\,(-t)^{\beta} \qquad \text{is bounded in } Q_1,$$

where $\beta = 1/p - 1$ *if* $p \leqslant (n+2)/(n-2)$ *or if* $n \leqslant 2$, *let* $u_\lambda(x, t) = \lambda^{2\beta} u(\lambda x, \lambda^2 t)$, *then*

$$\lim_{\lambda \to 0} (-t)^{\beta}\, u_\lambda(x, t) \qquad \text{equals} \quad \pm\beta^{\beta} \quad \text{or} \quad 0.$$

For each $c > 0$, *the limit exists uniformly for* $(x, t) \in Q_1$ *such that* $|x| < c(-t)^{1/2}$.

The assumption on the boundedness of $|u(x, t)|\,(-t)^{\beta}$ was proved by Weissler under some conditions on initial data. The zero alternate of limit value was later shown by Giga and Kohn to imply that singularity is removable.

We claim that if in addition we have

$$u(x, t) \geqslant 0 \qquad \text{for all} \quad (x, t) \in Q_1,$$

then u is not a distribution when $t = 0$.

COROLLARY 1. *Suppose a nonnegative solution* $u(x, t)$ *of semilinear heat equation satisfies conditions of Theorem* 5 *and blows up when* $t = 0$ *at* $x = 0$. *Suppose* $p < 3$, *then for any test function* $\phi \in C_0^{\infty}(B_1)$, $\phi \geqslant 0$, $\phi(0) > 0$, *we have*

$$\lim_{t \to 0^-} (u(x, t), \qquad \phi(x)\,) = \lim_{t \to 0^-} \int_{-\infty}^{+\infty} u(x, t)\, \phi(x)\, dx = +\infty.$$

Proof. By assumption, there is a positive number $a > 0$, such that $\phi(x) > \phi(0)/2$ for all $x \in (-a, a)$, then

$$\int_{-\infty}^{+\infty} u(x, t)\, \phi(x)\, dx \geqslant \frac{\phi(0)}{2} \int_{-a<x<a} u(x, t)\, dx$$

$$\geqslant c_1 \frac{\phi(0)}{2} \int_{|x| < c(-t)^{1/2}} \frac{1}{(-t)^{\beta}}\, dx$$

$$= c_2 \frac{\phi(0)}{2} \cdot (-t)^{1/2-\beta},$$

c_1, c_2 are some positive constants. So for $\beta > \frac{1}{2}$ $(p < 3)$,

$$(-t)^{(1/2)-\beta} \to \infty, \qquad \text{as} \quad t \to \infty.$$

Discussion of Continuation for Solutions beyond Critical Time

For explicit solutions in (2.7), (2.10), the poles move to the upper half plane when t is greater than $t_{crit.}$. As a result relation (2.5) fails to hold. Following discussions are presented in the context of differential equations (2.6) and (2.9) in themselves.

We observe first that both solution formulas (2.7) and (2.10) make sense except at $t = t_{crit.}$. It is easy to verify that, for $t < t_{crit.}$ and $t > t_{crit.}$, they satisfy the complex ODE and complex semilinear heat equation respectively. We notice that the movement of poles in to the upper half plane, which leads to the violation of (2.5), is reflected here in another way as resulting in the discontinuities of solutions in the sense of distributions when $t \to t_{crit.} - 0$ and $t \to t_{crit.} + 0$. For convenience, we will illustrate this phenomenon with the typical singular terms $1/(x + ti)$ and $1/(x + ti)^2$ that appear in these solutions. The reason is, although $\mathrm{Re}\{1/(x + ti)\}$ tends to $1/x$ as $t \to 0$, the imaginary part of $1/(x + ti)$ converges to different distributions depending on $t \to 0^+$ or $t \to 0^-$. A similar situation happens for $1/(x + ti)^2$. Let ϕ be $C_0^{\infty}(R)$. We now evaluate the action of the distribution on ϕ. Let $\alpha(x)$ be the characteristic function of positive x-axis, then

$$\lim_{t \to 0^+} \int_{-\infty}^{\infty} \frac{1}{(x + ti)}\, \varphi(x)\, dx = \lim_{t \to 0^+} \int_{-\infty}^{\infty} (\ln(x + ti))_x\, \varphi(x)\, dx$$

$$= \lim_{t \to 0^+} -\int_{-\infty}^{\infty} \ln(x + ti)\, \varphi_x(x)\, dx$$

$$= \lim_{t \to 0^+} -\int_{-\infty}^{\infty} (\ln |x + ti|)\, \varphi_x(x)\, dx$$

$$- i \int_{-\infty}^{\infty} \pi(1 - \alpha(x))\, \varphi_x(x)\, dx$$

$$= -\int_{-\infty}^{\infty} (\ln |x|)\, \varphi_x(x)\, dx - \varphi(0)\, \pi i.$$

Similarly we have

$$\lim_{t \to 0^-} \int_{-\infty}^{\infty} \frac{1}{(x + ti)}\, \varphi(x)\, dx = \lim_{t \to 0^-} \int_{-\infty}^{\infty} (\ln (x + ti))_x\, \varphi(x)\, dx$$

$$= \lim_{t \to 0^-} -\int_{-\infty}^{\infty} (\ln |x + ti|)\, \varphi_x(x)\, dx$$

$$- i \int_{-\infty}^{\infty} \pi\alpha(x)\, \varphi_x(x)\, dx$$

$$= -\int_{-\infty}^{\infty} (\ln |x|\ )\, \varphi_x(x)\, dx + \varphi(0)\, \pi i.$$

For the typical singular term in Schochet's explicit solutions, we do integration by parts one more times to get

$$\lim_{t \to 0^+} \int_{-\infty}^{\infty} \frac{1}{(x + ti)^2}\, \varphi(x)\, dx = -\int_{-\infty}^{\infty} (\ln |x|)\, \varphi_{xx}(x)\, dx - \varphi_x(0)\, \pi i,$$

$$\lim_{t \to 0^-} \int_{-\infty}^{\infty} \frac{1}{(x + ti)^2}\, \varphi(x)\, dx = -\int_{-\infty}^{\infty} (\ln |x|)\, \varphi_{xx}(x)\, dx + \varphi_x(0)\, \pi i.$$

Thus the limit when $t \to 0^-$ and $t \to 0^+$ of the inner product of

$$\frac{1}{x + ti}, \qquad \frac{1}{(x + ti)^2}$$

with φ are not the same whenever $\varphi(0) \neq 0$ and $\varphi_x(0) \neq 0$. We can use several steps in the computations above to show that the trivial continuation by (2.10) does not satisfy the viscous equation in the weak sense. Let $\phi(x, t)$ be a C_0^∞ test function. Assume $x = 0$ is the singular pole of viscous solution (2.10). Let B be a small rectangle centered at $(0, t_{crit.})$, $B = (-\delta, \delta) \times (t_{crit.} - \varepsilon, t_{crit.} + \varepsilon)$, δ, ε small positive numbers. We have

$$\int_{R \times (0, \infty) - B} \left(Q_t - Q_{xx} + \frac{i}{2} Q^2 \right) \phi(x, t)\, dx\, dt = 0$$

Integration by parts gives

$$\int_{R\times(0,\,\infty)-B}\left(-Q\phi_t-Q\phi_{xx}+\frac{i}{2}Q^2\phi\right)dx\,dt=-I_h(\varepsilon,\delta)-I_v(\varepsilon,\delta),$$

where

$$I_h(\varepsilon,\delta)=\int_{-\delta}^{+\delta}(-Q\phi(x,t_{crit.}+\varepsilon)+Q\phi(x,t_{crit.}-\varepsilon))\,dx,$$

and

$$I_v(\varepsilon,\delta)=\int_{t_{crit.}-\varepsilon}^{t_{crit.}+\varepsilon}\{(Q_x\phi-Q\phi_x)(\delta,t)-(Q_x\phi-Q\phi_x)(-\delta,t)\}\,dt.$$

It is easy to see that for any $\varepsilon>0$, δ can be chosen sufficiently small to make $I_v(\varepsilon,\delta)\sim 0$. To show $I_h(\varepsilon,\delta)$ does not tend to zero as $\varepsilon,\delta\to 0$, which implies that Q is not a weak solution around $(0,t_{crit.})$, one notes that for the typical singular term $1/(x+ti)^2$ in (2.10), similar computations lead to

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{-\delta}^{\delta}&\left(\frac{1}{(x+(t_{crit.}+\varepsilon)i)^2}-\frac{1}{(x+(t_{crit.}-\varepsilon)i)^2}\right)\phi(x,t)\,dx\\&=-2\phi_x(0,t_{crit.})\,\pi i.\end{aligned}$$

5. A Global Existence Theorem

Here we study equations (2.9) for some class of initial data where explicit solution formula is not available.

The nonlinear term here $(-(i/2)\,Q^2)$ of equation (2.9) demonstrates qualitatively different behavior in comparison to its real counterpart in semilinear heat equations (4.1), $|u|^{p-1}u$, which is monotone as a function of positive u. While formula (2.10) predict blowing up in finite time for solutions with initial data intersecting the imaginary y-axis in complex plane, we pay attention here to two other regions: right half plane $x>0$ and left half plane $x<0$. Roughly speaking, our result in Theorem 6 says that, while viscosity can hasten the development of singularities for the modeling flow as shown by Schochet, it does not produce new singularities when the inviscid solution exists for all time.

Theorem 6. *Suppose initial data ω_0, $H(\omega_0)\in L^2\cap L^\infty$, decay at infinity, and $\omega_0(x)>0$ for all x or $\omega_0(x)<0$ for all x. Then the initial value problem of viscous model equation* (2.9) *has global smooth solution.*

It is easy to see that initial data

$$Q_0(x) = \frac{bi}{x - x_0 + ai},$$

where a, b are non-zero real numbers, satisfies our condition here.

In the following, we treat our viscous model equation as a semi-linear parabolic system. The proof requires a closer look at our model nonlinearity and is based on the method of invariant region, see [7].

First we study the orbits of ODE $Q_t = -(i/2)\,Q^2$. Write $Q = \omega + iH$. Then

$$Q(t) = \frac{Q(0)}{1 + (t/2)\,iQ(0)} = \frac{\omega_0 + iH_0}{(1 - (t/2)\,H_0) + (t/2)\,i\omega_0}.$$

Now look at those initial data of ODE such that $H_0 = 0$ and $\omega_0 \neq 0$. We have

$$Q(t) - \frac{\omega_0}{2} = \frac{\omega_0}{1 + t\omega_0 i/2} - \frac{\omega_0}{2} = \left(\frac{\omega_0}{2}\right)\left(\frac{1 - t\omega_0 i/2}{1 + t\omega_0 i/2}\right).$$

Hence $|Q(t) - \omega_0/2| = |\omega_0|/2$ for all t. Thus all the circles in the left or right half plane which are centered on the x-axis and tangent to the y-axis at origin are ODE orbits. To prove Theorem 6, we need only find a closed curve such that the set $\{Q_v(x, t) \mid -\infty < x < \infty\}$, where $Q_v(x, t) = \omega_v(x, t) + H(\omega_v)(x, t)\, i$ is the solution of (2.9), sits inside for all time. For this purpose, a sufficient condition is that the curve should bound a region large enough to include the initial data $\{Q_0(x) \mid x \in R^1\}$ and the curve should be convex with the nonlinear term of equation (2.9), $(-(i/2)\,Q^2)$, pointing inward everywhere on this curve. Such a curve is not obvious to get here, but another general statement also holds. For a closed convex curve where the initial data of equation (2.9) lies inside, the solution can not first cross the part of the curve where the nonlinear term is pointing inward. Our ODE orbits above almost do except for two points: the direction of term $(-(i/2)\,Q^2)$ is tangent to the circle instead of pointing inside on the orbit; and these orbits are actually not closed because the origin is reached only at $t = \pm\infty$. Nevertheless, some perturbations on the orbits work. We can approximate a orbit circle by convex closed curves with the property that $(-(i/2)\,Q^2)$ points inside except near the origin.

Such a closed convex curve can be constructed by simply moving the center of the circle below the real axis an arbitrarily small amount, on which the vector field points inward on the boundary except near the origin. Now the vector field points to the right side of the lines in the first quadrant passing through the origin, and points to the right side of the curve along the curve $(\omega - c)^2 + H^4 = c^2$ in the fourth quadrant near the origin, where it intersects the bottom half of the translated circle. At the same time, we can clearly choose the slope of a line in the first quadrant to make it intersect the top half of the translated circle arbitrarily near the origin. Replacing an arc of the translated circle by the union of those two curves yields a convex region arbitrarily close to the original circle, on whose boundary the vector field points inward everywhere except at the origin. We thus conclude that a solution of (2.9) with initial data inside a circle can not cross it without first hitting the origin.

Next we show that the solution can not first hit the origin. For example, look at the situation on the right half plane. As long as it is bounded by the circle, we have $|Q_v(x, t)| \leqslant 2a$ where a is the radius. By similar arguments as in Section 3, we have $\omega_v(x, t) \geqslant \omega^*(t)$, where $\omega_v(x, t)$ is the real part of the solution of (2.9) and $\omega^*(t)$ is solution of ODE

$$\omega_t^*(t) = -2a\omega^*(t)$$

with initial data

$$\omega^*(0) = \inf_{-\infty < x < +\infty} \operatorname{Re}\{Q_0(x)\}.$$

Obviously $\omega^*(t) = \omega^*(0)\, e^{-2at} > 0$ for all t. Since any given initial data in the right half plane or the left half plane can be bounded by a ODE circle, we have proved Theorem 6.

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