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Extremal inverse eigenvalue problem for bordered diagonal matrices ☆

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Abstract

The following inverse eigenvalue problem was introduced and discussed in [J. Peng, X.Y. Hu, L. Zhang, Two inverse eigenvalue problems for a special kind of matrices, Linear Algebra Appl. 416 (2006) 336–347]: to construct a real symmetric bordered diagonal matrix A from the minimal and maximal eigenvalues of all its leading principal submatrices. However, the given formulae in [4, Theorem 1] to compute the matrix A may lead us to a matrix, which does not satisfy the requirements of the problem. In this paper, we rediscuss the problem to give a sufficient condition for the existence of such a matrix and necessary and sufficient conditions for the existence of a nonnegative such a matrix. Results are constructive and generate an algorithmic procedure to construct the matrices.

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Keywords: Symmetric bordered diagonal matrices; Matrix inverse eigenvalue problem

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1. Introduction

In this paper, we consider the problem of constructing a symmetric bordered diagonal matrix of the form:

$$A = \begin{pmatrix} a_1 & b_1 & b_2 & \cdots & b_{n-1} \\ b_1 & a_2 & 0 & \cdots & 0 \\ b_2 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-1} & 0 & 0 & \cdots & a_n \end{pmatrix}, \tag{1}$$

where $a_j, b_j \in \mathbb{R}$.

This class of matrices appears in certain symmetric inverse eigenvalue and inverse Sturm-Liouville problems, which arise in many applications, including control theory and vibration analysis [1–4].

We denote as I_j the identity matrix of order j; as A_j the $j \times j$ leading principal submatrix of A; as $P_j(\lambda)$ the characteristic polynomial of A_j and as $\lambda_1^{(j)} \leqslant \lambda_2^{(j)} \leqslant \cdots \leqslant \lambda_j^{(j)}$ the eigenvalues of A_j.

Our work is motivated by the results in [4]. There, the authors introduced two inverse eigenvalue problems, where a special spectral information is considered. An inverse eigenvalue problem for tridiagonal matrices with the same spectral information is also considered in [5]. One of the problems in [4], Problem I, is of our interest here:

Problem I [4]. For $2n-1$ given real numbers $\lambda_1^{(n)} < \lambda_1^{(n-1)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_n^{(n)}$, find an $n \times n$ matrix A of the form (1), with the a_i all distinct for $i = 2, 3, \ldots, n$ and the b_i all positive, such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are, respectively, the minimal and the maximal eigenvalue of A_j for all $j = 1, 2, \ldots, n$.

In [4, Theorem 1] is said that there is a unique solution of Problem I if and only if

$$\widetilde{h}_j = (-1)^{j-1}\left[P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) - P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right)\right] > 0. \tag{2}$$

We observe that the condition (2) is always satisfied under the hyphotesis of Problem I. Moreover, the formulae to compute the a_i and the b_i, given in [4, Theorem 1] may lead us to a matrix, which does not satisfy the requirements: the given real numbers $\lambda_1^{(4)} = 1$, $\lambda_1^{(3)} = 2$, $\lambda_1^{(2)} = 3$, $\lambda_1^{(1)} = 4$, $\lambda_2^{(2)} = 5$, $\lambda_3^{(3)} = 6$, $\lambda_4^{(4)} = 7$, satisfy the condition (2). However, the resulting matrix is

$$A = \begin{pmatrix} 4 & 1 & \sqrt{3} & \sqrt{5} \\ 1 & 4 & 0 & 0 \\ \sqrt{3} & 0 & 4 & 0 \\ \sqrt{5} & 0 & 0 & 4 \end{pmatrix}, \tag{3}$$

where the diagonal entries are not distinct.

In this paper, we consider the following more general problem:

Problem II. Given the $2n-1$ real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, find an $n \times n$ matrix A of the form (1) such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are, respectively, the minimal and the maximal eigenvalue of A_j, $j = 1, 2, \ldots, n$.

The paper is organized as follows: In Section 2, we solve Problem II by giving a necessary and sufficient condition for the existence of the matrix A in (1) and also solve the case in which the matrix A, in Problem II, is required to have all its entries b_i positive. In Section 3, we discuss Problem I in [4] and give a sufficient condition for its solution. In Section 4, we study the nonnegative case by giving a necessary and sufficient condition for the existence of a nonnegative matrix A of the form (1) such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are, respectively, the minimal and the maximal eigenvalue of A_j for all $j = 1, 2, \ldots, n$. Finally, in Section 5 we show some examples to illustrate the results.

2. Solution of Problem II

We start this section by recalling the following lemmas:

Lemma 1. *Let A be a matrix of the form (1). Then the sequence of characteristic polynomials $\{P_j(\lambda)\}_{j=1}^n$ satisfies the recurrence relation:*

$$\begin{aligned} P_1(\lambda) &= (\lambda - a_1), \\ P_2(\lambda) &= (\lambda - a_2)P_1(\lambda) - b_1^2, \\ P_j(\lambda) &= (\lambda - a_j)P_{j-1}(\lambda) - b_{j-1}^2 \textstyle\prod_{i=2}^{j-1}(\lambda - a_i), \quad j = 3, 4, \ldots, n. \end{aligned} \tag{4}$$

Lemma 2. *Let $P(\lambda)$ be a monic polynomial of degree n with all real zeroes. If λ_1 and λ_n are, respectively, the minimal and the maximal zero of $P(\lambda)$, then*

1. *If $\mu < \lambda_1$, we have that $(-1)^n P(\mu) > 0$.*
2. *If $\mu > \lambda_n$, we have that $P(\mu) > 0$.*

Proof. Let $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$. Thus, if $\mu < \lambda_1$ and n is odd then $P(\mu) < 0$. If $\mu < \lambda_1$ and n is even then $P(\mu) > 0$. Hence, $(-1)^n P(\mu) > 0$. If $\mu > \lambda_n$, then clearly $P(\mu) > 0$. □

Observe that from the Cauchy interlacing property, the minimal and the maximal eigenvalue, $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, respectively, of each leading principal submatrix A_j, $j = 1, 2, \ldots, n$, of the matrix A in (1) satisfy the relations:

$$\lambda_1^{(n)} \leqslant \cdots \leqslant \lambda_1^{(3)} \leqslant \lambda_1^{(2)} \leqslant \lambda_1^{(1)} \leqslant \lambda_2^{(2)} \leqslant \lambda_3^{(3)} < \cdots \leqslant \lambda_n^{(n)} \tag{5}$$

and

$$\lambda_1^{(j)} \leqslant a_i \leqslant \lambda_j^{(j)}, \quad i = 1, \ldots, j; \quad j = 1, \ldots, n. \tag{6}$$

Lemma 3. *Let $\{P_j(\lambda)\}_{j=1}^n$ be the polynomials defined in (4), whose minimal and maximal zeroes, $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, respectively, satisfy the relation (5). Then*

$$\widetilde{h}_j = (-1)^{j-1}\left[P_{j-1}(\lambda_1^{(j)})\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) - P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right)\right] \geqslant 0, \quad j = 2, 3, \ldots, n. \tag{7}$$

Proof. From Lemma 2, we have

$$(-1)^{j-1}P_{j-1}\left(\lambda_1^{(j)}\right) \geqslant 0 \quad \text{and} \quad P_{j-1}\left(\lambda_j^{(j)}\right) \geqslant 0. \tag{8}$$

Moreover, from (6)

$$\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) \geqslant 0 \tag{9}$$

and

$$(-1)^{j-1}\prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right) \leqslant 0. \tag{10}$$

Clearly $\widetilde{h}_j \geqslant 0$ follows from (8)–(10). □

The following theorem solves Problem II. In particular the theorem shows that the condition (5) is necessary and sufficient for the existence of the matrix A in (1).

Theorem 1. *Let the real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, be given. Then there exists an $n \times n$ matrix A of the form* (1), *such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are, respectively, the minimal and the maximal eigenvalue of its leading principal submatrix A_j, $j = 1, 2, \ldots, n$, if and only if*

$$\lambda_1^{(n)} \leqslant \cdots \leqslant \lambda_1^{(3)} \leqslant \lambda_1^{(2)} \leqslant \lambda_1^{(1)} \leqslant \lambda_2^{(2)} \leqslant \lambda_3^{(3)} \leqslant \cdots \leqslant \lambda_n^{(n)}. \tag{11}$$

Proof. Let $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, satisfying (11). Observe that

$$A_1 = [a_1] = \left[\lambda_1^{(1)}\right]$$

and $P_1(\lambda) = \lambda - a_1$. To show the existence of A_j, $j = 2, 3, \ldots, n$ with $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ as its minimal and maximal eigenvalues, respectively, is equivalent to show that the system of equations

$$\left.\begin{aligned} P_j\left(\lambda_1^{(j)}\right) &= \left(\lambda_1^{(j)} - a_j\right)P_{j-1}\left(\lambda_1^{(j)}\right) - b_{j-1}^2\prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right) = 0 \\ P_j\left(\lambda_j^{(j)}\right) &= \left(\lambda_j^{(j)} - a_j\right)P_{j-1}\left(\lambda_j^{(j)}\right) - b_{j-1}^2\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) = 0 \end{aligned}\right\} \tag{12}$$

has real solutions a_j and b_{j-1}, $j = 2, 3, \ldots, n$. If the determinant

$$h_j = P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) - P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right) \tag{13}$$

of the coefficients matrix of the system (12) is nonzero then the system has unique solutions a_j and b_{j-1}^2, $j = 2, 3, \ldots, n$. In this case, from Lemma 3 we have $\widetilde{h}_j > 0$. By solving the system (12) we obtain

$$a_j = \frac{\lambda_1^{(j)}P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) - \lambda_j^{(j)}P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right)}{h_j} \tag{14}$$

and

$$b_{j-1}^2 = \frac{\left(\lambda_j^{(j)} - \lambda_1^{(j)}\right) P_{j-1}\left(\lambda_1^{(j)}\right) P_{j-1}\left(\lambda_j^{(j)}\right)}{h_j}. \tag{15}$$

Since

$$(-1)^{j-1}\left(\lambda_j^{(j)} - \lambda_1^{(j)}\right) P_{j-1}\left(\lambda_1^{(j)}\right) P_{j-1}\left(\lambda_j^{(j)}\right) \geqslant 0,$$

then b_{j-1} is a real number and therefore, there exists A with the spectral properties required.

Now we will show that if $h_j = 0$, the system (12) still has a solution. We do this by induction by showing that the rank of the coefficients matrix is equal to the rank of the augmented matrix.

Let $j = 2$. If $h_2 = 0$ then

$$\begin{aligned}\widetilde{h}_2 &= (-1)^1 h_2 \\ &= (-1)^1\left[P_1\left(\lambda_1^{(2)}\right) - P_1\left(\lambda_2^{(2)}\right)\right] = 0,\end{aligned}$$

which, from Lemma 2, is equivalent to

$$P_1\left(\lambda_1^{(2)}\right) = 0 \quad \wedge \quad P_1\left(\lambda_2^{(2)}\right) = 0$$

and therefore

$$\lambda_1^{(2)} = \lambda_1^{(1)} \quad \wedge \quad \lambda_1^{(1)} = \lambda_2^{(2)}. \tag{16}$$

In this case the augmented matrix is

$$\left[\begin{array}{cc|c} P_1\left(\lambda_1^{(2)}\right) & 1 & \lambda_1^{(2)} P_1(\lambda_1^{(2)}) \\ P_1\left(\lambda_2^{(2)}\right) & 1 & \lambda_2^{(2)} P_1(\lambda_2^{(2)}) \end{array}\right]$$

and the ranks of both matrices, the coefficient matrix and the augmented matrix, are equal. Hence A_2 exists and has the form

$$A_2 = \begin{bmatrix} \lambda_1^{(1)} & 0 \\ 0 & \lambda_1^{(1)} \end{bmatrix}.$$

Now we consider $j \geqslant 3$. If $h_j = 0$ then

$$\begin{aligned}\widetilde{h}_j &= (-1)^{j-1} h_j \\ &= (-1)^{j-1}\left[P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) - P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right)\right] = 0.\end{aligned}$$

From Lemma 2

$$P_{j-1}\left(\lambda_1^{(j)}\right) = 0 \quad \vee \quad \prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) = 0$$

and

$$P_{j-1}\left(\lambda_j^{(j)}\right) = 0 \quad \vee \quad \prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right) = 0.$$

Then $h_j = 0$ leads us to the following cases:

(i) $\lambda_1^{(j)} = \lambda_1^{(j-1)} \quad \wedge \quad \lambda_{j-1}^{(j-1)} = \lambda_j^{(j)}$,

(ii) $\lambda_1^{(j)} = \lambda_1^{(j-1)} \quad \wedge \quad \prod_{i=2}^{j-1} \left(\lambda_1^{(j)} - a_i\right) = 0$,

(iii) $\prod_{i=2}^{j-1} \left(\lambda_j^{(j)} - a_i\right) = 0 \quad \wedge \quad \lambda_{j-1}^{(j-1)} = \lambda_j^{(j)}$,

(iv) $\prod_{i=2}^{j-1} \left(\lambda_j^{(j)} - a_i\right) = 0 \quad \wedge \quad \prod_{i=2}^{j-1} \left(\lambda_1^{(j)} - a_i\right) = 0$

and the augmented matrix is

$$\left[\begin{array}{cc|c} P_{j-1}\left(\lambda_1^{(j)}\right) & \prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right) & \lambda_1^{(j)} P_{j-1}\left(\lambda_1^{(j)}\right) \\ P_{j-1}\left(\lambda_j^{(j)}\right) & \prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) & \lambda_j^{(j)} P_{j-1}\left(\lambda_j^{(j)}\right) \end{array}\right]. \tag{17}$$

By replacing conditions (i)–(iii) in (17), it is clear that the coefficients matrix and the augmented matrix have the same rank. From condition (iv), the system of equations (12) becomes

$$\left.\begin{array}{l} P_{j-1}\left(\lambda_1^{(j)}\right) a_j = \lambda_1^{(j)} P_{j-1}\left(\lambda_1^{(j)}\right) \\ P_{j-1}\left(\lambda_j^{(j)}\right) a_j = \lambda_j^{(j)} P_{j-1}\left(\lambda_j^{(j)}\right) \end{array}\right\}.$$

If $P_{j-1}(\lambda_1^{(j)}) \neq 0$ and $P_{j-1}\left(\lambda_j^{(j)}\right) \neq 0$ then $a_j = \lambda_1^{(j)} = \lambda_j^{(j)}$ and from (11)

$$\lambda_1^{(j)} = \lambda_1^{(j-1)} = \cdots = \lambda_1^{(1)} = \cdots = \lambda_{j-1}^{(j-1)} = \lambda_j^{(j)}.$$

Thus, $P_{j-1}\left(\lambda_1^{(j)}\right) = P_{j-1}\left(\lambda_j^{(j)}\right) = 0$, which is a contradiction. Hence, under condition (iv) $P_{j-1}\left(\lambda_1^{(j)}\right) = 0$ or $P_{j-1}\left(\lambda_j^{(j)}\right) = 0$ and therefore, the coefficients matrix and the augmented matrix have also the same rank. By taking $b_{j-1}^2 \geqslant 0$, there exists a $j \times j$ matrix A_j with the required spectral properties. The necessity comes from the Cauchy interlacing property. □

We have seen in the proof of Theorem 1 that if the determinant h_j of the coefficients matrix of the system (12) is nonzero, then the Problem II has a unique solution except for the sign of the b_i entries.

Now we solve the Problem II in the case that the b_i entries are required to be positive. We need the following Lemma:

Lemma 4. *Let A be a matrix of the form* (1) *with $b_i \neq 0, i = 1, \ldots, n-1$. Let $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, respectively, be the minimal and the maximal eigenvalue of the leading principal submatrix A_j, $j = 1, 2, \ldots, n$, of A. Then*

$$\lambda_1^{(j)} < \cdots < \lambda_1^{(3)} < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \lambda_3^{(3)} < \cdots < \lambda_j^{(j)} \tag{18}$$

and

$$\lambda_1^{(j)} < a_i < \lambda_j^{(j)}, \quad i = 2, 3, \ldots, j \tag{19}$$

for each $j = 2, 3, \ldots, n$.

Proof. For $j = 2$, we have from (4)

$$\begin{aligned} P_2(\lambda) &= (\lambda - a_2)P_1(\lambda) - b_1^2 \\ &= (\lambda - a_2)(\lambda - \lambda_1^{(1)}) - b_1^2. \end{aligned}$$

As $b_1 \neq 0$, then $P_2(\lambda_1^{(1)}) \neq 0$ and from (5), we have

$$\lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)}.$$

If $\lambda_1^{(2)} = a_2$ or $\lambda_2^{(2)} = a_2$ then

$$0 = P_2(a_2) = (a_2 - a_2)P_1(a_2) - b_1^2 = -b_1^2$$

contradicts $b_1 \neq 0$ and from (6) we have

$$\lambda_1^{(2)} < a_2 < \lambda_2^{(2)}. \tag{20}$$

Let $j = 3$. Then from (4)

$$\begin{aligned} P_3\left(\lambda_1^{(2)}\right) &= \left(\lambda_1^{(2)} - a_3\right) P_2\left(\lambda_1^{(2)}\right) - b_2^2\left(\lambda_1^{(2)} - a_2\right) \\ &= -b_2^2\left(\lambda_1^{(2)} - a_2\right) \neq 0. \end{aligned}$$

In the same way $P_3(\lambda_2^{(2)}) \neq 0$. Hence, $\lambda_1^{(2)}$ and $\lambda_2^{(2)}$ are not zeroes of $P_3(\lambda)$ and from (5)

$$\lambda_1^{(3)} < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \lambda_3^{(3)}. \tag{21}$$

Now, suppose that $\lambda_1^{(3)} = a_3$. Then

$$\begin{aligned} 0 &= P_3(a_3) = (a_3 - a_3)P_2(a_3) - b_2^2(a_3 - a_2) \\ &= -b_2^2(a_3 - a_2) = -b_2^2\left(\lambda_1^{(3)} - a_2\right) \end{aligned}$$

contradicts the inequalities (20) and (21). Same occurs if we assume that $\lambda_3^{(3)} = a_3$. Then from (6) we have

$$\lambda_1^{(3)} < a_i < \lambda_3^{(3)}, \quad i = 2, 3.$$

Now, suppose that (18) and (19) hold for $4 \leqslant j \leqslant n-1$ and consider

$$P_{j+1}(\lambda) = (\lambda - a_{j+1})P_j(\lambda) - b_j^2 \prod_{i=2}^{j}(\lambda - a_i).$$

Since $b_j \neq 0$ and $\lambda_1^{(j)} < a_i < \lambda_j^{(j)}, i = 2, 3, \ldots, j,$ then $\prod_{i=2}^{j}\left(\lambda_1^{(j)} - a_i\right) \neq 0$ and $\prod_{i=2}^{j}\left(\lambda_j^{(j)} - a_i\right) \neq 0$. Hence $\lambda_1^{(j)}$ nor $\lambda_j^{(j)}$ are zeroes of $P_{j+1}(\lambda)$. Then from (5) we have

$$\lambda_1^{(j+1)} < \lambda_1^{(j)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_j^{(j)} < \lambda_{j+1}^{(j+1)}. \tag{22}$$

Finally, if $\lambda_1^{(j+1)} = a_{j+1}$ then

$$0 = P_{j+1}(a_{j+1}) = (a_{j+1} - a_{j+1})P_j(a_{j+1}) - b_j^2 \prod_{i=2}^{j}(a_{j+1} - a_i)$$
$$= -b_j^2 \prod_{i=2}^{j}(a_{j+1} - a_i) = -b_j^2 \prod_{i=2}^{j}\left(\lambda_1^{(j+1)} - a_i\right)$$

contradicts (22). Then from (6)

$$\lambda_1^{(j+1)} < a_i < \lambda_{j+1}^{(j+1)}, \quad i = 2, 3, \ldots, j+1. \qquad \square$$

The following corollary solves Problem II with $b_j > 0$.

Corollary 1. *Let the real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, be given. Then there exists a unique $n \times n$ matrix A of the form (1), with $a_j \in \mathbb{R}$ and $b_j > 0$, such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are, respectively, the minimal and the maximal eigenvalue of the leading principal submatrix A_j, $j = 1, \ldots, n$, of A, if and only if*

$$\lambda_1^{(n)} < \cdots < \lambda_1^{(3)} < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \lambda_3^{(3)} < \cdots < \lambda_n^{(n)}. \tag{23}$$

Proof. The proof is quite similar to the proof of Theorem 1: Let $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 2, \ldots, n$, satisfying (23). To show the existence of A_j, $j = 2, 3, \ldots, n$, with the required spectral properties, is equivalent to show that the system of equations (12) has real solutions a_j and b_{j-1}, with $b_{j-1} > 0$, $j = 2, 3, \ldots, n$. To do this it is enough to show that the determinant of the coefficients matrix

$$h_j = P_{j-1}\left(\lambda_1^{(j)}\right) \prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) - P_{j-1}\left(\lambda_j^{(j)}\right) \prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right) \tag{24}$$

be nonzero.

From Lemmas 3 and 4 it follows that $\widetilde{h}_j = (-1)^1 h_j > 0$. Hence $h_j \neq 0$ and the system (12) has real and unique solutions:

$$a_j = \frac{\lambda_1^{(j)} P_{j-1}\left(\lambda_1^{(j)}\right) \prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) - \lambda_j^{(j)} P_{j-1}\left(\lambda_j^{(j)}\right) \prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right)}{h_j} \tag{25}$$

and

$$b_{j-1}^2 = \frac{\left(\lambda_j^{(j)} - \lambda_1^{(j)}\right) P_{j-1}\left(\lambda_1^{(j)}\right) P_{j-1}\left(\lambda_j^{(j)}\right)}{h_j}, \tag{26}$$

where

$$(-1)^{j-1}\left(\lambda_j^{(j)} - \lambda_1^{(j)}\right) P_{j-1}\left(\lambda_1^{(j)}\right) P_{j-1}\left(\lambda_j^{(j)}\right) > 0.$$

Then it is clear that $b_{j-1}^2 > 0$. Therefore, the b_{j-1} can be chosen positive and then there exists a unique matrix A_j with the required spectral properties. The necessity of the result comes from Lemma 4. $\square$

3. Partial solution to Problem I

As it was observed in Section 1, Problem I in [4] has not been solved. In fact, the matrix A in (3) shows that to apply the formulae in [4, Theorem 1] may lead us to a matrix, which does not satisfy the requirements. In this section, we give a sufficient condition to solve Problem I. Previously, we give conditions under which we may construct a matrix of the form (1) with $a_i = a \in \mathbb{R}$, $i = 1, \ldots, n$ and $b_i \neq 0$. We start with the following:

Lemma 5. *Let A be a matrix of the form*

$$\widetilde{A} = \begin{pmatrix} 0 & b_1 & b_2 & \cdots & b_{n-1} \\ b_1 & 0 & 0 & \cdots & 0 \\ b_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-1} & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{with } b_j \neq 0, \quad 1 \leqslant j \leqslant n-1. \tag{27}$$

Let $\widetilde{P}_j(\lambda)$ be the characteristic polynomial of the leading principal submatrix $\widetilde{A}_j$ of $\widetilde{A}$, $j = 1, \ldots, n$. Then, if j is even, $\widetilde{P}_j(\lambda)$ is an even polynomial and if j is odd, $\widetilde{P}_j(\lambda)$ is a odd polynomial.

Proof. If $a_j = 0$, $j = 1, 2, \ldots, n$, then the recurrence relation (4) become

$$\begin{aligned} &\widetilde{P}_1(\lambda) = \lambda, \\ &\widetilde{P}_2(\lambda) = \lambda^2 - b_1^2, \\ &\widetilde{P}_j(\lambda) = \lambda \widetilde{P}_{j-1}(\lambda) - b_{j-1}^2(\lambda)^{j-2}, \quad j = 3, \ldots, n. \end{aligned} \tag{28}$$

Clearly, $\widetilde{P}_1(\lambda)$ is a odd polynomial, while $\widetilde{P}_2(\lambda)$ is an even polynomial. Now, suppose that $\widetilde{P}_j(\lambda)$ is even for an even j and that $\widetilde{P}_j(\lambda)$ is odd for a odd j. Let $j + 1$ be even. Then j is odd with $\widetilde{P}_j(\lambda)$ odd and $j - 1$ is even with $\widetilde{P}_{j-1}(\lambda)$ even. From (4), we have

$$\begin{aligned} \widetilde{P}_{j+1}(-\lambda) &= -\lambda \widetilde{P}_j(-\lambda) - b_j^2(-\lambda)^{j-1} \\ &= \lambda \widetilde{P}_j(\lambda) - b_j^2(\lambda)^{j-1} \\ &= \widetilde{P}_{j+1}(\lambda). \end{aligned}$$

Hence $\widetilde{P}_{j+1}(\lambda)$ is an even polynomial. Analogously if $j + 1$ is odd, $\widetilde{P}_{j+1}(-\lambda) = -\widetilde{P}_{j+1}(\lambda)$. □

Definition 1. We say that $\Gamma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is a balanced set if $\lambda_i = -\lambda_{n-i+1}$ with $\lambda_{\frac{n+1}{2}} = 0$ for odd n.

Thus, if $\lambda_1^{(1)} = 0$ and $\lambda_1^{(j)} = -\lambda_j^{(j)}$, $j = 2, 3, \ldots, n$, then the minimal and maximal eigenvalues $\{\lambda_1^{(j)}, \lambda_j^{(j)}\}$ of all leading principal submatrices $\widetilde{A}_j$ of $\widetilde{A}$ form a balanced set.

Corollary 2. *Let $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, be real numbers satisfying (23). Then there exists a unique $n \times n$ matrix $A = \widetilde{A} + aI$, $a \in \mathbb{R}$, where $\widetilde{A}$ is of the form (27), such that $\lambda_1^{(j)}$ and*

$\lambda_j^{(j)}$ *are, respectively, the minimal and the maximal eigenvalue of the leading principal submatrix* A_j *of* A, *if and only if*

$$\lambda_1^{(j)} + \lambda_j^{(j)} = 2\lambda_1^{(1)}, \quad j = 2, \ldots, n. \tag{29}$$

Proof. Let $\lambda_1^{(j)} + \lambda_j^{(j)} = 2\lambda_1^{(1)}$, $j = 2, \ldots, n$. It is enough to prove the result for a balanced set, that is, for $\lambda_1^{(1)} = 0$. Otherwise, if $\lambda_1^{(1)} \neq 0$, then define $\mu_i^{(j)} = \lambda_i^{(j)} - \lambda_1^{(1)}$, $j = 1, 2, \ldots, n$, $i = 1, j$ to obtain $\mu_1^{(1)} = 0$, $\mu_1^{(j)} = -\mu_j^{(j)}$, $j = 2, \ldots, n$. Hence, if there exists a unique matrix $\widetilde{A}$ of the form (27) such that $\mu_1^{(j)}$ and $\mu_j^{(j)}$ are, respectively, the minimal and the maximal eigenvalue of the leading principal submatrix $\widetilde{A}_j$, $j = 1, \ldots, n$, of $\widetilde{A}$, then $A = \widetilde{A} + \lambda_1^{(1)} I$ is the unique symmetric bordered diagonal matrix with the required spectral properties.

Let $\lambda_1^{(1)} = 0$ and $\lambda_1^{(j)} = -\lambda_j^{(j)}$, $j = 2, \ldots, n$. Since (23) holds, then from Corollary 1 there exists a unique matrix A of the form (1) with the required spectral properties. It only remains to show that $a_j = 0$, $j = 1, 2, \ldots, n$.

Clearly, $a_1 = \lambda_1^{(1)} = 0$ and $a_1 + a_2 = a_2 = \lambda_1^{(2)} + \lambda_2^{(2)} = 0$. Suppose that $a_k = 0$, $k = 1, 2, \ldots, j$; $j < n$. Let $k + 1$ be even. Then from Lemma 5, $P_k(\lambda)$ is odd and the numerator in (25) is

$$\begin{aligned}
&\lambda_1^{(k+1)} P_k\left(\lambda_1^{(k+1)}\right)\left(\lambda_{k+1}^{(k+1)}\right)^{k-1} - \lambda_{k+1}^{(k+1)} P_k\left(\lambda_{k+1}^{(k+1)}\right)\left(\lambda_1^{(k+1)}\right)^{k-1} \\
&= -\lambda_{k+1}^{(k+1)} P_k\left(-\lambda_{k+1}^{(k+1)}\right)\left(\lambda_{k+1}^{(k+1)}\right)^{k-1} - \lambda_{k+1}^{(k+1)} P_k\left(\lambda_{k+1}^{(k+1)}\right)\left(-\lambda_{k+1}^{(k+1)}\right)^{k-1} \\
&= \lambda_{k+1}^{(k+1)} P_k\left(\lambda_{k+1}^{(k+1)}\right)\left(\lambda_{k+1}^{(k+1)}\right)^{k-1} - \lambda_{k+1}^{(k+1)} P_k\left(\lambda_{k+1}^{(k+1)}\right)\left(\lambda_{k+1}^{(k+1)}\right)^{k-1} \\
&= 0
\end{aligned}$$

from where $a_{k+1} = 0$. Similarly, it can be shown that $a_{k+1} = 0$ when $k + 1$ is odd.

Now, let A be the unique $n \times n$ matrix of the form (1) with $a_j = a$, $j = 1, 2, \ldots, n$, $b_j \neq 0$, such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are, respectively, the minimal and the maximal eigenvalue of the leading principal submatrix A_j, $j = 1, 2, \ldots, n$, of A. Then $A = \widetilde{A} + aI$, with $a = \lambda_1^{(1)}$ and $\widetilde{A}$ of the form (27) having leading principal submatrices $\widetilde{A}_j$ with characteristic polynomials $\widetilde{P}_j(\lambda)$, $j = 1, 2, \ldots, n$. Since $\widetilde{P}_j(\lambda)$ even or $\widetilde{P}_j(\lambda)$ odd imply $\widetilde{P}_j(-\lambda) = 0$, then the eigenvalues $\mu_1^{(j)} < \mu_2^{(j)} < \cdots < \mu_{j-1}^{(j)} < \mu_j^{(j)}$ of $\widetilde{A}_j$ satisfy the relation $\mu_i^{(j)} + \mu_{j-i+1}^{(j)} = 0$. It is clear that the minimal and maximal eigenvalues of $\widetilde{A}_j$ are, respectively, $\lambda_1^{(j)} - \lambda_1^{(1)}$ and $\lambda_j^{(j)} - \lambda_1^{(1)}$, $j = 1, 2, \ldots, n$. Hence $\left(\lambda_1^{(j)} - \lambda_1^{(1)}\right) + \left(\lambda_j^{(j)} - \lambda_1^{(1)}\right) = 0$ and consequently, $\lambda_1^{(j)} + \lambda_j^{(j)} = 2\lambda_1^{(1)}$, $j = 1, 2, \ldots, n$. The proof is completed. □

The following result gives a sufficient condition in order Problem I to have a solution.

Theorem 2. *Let* $\lambda_1^{(j)}$ *and* $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, *be real numbers satisfying*

$$\lambda_1^{(n)} < \cdots < \lambda_1^{(3)} < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \lambda_3^{(3)} < \cdots < \lambda_n^{(n)}. \tag{30}$$

Then, there exists a unique $n \times n$ *matrix* A *of the form* (1), *with* $a_i \neq a_j$ *for* $i \neq j$ $(i, j = 1, 2, \ldots, n)$ *and* $b_i > 0$, *such that* $\lambda_1^{(j)}$ *and* $\lambda_j^{(j)}$ *are, respectively, the minimal and the maximal eigenvalue of the leading principal submatrix* A_j, $j = 1, 2, \ldots, n$, *of* A *if*

$$\lambda_1^{(2)} + \lambda_2^{(2)} \neq 2\lambda_1^{(1)} \tag{31}$$

and

$$\frac{\lambda_{j-1}^{(j-1)} - \lambda_j^{(j)}}{\lambda_{j-1}^{(j-1)} - \lambda_1^{(j)}} > \frac{P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right)}{P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right)} \tag{32}$$

or

$$\frac{\lambda_1^{(j-1)} - \lambda_j^{(j)}}{\lambda_1^{(j-1)} - \lambda_1^{(j)}} < \frac{P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right)}{P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right)}, \tag{33}$$

$j = 3, 4, \ldots, n$.

Proof. From Corollary 1, condition (30) guarantees the existence of a unique matrix A of the form (1) with $b_i > 0$ and the required spectral properties, and from Corollary 2, condition (31) is necessary and sufficient in order that $a_1 \neq a_2$. Now, let

$$u_j = P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right)$$

and

$$v_j = P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right),$$

$j = 3, 4, \ldots, n$ (the numerator and the denominator in the right side of (32)). Suppose that (32) holds for $j = 3$, that is

$$\frac{\lambda_2^{(2)} - \lambda_3^{(3)}}{\lambda_2^{(2)} - \lambda_1^{(3)}} > \frac{(-1)^2 u_3}{(-1)^2 v_3}.$$

Then

$$(-1)^2\left[\lambda_1^{(3)} u_3 - \lambda_3^{(3)} v_3\right] > \lambda_2^{(2)}(-1)^2[u_3 - v_3]$$

and

$$a_3 = \frac{(-1)^2[\lambda_1^{(3)} u_3 - \lambda_3^{(3)} v_3]}{(-1)^2[u_3 - v_3]} > \lambda_2^{(2)}.$$

From Lemma 4, we have $\lambda_1^{(2)} < a_1, a_2 < \lambda_2^{(2)}$. Hence, $a_3 \neq a_2 \neq a_1$.
Similarly, if (33) holds for $j = 3$, then $a_3 < \lambda_1^{(2)}$, and therefore, $a_3 \neq a_2 \neq a_1$.

Now, suppose that the a_i are all different, $i = 1, \ldots, j-1$ and (32) holds, that is

$$\frac{\lambda_{j-1}^{(j-1)} - \lambda_j^{(j)}}{\lambda_{j-1}^{(j-1)} - \lambda_1^{(j)}} > \frac{(-1)^{j-1} u_j}{(-1)^{j-1} v_j}.$$

Then

$$(-1)^{j-1}\left[\lambda_1^{(j)} u_j - \lambda_j^{(j)} v_j\right] > \lambda_{j-1}^{(j-1)}(-1)^{j-1}[u_j - v_j]$$

and therefore

$$a_j = \frac{(-1)^{j-1}\left[\lambda_1^{(j)}u_j - \lambda_j^{(j)}v_j\right]}{(-1)^{j-1}[u_j - v_j]} > \lambda_{j-1}^{(j-1)}.$$

From Lemma 4, we have $\lambda_1^{(j-1)} < a_i < \lambda_{j-1}^{(j-1)}$, $i = 1, \ldots, j-1$ and then $a_j \neq a_{j-1} \neq \cdots \neq a_2 \neq a_1$. Similarly if (33) holds then we obtain $a_j < \lambda_1^{(j-1)}$ and then $a_j \neq a_{j-1} \neq \cdots \neq a_2 \neq a_1$ again. □

We observe that a sufficient condition for Problem I can be obtained from (30) together with (32) or (33).

4. The nonnegative case

In this section, we look for conditions for the existence of a matrix A of the form (1) with $a_j \geqslant 0$, $b_j \geqslant 0$ and such that the given real numbers $\lambda_1^{(j)}, \lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, are, respectively, the minimal and the maximal eigenvalue of the leading principal submatrix A_j, $j = 1, 2, \ldots, n$, of A. We start by giving a necessary and sufficient condition for the existence of such a matrix, when $\lambda_1^{(j)}, \lambda_j^{(j)}$ are all distinct.

Corollary 3. *Let $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, be real numbers satisfying*

$$\lambda_1^{(n)} < \cdots < \lambda_1^{(3)} < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \lambda_3^{(3)} < \cdots < \lambda_n^{(n)}. \tag{34}$$

Then, there exists a unique $n \times n$ nonnegative matrix A of the form (1), such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are, respectively, the minimal and the maximal eigenvalue of the leading principal submatrix A_j, $j = 1, 2, \ldots, n$, of A if and only if

$$\lambda_1^{(1)} \geqslant 0 \tag{35}$$

and

$$\frac{\lambda_1^{(j)}}{\lambda_j^{(j)}} \geqslant \frac{P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right)}{P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right)}, \quad j = 2, 3, \ldots, n. \tag{36}$$

Proof. Corollary 1 guarantees the existence of a unique matrix A of the form (1) with $b_i > 0$, $i = 1, \ldots, n-1$. It remains to show that the diagonal elements a_i are nonnegative. From (35), $a_1 = \lambda_1^{(1)} \geqslant 0$ and from (36)

$$\frac{\lambda_1^{(j)}}{\lambda_j^{(j)}} \geqslant \frac{(-1)^{j-1}P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right)}{(-1)^{j-1}P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right)}, \quad j = 2, 3, \ldots, n.$$

Since $0 \leqslant \lambda_1^{(1)} < \lambda_j^{(j)}$ then from Lemmas 2 and 3

$$(-1)^{j-1}P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) > 0.$$

Then

$$\lambda_1^{(j)}(-1)^{j-1}P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)}-a_i\right) \geqslant \lambda_j^{(j)}(-1)^{j-1}P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)}-a_i\right)$$

or

$$\begin{aligned}\widetilde{g}_j = (-1)^{j-1}\Bigg[&\lambda_1^{(j)}P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)}-a_i\right)\\ &-\lambda_j^{(j)}P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)}-a_i\right)\Bigg] \geqslant 0.\end{aligned}$$

Hence, from the proof of Corollary 1, we obtain

$$a_j = \frac{\tilde{g}_j}{\tilde{h}_j} \geqslant 0.$$

Now, let us assume that there exists a unique $n \times n$ nonnegative matrix A of the form (1) with $b_i > 0, \lambda_1^{(j)}, \lambda_j^{(j)}, \ j = 1, \ldots, n$, satisfying (34) and being the minimal and the maximal eigenvalue of each leading principal submatrix A_j of A. From Lemma 4 the condition (30) is satisfied. Moreover, from the proof of Corollary 1, the diagonal elements of A are of the form

$$\begin{aligned} a_j &= \frac{\left[\lambda_1^{(j)}P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)}-a_i\right)-\lambda_j^{(j)}P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)}-a_i\right)\right]}{h_j}\\ &= \frac{(-1)^{j-1}\left[\lambda_1^{(j)}P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)}-a_i\right)-\lambda_j^{(j)}P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)}-a_i\right)\right]}{\widetilde{h}_j}\\ &\geqslant 0.\end{aligned}$$

with $\widetilde{h}_j > 0$. Then

$$(-1)^{j-1}\lambda_1^{(j)}P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)}-a_i\right) \geqslant (-1)^{j-1}\lambda_j^{(j)}P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)}-a_i\right),$$

that is

$$\begin{aligned}\frac{\lambda_1^{(j)}}{\lambda_j^{(j)}} &\geqslant \frac{(-1)^{j-1}P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)}-a_i\right)}{(-1)^{j-1}P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)}-a_i\right)}\\ &= \frac{P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)}-a_i\right)}{P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)}-a_i\right)}\end{aligned}$$

and the proof is completed. □

Now we discuss the case in which some of the given real numbers $\lambda_1^{(j)}, \lambda_j^{(j)}, \ j = 1, 2, \ldots, n$, are equal. It is clear that if $\lambda_1^{(n)} = \lambda_n^{(n)} = \alpha$, then $\lambda_1^{(j)} = \lambda_j^{(j)} = \alpha, \ j = 1, 2, \ldots, n$, and therefore, $A = \alpha I$.

Suppose that the determinant h_j of the coefficients matrix of the system (12) is nonzero, that is

$$h_j = P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) - P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right) \neq 0.$$

In this case the solution matrix A is unique, except for the sign of b_{j-1}, which we may choose as nonnegative. Then we examine conditions for the nonnegativity of

$$a_j = \frac{\left[\lambda_1^{(j)} P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) - \lambda_j^{(j)} P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right)\right]}{P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) - P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right)}.$$

Since $h_j \neq 0$, from Lemma 3 $\tilde{h}_j = (-1)^{j-1} h_j > 0$. Then from Lemma 2 we have the following cases:

$$\text{(i) } P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) \neq 0 \quad \text{and} \quad P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right) = 0.$$

Then $a_j = \lambda_1^{(j)}$ and $a_j \geqslant 0$ if $\lambda_1^{(j)} \geqslant 0$.

$$\text{(ii) } P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) = 0 \quad \text{and} \quad P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right) \neq 0.$$

Then $a_j = \lambda_j^{(j)}$ and $a_j \geqslant 0$ always occurs since $\lambda_1^{(1)} \leqslant \lambda_j^{(j)}$ and $0 \leqslant \lambda_1^{(1)}$ is a necessary condition:

$$\text{(iii) } P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) \neq 0 \quad \text{and} \quad P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right) \neq 0.$$

Then $\lambda_1^{(j)} < \lambda_1^{(j-1)}$ and $\lambda_{j-1}^{(j-1)} < \lambda_j^{(j)}$ and a necessary and sufficient condition for $a_j \geqslant 0$ is given by (35) and (36) of Corollary 3.

Now, suppose that $h_j = 0$. From Lemma 2 we have the following cases:

(i) $\lambda_1^{(j)} = \lambda_1^{(j-1)} \quad \text{and} \quad \lambda_{j-1}^{(j-1)} = \lambda_j^{(j)}$.

From (12) a_j can take any real value. Then we may choose $a_j \geqslant 0$. On the other hand

$$b_{j-1}^2 = 0 \vee \left(\prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right) = 0 \wedge \prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) = 0\right).$$

Thus $b_{j-1} = 0$ or b_{j-1} can be chosen as nonnegative.

$$\text{(ii) } \lambda_1^{(j)} = \lambda_1^{(j-1)} \wedge \prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right) = 0.$$

If $P_{j-1}(\lambda_j^{(j)}) = 0$, then a_j can take any real value. In particular, $a_j \geqslant 0$ and $b_{j-1} = 0$ or $b_{j-1} \geqslant 0$. If $P_{j-1}(\lambda_j^{(j)}) \neq 0$, then

$$a_j = \frac{\lambda_j^{(j)} P_{j-1}\left(\lambda_j^{(j)}\right) - \prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) b_{j-1}^2}{P_{j-1}\left(\lambda_j^{(j)}\right)}.$$

From Lemma 2, $P_{j-1}(\lambda_j^{(j)}) > 0$, and from (6), $\prod_{i=2}^{j-1}(\lambda_j^{(j)} - a_i) \geqslant 0$. Moreover, if $b_{j-1} \geqslant 0$, then $a_j \geqslant 0$ if

$$\lambda_j^{(j)} \geqslant \frac{\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) b_{j-1}^2}{P_{j-1}\left(\lambda_j^{(j)}\right)}.$$

(iii) $\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) = 0 \wedge \lambda_{j-1}^{(j-1)} = \lambda_j^{(j)}$.

If $P_{j-1}(\lambda_1^{(j)}) = 0$, a_j can be taken as nonnegative. Moreover, $b_{j-1} = 0$ or $b_{j-1} \geqslant 0$. If $P_{j-1}(\lambda_1^{(j)}) \neq 0$, then

$$a_j = \frac{\lambda_1^{(j)} P_{j-1}\left(\lambda_1^{(j)}\right) - \prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right) b_{j-1}^2}{P_{j-1}\left(\lambda_1^{(j)}\right)}.$$

From Lemma 2, $(-1)^{j-1} P_{j-1}(\lambda_1^{(j)}) > 0$, and from (6), $(-1)^{j-1}\prod_{i=2}^{j-1}(\lambda_1^{(j)} - a_i) \geqslant 0$. Moreover, if $b_{j-1} \geqslant 0$, then $a_j \geqslant 0$ if

$$\lambda_1^{(j)} \geqslant \frac{\prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right) b_{j-1}^2}{P_{j-1}\left(\lambda_1^{(j)}\right)}.$$

(iv) $\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) = 0 \wedge \prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right) = 0$.

In this case the system (12) reduces to

$$\left.\begin{aligned} P_{j-1}\left(\lambda_1^{(j)}\right) a_j + 0 b_{j-1}^2 &= \lambda_1^{(j)} P_{j-1}\left(\lambda_1^{(j)}\right) \\ P_{j-1}\left(\lambda_j^{(j)}\right) a_j + 0 b_{j-1}^2 &= \lambda_j^{(j)} P_{j-1}\left(\lambda_j^{(j)}\right) \end{aligned}\right\}.$$

We assume that $b_{j-1} \geqslant 0$. Then, if $P_{j-1}(\lambda_1^{(j)}) = 0$ and $P_{j-1}(\lambda a_j^{(j)}) = 0$, we may choose $a_j \geqslant 0$. If $P_{j-1}(\lambda_1^{(j)}) = 0$ and $P_{j-1}(\lambda_j^{(j)}) \neq 0$, then $a_j = \lambda_j^{(j)} \geqslant \lambda_1^{(1)} \geqslant 0$. If $P_{j-1}(\lambda_1^{(j)}) \neq 0$ and $P_{j-1}(\lambda_j^{(j)}) = 0$, then $a_j = \lambda_1^{(j)} \geqslant 0$ if $\lambda_1^{(j)} \geqslant 0$. Finally, the case $P_{j-1}(\lambda_1^{(j)}) \neq 0$ and $P_{j-1}(\lambda_j^{(j)}) \neq 0$ cannot occur.

5. Examples

Example 1. The following numbers:

$\lambda_1^{(5)}$	$\lambda_1^{(4)}$	$\lambda_1^{(3)}$	$\lambda_1^{(2)}$	$\lambda_1^{(1)}$	$\lambda_2^{(2)}$
−11.2369	−11.1921	−10.9106	−8.7760	−6.0043	−2.6295
$\lambda_3^{(3)}$	$\lambda_4^{(4)}$	$\lambda_5^{(5)}$			
1.8532	8.4266	10.4020			

satisfy the sufficient conditions (30)–(32) of the Theorem 2. Then the bordered diagonal matrix with $b_i > 0$ and $a_i \neq a_j$, $i \neq j$ is

$$A = \begin{pmatrix} -6.0043 & 3.0584 & 5.2453 & 2.9624 & 1.2602 \\ 3.0584 & -5.4011 & & & \\ 5.2453 & & -2.3357 & & \\ 2.9624 & & & 7.6429 & \\ 1.2602 & & & & 10.2504 \end{pmatrix}.$$

Example 2. We modify the previous example, in order that some given eigenvalues be equal:

$$\begin{array}{cccccc} \lambda_1^{(5)} & \lambda_1^{(4)} & \lambda_1^{(3)} & \lambda_1^{(2)} & \lambda_1^{(1)} & \lambda_2^{(2)} \\ -11.2369 & -10.9106 & -10.9106 & -8.7760 & -6.0043 & -6.0043 \\ \lambda_3^{(3)} & \lambda_4^{(4)} & \lambda_5^{(5)} & & & \\ 1.8532 & 8.4266 & 10.4020 & & & \end{array}$$

These numbers satisfy (11). One solution of Problem II is the matrix

$$A = \begin{pmatrix} -6.0043 & 0 & 6.2090 & 0 & 3.2977 \\ 0 & -8.7760 & & & \\ 6.2090 & & -3.0531 & & \\ 0 & & & 8.4266 & \\ 3.2977 & & & & 9.5989 \end{pmatrix}.$$

Example 3. The numbers

$$\begin{array}{cccccc} \lambda_1^{(5)} & \lambda_1^{(4)} & \lambda_1^{(3)} & \lambda_1^{(2)} & \lambda_1^{(1)} & \lambda_2^{(2)} \\ -3.8467 & -3.4048 & -3.3900 & -1.5635 & 0.2233 & 6.0818 \\ \lambda_3^{(3)} & \lambda_4^{(4)} & \lambda_5^{(5)} & & & \\ 9.4090 & 11.7029 & 15.3806 & & & \end{array}$$

satisfy relations (30)–(32), and relations (35) and (36). Then we obtain the nonnegative bordered diagonal matrix

$$A = \begin{pmatrix} 0.2233 & 3.2354 & 4.6803 & 0.5594 & 3.3490 \\ 3.2354 & 4.2950 & & & \\ 4.6803 & & 6.3405 & & \\ 0.5594 & & & 11.6505 & \\ 3.3490 & & & & 14.4225 \end{pmatrix}$$

with the required spectral properties.

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