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Wheeler–DeWitt equation in AdS/CFT correspondence

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Abstract

We discuss a quantum extension of the holographic RG flow equation obtained previously from the classical Hamiltonian constraint in the bulk AdS supergravity. The Wheeler–DeWitt equation is proposed to generate the extended RG flow and to produce $1/N$ subleading corrections systematically. Our formulation in five dimensions is applied to the derivation of the Weyl anomaly of boundary $\mathcal{N}=4$ $SU(N)$ super-Yang–Mills theory beyond the large N limit. It is shown that subleading $1/N^2$ corrections arising from fields in AdS_5 supergravity agree with those obtained recently by Mansfield et al. using their Schrödinger equation, thereby guaranteeing to reproduce the exact form of the boundary Weyl anomaly after summing up all of the KK modes.

Since the proposal of the AdS/CFT correspondence by Maldacena [1], a lot of effort has been made to test his conjecture. Among several others, Henningson and Skenderis derived the boundary theory Weyl anomaly in the large N limit by evaluating the tree-level on-shell action of the bulk AdS supergravity [2]. (See also [3].) The same result was also obtained by Mansfield and Nolland [4], where they constructed a functional Schrödinger equation for the partition function on AdS space and solved it at the tree-level. Recently the Schrödinger equation was fully considered to obtain a subleading correction to the large N result, and the exact form of the boundary Weyl anomaly has been successfully derived [5,6].

On the other hand, an alternative formulation of the bulk theory was given by de Boer, Verlinde and Verlinde [7], who gave attention to the fact that the radial flow in AdS space transverse to boundary directions corresponds to the renormalization group (RG) flow at the boundary. The classical Hamiltonian constraint arising from the reparameterization invariance of the bulk supergravity with respect to (w.r.t.) the radial direction was shown to be cast into the Callan–Symanzik RG flow equation on the boundary space. In this formulation, the derivation of the anomaly, at least in the large N limit, was seen to be performed more simply and quickly than with the other methods.

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The purpose of this Letter is to extend the classical Hamiltonian constraint to a quantum equation on the bulk AdS and to present a generalized holographic RG flow equation, which systematically produces subleading corrections in the $1/N$ expansion. Obviously, the Wheeler–DeWitt equation is a promising candidate for it and suggests that the subleading correction arises from terms with second-order functional derivatives w.r.t. fields of AdS supergravity, as happens in the ordinary WKB method of quantum mechanics. We demonstrate that the Wheeler–DeWitt equation applied to the $\mathrm{AdS}_5/\mathrm{CFT}_4$ case works indeed to derive the exact $1/N^2$ correction to the Weyl anomaly of the boundary $\mathcal{N}=4$ $SU(N)$ super-Yang–Mills (SYM) theory.

We start with a brief review of the tree-level (large N) calculation of the boundary Weyl anomaly via the holographic RG flow equation [7–9], in which the temporal gauge is used for the metric in $d+1$ dimensions,

$$ds^2 = g_{ab}\,dx^a\,dx^b = dr^2 + g_{\mu\nu}(x,r)\,dx^\mu\,dx^\nu \quad (\mu,\nu = 1,\ldots,d), \tag{1}$$

where $g_{\mu\nu}$ is dynamical in the bulk but supposed to tend to the form of an AdS_{d+1} metric asymptotically at the boundary $r\to\infty$ [10],

$$g_{\mu\nu}(x,r) = e^{2r/l}\left[\hat{g}_{\mu\nu}(x) - \frac{l^2}{d-2}\left(\widehat{R}_{\mu\nu} - \frac{1}{2(d-1)}\widehat{R}\hat{g}_{\mu\nu}\right)e^{-2r/l} + \mathcal{O}\big(\big(e^{-2r/l}\big)^2\big)\right]. \tag{2}$$

The curvatures $\widehat{R}_{\mu\nu}$, $\widehat{R}$ are defined with the boundary metric $\hat{g}_{\mu\nu}(x)$, and l is the AdS_{d+1} radius. As the AdS metric diverges at the boundary, we will take the large cut-off $r=r_0$. We consider the gravitational Lagrangian coupled to a massive scalar field in the bulk,

$$\mathcal{L}_{d+1} = \kappa^{-2}\sqrt{g}\left[-R_{d+1} + 2\Lambda + \frac{1}{2}g^{ab}\partial_a\phi\partial_b\phi + \frac{1}{2}m^2\phi^2\right], \tag{3}$$

where the cosmological constant $\Lambda = -d(d-1)/2l^2$. Then the Hamiltonian reads

$$H = \int d^dx\,\mathcal{H} = \int d^dx\left[-\kappa^2 g^{-1/2}P_{\mu\nu\lambda\sigma}(g)\pi^{\mu\nu}\pi^{\lambda\sigma} - \frac{1}{2}\kappa^2 g^{-1/2}\pi^2 + \mathcal{L}_d\right], \tag{4}$$

where $P_{\mu\nu\lambda\sigma}(g) = \frac{1}{2}(g_{\mu\lambda}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\lambda}) - \frac{1}{d-1}g_{\mu\nu}g_{\lambda\sigma}$ and

$$\mathcal{L}_d = \kappa^{-2}\sqrt{g}\left[-R + 2\Lambda + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}m^2\phi^2\right]. \tag{5}$$

Introducing the Hamilton–Jacobi functional $W(g,\phi)$ as $\pi^{\mu\nu} = \delta W/\delta g_{\mu\nu}$ and $\pi = \delta W/\delta\phi$, and inserting them into the Hamiltonian constraint $\mathcal{H}\approx 0$, we obtain the holographic RG flow equation [7],

$$\mathcal{H} = -\{W,W\} + \mathcal{L}_d = 0, \tag{6}$$

where

$$\{W,W\} = \kappa^2 g^{-1/2}\left[\left(\frac{\delta W}{\delta g_{\mu\nu}}\right)^2 - \frac{1}{d-1}\left(g_{\mu\nu}\frac{\delta W}{\delta g_{\mu\nu}}\right)^2 + \frac{1}{2}\left(\frac{\delta W}{\delta\phi}\right)^2\right]. \tag{7}$$

Note that the RG flow equation is defined on the surface $r=r_0$. In $d=4$, we decompose the functional W into the sum of S_{loc} and Γ,

$$S_{\mathrm{loc}} = \kappa^{-2}\int d^4x\,\sqrt{g}\left[U(\phi) - \Phi(\phi)R + \frac{1}{2}M(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right], \tag{8}$$

where $U(\phi)$, $\Phi(\phi)$, $M(\phi)$ are functions of ϕ expanded as

$$\begin{aligned} &U(\phi) = U_0 + U_1\phi + \frac{1}{2}U_2\phi^2 + \mathcal{O}\big(\phi^3\big), \qquad \Phi(\phi) = \Phi_0 + \Phi_1\phi + \frac{1}{2}\Phi_2\phi^2 + \mathcal{O}\big(\phi^3\big),\\ &M(\phi) = M_0 + \mathcal{O}(\phi). \end{aligned} \tag{9}$$

The terms in $S_{\rm loc}$ are the first three terms appearing in the derivative expansion of W. Inserting $W = S_{\rm loc} + \Gamma$ into (6) and comparing terms with the same weight ω (the number of differentiations) [8], we have a series of equations

$$\begin{aligned} \omega = 0: &\quad -\frac{U^2}{3} + \frac{1}{2}(U')^2 = -\frac{12}{l^2} + \frac{1}{2}m^2\phi^2, \\ \omega = 2: &\quad \left(\frac{U\Phi}{3} - U'\Phi'\right)R - \frac{ca}{6}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + U'\left(-M\Box\phi - \nabla^\mu M\nabla_\mu\phi + \frac{1}{2}M'g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) \\ &\quad = -R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi, \end{aligned} \tag{10}$$

where $'$ means the differentiation w.r.t. ϕ, which leads to

$$\begin{aligned} &U_0 = -6l^{-1}, \qquad U_1 = 0, \qquad U_2 = l^{-1}(\Delta_s - 4), \\ &\Phi_0 = \frac{l}{2}, \qquad \Phi_1 = 0, \qquad \Phi_2 = \frac{l(\Delta_s - 4)}{12(\Delta_s - 3)}, \quad M_0 = \frac{l}{2(\Delta_s - 3)}, \end{aligned} \tag{11}$$

where $\Delta_s = 2 + \sqrt{m^2l^2 + 4}$ is the scaling dimension of a boundary operator associated with ϕ, and

$$\omega = 4: \quad \Phi_0^2\kappa^{-2}\sqrt{g}\left(R^{\mu\nu}R_{\mu\nu} - \frac{1}{3}R^2\right) - \frac{U_0}{3}g_{\mu\nu}\frac{\delta\Gamma}{\delta g_{\mu\nu}} + \mathcal{O}(\phi^2) = 0. \tag{12}$$

Finally, taking the boundary values of the metric $g_{\mu\nu}(x, r_0) \to e^{2r_0/l}\hat{g}_{\mu\nu}(x)$ and the scalar field $\phi(x, r_0) \to 0$ as $r_0 \to \infty$, and using the relations $\kappa^{-2} = 1/16\pi G_5 = \mathrm{Vol}(S^5)/16\pi G_{10} = l^5\pi^3/16\pi(8\pi^6 g_s^2 l_s^8)$ and $l^8 = (4\pi)^2(g_sN)^2 l_s^8$ in (12), we obtain the Weyl anomaly of the boundary theory on the curved background $\hat{g}_{\mu\nu}$,

$$\langle T_\mu{}^\mu\rangle = -\frac{2}{\sqrt{\hat{g}}}\hat{g}_{\mu\nu}\frac{\delta\Gamma}{\delta\hat{g}_{\mu\nu}} = \frac{N^2}{32\pi^2}\left(\widehat{R}^{\mu\nu}\widehat{R}_{\mu\nu} - \frac{1}{3}\widehat{R}^2\right), \tag{13}$$

which exactly reproduces the Weyl anomaly of $\mathcal{N} = 4$ $SU(N)$ SYM at leading (large N) order. Note that the scalar field does not contribute to the final result (13) since, at leading order, there only appear the first-order derivatives of $S_{\rm loc}$ w.r.t. ϕ, which tend to zero when taking $\phi(x, r_0) \to 0$ as $r_0 \to \infty$. However, at subleading order, we have second-order derivatives w.r.t. the scalar field (and also w.r.t. all the other Kaluza–Klein (KK) modes appearing in the bulk supergravity [11]), which generally contribute to the subleading Weyl anomaly even when they take their vanishing boundary values, as will be shown below.

As is well known, the exact form of the boundary Weyl anomaly is given by (13) with the replacement $N^2 \to N^2 - 1$. The factor -1 represents a $1/N^2$ correction to the leading result and is expected to be derived from the quantum (one-loop) calculation of the bulk supergravity. A generalized version of the Hamiltonian constraint (6) responsible for the quantum case is the Wheeler–DeWitt equation, which is a quantum mechanical realization of (6) where a physical state Ψ has to be annihilated by the quantum operator $\mathcal{H}$, which guarantees the 'time' $r = r_0$ reparameterization invariance of Ψ as $\partial_{r_0}\Psi = -\int d^dx\,\mathcal{H}\Psi = 0$. The physical state, when expressed in terms of path integral, would be interpreted as the partition function with boundary values, $\Psi(g, \phi)$,

$$\mathcal{H}\Psi = -\kappa^2 g^{-1/2}\left(P_{\mu\nu\lambda\sigma}(g)\frac{\delta}{\delta g_{\mu\nu}}\frac{\delta}{\delta g_{\lambda\sigma}} + \frac{1}{2}\frac{\delta^2}{\delta\phi^2}\right)\Psi + \mathcal{L}_d\Psi = 0, \tag{14}$$

or equivalently with $\Psi(g, \phi) = e^{-W(g,\phi)}$,

$$\mathcal{H}\Psi/\Psi = -\{W, W\} + \mathcal{L}_d + \kappa^2 g^{-1/2}\left(P_{\mu\nu\lambda\sigma}(g)\frac{\delta^2 W}{\delta g_{\mu\nu}\delta g_{\lambda\sigma}} + \frac{1}{2}\frac{\delta^2 W}{\delta\phi^2}\right) = 0, \tag{15}$$

where the last two second-order derivative terms are of order $\kappa^2 \sim N^{-2}$ and thus subleading corrections to (6). We argue that this naive expression, however, leads to a misleading result and have to be more careful to define the quantum Hamiltonian operator associated with the time reparameterization.

To explain what is actually missing in (15), let us consider the scalar part of the partition function Ψ_s, with the boundary conditions, $\phi(x,r_i)=\phi_i(x)$ and $\phi(x,r_f)=\phi_f(x)$,

$$\Psi_s=\int \mathcal{D}\phi\, e^{\int_{r_i}^{r_f} dr \int d^d x\, \mathcal{L}_s(\phi,g)}, \quad \mathcal{L}_s=\kappa^{-2}\sqrt{g}\left[\frac{1}{2}g^{ab}\partial_a\phi\partial_b\phi+\frac{1}{2}m^2\phi^2\right], \tag{16}$$

where the path integral measure $\mathcal{D}\phi$ is induced by the reparameterization-invariant inner product $\|\delta\phi\|^2=\int d^{d+1}x\,\sqrt{g}\,\delta\phi^2$. By the standard canonical path integration, it is easily verified that Ψ_s is expressed in the operator formalism as

$$\langle\phi_f|T\exp\left[-\int_{r_i}^{r_f} dr\int d^d x\,\mathcal{H}_s\right]|\phi_i\rangle=\mathrm{g}_f^{1/8}\Psi_s\mathrm{g}_i^{1/8}, \quad \mathrm{g}_{i(f)}=\prod_x g(x,r_{i(f)}). \tag{17}$$

The extra factor $\mathrm{g}_f^{1/8}\mathrm{g}_i^{1/8}$ modifies the ordinary Hamiltonian operator $\mathcal{H}_s$ into $\widetilde{\mathcal{H}}_s$ in the Schrödinger equation, $\partial_{r_f}\Psi_s=-\int d^d x\,\widetilde{\mathcal{H}}_s\Psi_s$,

$$\widetilde{\mathcal{H}}_s=\frac{1}{2}\kappa^{-2}\sqrt{g_f}\left[-\kappa^4 g_f^{-1}\frac{\delta^2}{\delta\phi_f^2}+g_f^{\mu\nu}\partial_\mu\phi_f\partial_\nu\phi_f+m^2\phi_f^2\right]+\partial_{r_f}\ln g_f^{-1/8}\delta^d(0). \tag{18}$$

Note that the Schrödinger equation is identical to that derived in [4] when the metric is replaced with AdS background metric. Comparing (14), (15) and (18), we see that the extra subleading term $\partial_{r_0}\ln g^{-1/8}\delta^d(0)$ is needed in the scalar part of the Wheeler–DeWitt equation. It is straightforward to extend the argument to higher spin cases such as vector, tensor and fermionic fields [12]; for example, a similar calculation for a $(d+1)$-dimensional vector leads to its quantum Hamiltonian with the extra term $\partial_{r_0}\ln g^{-(d-2)/8d}\delta^\mu_\mu\delta^d(0)$ and so on. The Wheeler–DeWitt equation for the total partition function Ψ is thus defined with the subleading $\delta^d(0)$ term for each KK particle appearing in AdS_{d+1} supergravity.

Let us estimate the subleading contribution to the boundary anomaly arising from a five-dimensional scalar field, where the extra term combined with the last term in (15) gives

$$\begin{aligned}&\frac{1}{2}\kappa^2 g^{-1/2}\frac{\delta^2 W}{\delta\phi^2}+\partial_{r_0}\ln g^{-1/8}\delta^4(0)\\&\quad=\frac{1}{2}\left[M_0(-\Box)-\Phi_2 R+U_2+\partial_{r_0}\ln g^{-1/4}\right]\delta^4(0)+\frac{1}{2}\kappa^2 g^{-1/2}\frac{\delta^2\Gamma}{\delta\phi^2}+\mathcal{O}(\phi),\end{aligned} \tag{19}$$

where we ignore the last two terms on the right-hand side (RHS) since the second term is of weight $\omega=8$, while ϕ-dependent terms vanish as $\phi\to 0$. The operator in the first term stands in need of regularization; for example, it is carried out by the zeta-function regularization in which the operator is represented by a generalized zeta-function $\zeta(-1)$ as described in detail in [12]. After taking $r_0\to\infty$ and removing regularization-dependent divergent terms, we have a finite term given by the heat-kernel coefficient $a_2(x,x)$ of $\omega=4$ in the DeWitt–Schwinger proper time representation [13], showing that the subleading contribution does not modify the result (11). In the operator, terms with $-\Box$ and R are of next-to-leading order $\sim\mathcal{O}(e^{-2r_0/l})$ in the vicinity of the boundary, compared with $U_2\sim\mathcal{O}(1)$. We thus need leading and next-to-leading terms of the asymptotic AdS metric (2) to evaluate the operator

$$\frac{1}{2}\left[M_0\left(-\Box+\frac{1}{6}\widehat{R}\right)+(\Delta_s-2)l^{-1}\right]\delta^4(0)=\frac{\sqrt{\hat{g}}}{32\pi^2}(\Delta_s-2)l^{-1}a_2^{\xi=1/6}(x,x), \tag{20}$$

which shows that a 5D minimally coupled scalar gives the heat-kernel coefficient $a_2(x,x)$ of a 4D conformally coupled scalar. As the RHS of (20) comes in the RHS of the $\omega=4$ Eq. (12), we have, at the boundary $r_0\to\infty$,

$$\langle T_\mu{}^\mu\rangle=\frac{N^2}{32\pi^2}\left(\widehat{R}^{\mu\nu}\widehat{R}_{\mu\nu}-\frac{1}{3}\widehat{R}^2\right)-\frac{\Delta_s-2}{32\pi^2}a_2^{\xi=1/6}(x,x). \tag{21}$$

Interestingly, similar calculations for higher-spin fields in five dimensions exhibit that their extra $\delta^4(0)$ terms lead to four-dimensional conformally covariant operators, as the scalar case [12]. We see that the subleading correction (21) for the scalar and those for the higher-spin fields to the leading N^2 result are exactly the same as those previously obtained in the Schrödinger method [5,6], which guarantees the desired shift $N^2 \to N^2 - 1$ when summing up contributions from all of the KK particles in AdS supergravity,

$$\langle T_\mu{}^\mu \rangle = \frac{N^2}{32\pi^2}\left(\widehat{R}^{\mu\nu}\widehat{R}_{\mu\nu} - \frac{1}{3}\widehat{R}^2\right) - \sum_I \frac{\Delta_I - 2}{32\pi^2} a_2^I(x,x) = \frac{N^2-1}{32\pi^2}\left(\widehat{R}^{\mu\nu}\widehat{R}_{\mu\nu} - \frac{1}{3}\widehat{R}^2\right). \tag{22}$$

It is straightforward to generalize the above argument to $d =$ even dimensional case in which S_{loc} in (8) is given by the sum of all possible local terms with weight $\omega = 0$ to $d - 2$. For the leading large N result, see [2,8]. In the massive scalar theory (3), the subleading correction is given by the heat-kernel coefficient $a_{d/2}(x,x)$ for d-dimensional conformally coupled operator $-\Box + \xi_d \widehat{R}$, with $\xi_d = (d-2)/4(d-1)$,

$$\langle T_\mu{}^\mu \rangle_{\text{subleading}} = -\frac{\Delta_s - d/2}{2(4\pi)^{d/2}} a_{d/2}^{\xi_d}(x,x), \tag{23}$$

where $\Delta_s - d/2 = \sqrt{l^2m^2 + (d/2)^2}$. It will be discussed elsewhere how this result and those for higher-spin fields contribute to the boundary Weyl anomaly at subleading order in $\text{AdS}_3/\text{CFT}_2$ and $\text{AdS}_7/\text{CFT}_6$ cases.

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