Extremal inverse eigenvalue problem for bordered diagonal matrices

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Abstract

The following inverse eigenvalue problem was introduced and discussed in [J. Peng, X.Y. Hu, L. Zhang, Two inverse eigenvalue problems for a special kind of matrices, Linear Algebra Appl. 416 (2006) 336–347]: to construct a real symmetric bordered diagonal matrix A from the minimal and maximal eigenvalues of all its leading principal submatrices. However, the given formulae in [4, Theorem 1] to compute the matrix A may lead us to a matrix, which does not satisfy the requirements of the problem. In this paper, we rediscuss the problem to give a sufficient condition for the existence of such a matrix and necessary and sufficient conditions for the existence of a nonnegative such a matrix. Results are constructive and generate an algorithmic procedure to construct the matrices.

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1. Introduction

In this paper, we consider the problem of constructing a symmetric bordered diagonal matrix of the form:

\[
A = \begin{pmatrix}
a_1 & b_1 & b_2 & \cdots & b_{n-1} \\
b_1 & a_2 & 0 & \cdots & 0 \\
b_2 & 0 & a_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
b_{n-1} & 0 & 0 & \cdots & a_n
\end{pmatrix},
\]  
\( (1) \)

where \( a_j, b_j \in \mathbb{R} \).

This class of matrices appears in certain symmetric inverse eigenvalue and inverse Sturm-Liouville problems, which arise in many applications, including control theory and vibration analysis [1–4].

We denote as \( I_j \) the identity matrix of order \( j \); as \( A_j \) the \( j \times j \) leading principal submatrix of \( A \); as \( P_j(\lambda) \) the characteristic polynomial of \( A_j \) and as \( \lambda^{(j)}_1 \leq \lambda^{(j)}_2 \leq \cdots \leq \lambda^{(j)}_j \) the eigenvalues of \( A_j \).

Our work is motivated by the results in [4]. There, the authors introduced two inverse eigenvalue problems, where a special spectral information is considered. An inverse eigenvalue problem for tridiagonal matrices with the same spectral information is also considered in [5]. One of the problems in [4], Problem I, is of our interest here:

**Problem I** [4]. For \( 2n-1 \) given real numbers \( \lambda^{(n)}_1 < \lambda^{(n-1)}_1 < \cdots < \lambda^{(2)}_1 < \lambda^{(1)}_1 < \lambda^{(2)}_2 < \cdots < \lambda^{(n)}_n \), find an \( n \times n \) matrix \( A \) of the form (1), with the \( a_i \) all distinct for \( i = 2, 3, \ldots, n \) and the \( b_i \) all positive, such that \( \lambda^{(j)}_1 \) and \( \lambda^{(j)}_j \) are, respectively, the minimal and the maximal eigenvalue of \( A_j \) for all \( j = 1, 2, \ldots, n \).

In [4, Theorem 1] is said that there is a unique solution of Problem I if and only if

\[
\tilde{h}_j = (-1)^{j-1} \left[ P_{j-1} \left( \lambda^{(j)}_1 \right) \prod_{i=2}^{j-1} \left( \lambda^{(j)}_j - a_i \right) - P_{j-1} \left( \lambda^{(j)}_j \right) \prod_{i=2}^{j-1} \left( \lambda^{(j)}_1 - a_i \right) \right] > 0.  \tag{2}
\]

We observe that the condition (2) is always satisfied under the hypothesis of Problem I. Moreover, the formulae to compute the \( a_i \) and the \( b_i \), given in [4, Theorem 1] may lead us to a matrix, which does not satisfy the requirements: the given real numbers \( \lambda^{(4)}_1 = 1, \lambda^{(3)}_1 = 2, \lambda^{(2)}_1 = 3, \lambda^{(1)}_1 = 4, \lambda^{(2)}_2 = 5, \lambda^{(3)}_3 = 6, \lambda^{(4)}_4 = 7 \), satisfy the condition (2). However, the resulting matrix is

\[
A = \begin{pmatrix}
4 & 1 & \sqrt{3} & \sqrt{5} \\
1 & 4 & 0 & 0 \\
\sqrt{3} & 0 & 4 & 0 \\
\sqrt{5} & 0 & 0 & 4
\end{pmatrix},
\]  
\( (3) \)

where the diagonal entries are not distinct.

In this paper, we consider the following more general problem:

**Problem II**. Given the \( 2n-1 \) real numbers \( \lambda^{(j)}_1 \) and \( \lambda^{(j)}_j \), \( j = 1, 2, \ldots, n \), find an \( n \times n \) matrix \( A \) of the form (1) such that \( \lambda^{(j)}_1 \) and \( \lambda^{(j)}_j \) are, respectively, the minimal and the maximal eigenvalue of \( A_j \), \( j = 1, 2, \ldots, n \).
The paper is organized as follows: In Section 2, we solve Problem II by giving a necessary and sufficient condition for the existence of the matrix $A$ in (1) and also solve the case in which the matrix $A$, in Problem II, is required to have all its entries $b_i$ positive. In Section 3, we discuss Problem I in [4] and give a sufficient condition for its solution. In Section 4, we study the nonnegative case by giving a necessary and sufficient condition for the existence of a nonnegative matrix $A$ of the form (1) such that $\lambda(1)_{j}^{(j)}$ and $\lambda_{j}$ are, respectively, the minimal and the maximal eigenvalue of $A_j$ for all $j = 1, 2, \ldots, n$. Finally, in Section 5 we show some examples to illustrate the results.

2. Solution of Problem II

We start this section by recalling the following lemmas:

Lemma 1. Let $A$ be a matrix of the form (1). Then the sequence of characteristic polynomials $\{P_j(\lambda)\}_{j=1}^{n}$ satisfies the recurrence relation:

$$
P_1(\lambda) = (\lambda - a_1),
$$
$$
P_2(\lambda) = (\lambda - a_2) P_1(\lambda) - b_1^2,
$$
$$
P_j(\lambda) = (\lambda - a_j) P_{j-1}(\lambda) - b_{j-1}^2 \prod_{i=2}^{j-1} (\lambda - a_i), \quad j = 3, 4, \ldots, n.
$$

Lemma 2. Let $P(\lambda)$ be a monic polynomial of degree $n$ with all real zeroes. If $\lambda_1$ and $\lambda_n$ are, respectively, the minimal and the maximal zero of $P(\lambda)$, then

1. If $\mu < \lambda_1$, we have that $(-1)^n P(\mu) > 0$.
2. If $\mu > \lambda_n$, we have that $P(\mu) > 0$.

Proof. Let $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$. Thus, if $\mu < \lambda_1$ and $n$ is odd then $P(\mu) < 0$. If $\mu < \lambda_1$ and $n$ is even then $P(\mu) > 0$. Hence, $(-1)^n P(\mu) > 0$. If $\mu > \lambda_n$, then clearly $P(\mu) > 0$. □

Observe that from the Cauchy interlacing property, the minimal and the maximal eigenvalue, $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, respectively, of each leading principal submatrix $A_j$, $j = 1, 2, \ldots, n$, of the matrix $A$ in (1) satisfy the relations:

$$
\lambda^{(n)}_1 \leq \cdots \leq \lambda_1^{(3)} \leq \lambda_1^{(2)} \leq \lambda_1^{(1)} \leq \lambda_2^{(2)} \leq \lambda_3^{(3)} < \cdots < \lambda^{(n)}_n \quad (5)
$$

and

$$
\lambda_1^{(j)} \leq a_i \leq \lambda_j^{(j)}, \quad i = 1, \ldots, j; \quad j = 1, \ldots, n. \quad (6)
$$

Lemma 3. Let $\{P_j(\lambda)\}_{j=1}^{n}$ be the polynomials defined in (4), whose minimal and maximal zeroes, $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, respectively, satisfy the relation (5). Then

$$
\tilde{h}_j = (-1)^{j-1} \left[ P_{j-1}(\lambda_1^{(j)}) \prod_{i=2}^{j-1} (\lambda_j^{(j)} - a_i) - P_{j-1}(\lambda_1^{(j)}) \prod_{i=2}^{j-1} (\lambda_1^{(j)} - a_i) \right] \geq 0,
$$

$$
j = 2, 3, \ldots, n.
$$

(7)
Proof. From Lemma 2, we have
\[ (-1)^{-1} P_{j-1} \left( \lambda_1^{(j)} \right) \geq 0 \quad \text{and} \quad P_{j-1} \left( \lambda_j^{(j)} \right) \geq 0. \] (8)
Moreover, from (6)
\[ \prod_{i=2}^{j-1} \left( \lambda_j^{(j)} - a_i \right) \geq 0 \] (9)
and
\[ (-1)^{-1} \prod_{i=2}^{j-1} \left( \lambda_1^{(j)} - a_i \right) \leq 0. \] (10)
Clearly \( \tilde{h}_j \geq 0 \) follows from (8)–(10). □

The following theorem solves Problem II. In particular the theorem shows that the condition (5) is necessary and sufficient for the existence of the matrix \( A \) in (1).

**Theorem 1.** Let the real numbers \( \lambda_1^{(j)} \) and \( \lambda_j^{(j)} \), \( j = 1, 2, \ldots, n \), be given. Then there exists an \( n \times n \) matrix \( A \) of the form (1), such that \( \lambda_1^{(j)} \) and \( \lambda_j^{(j)} \) are, respectively, the minimal and the maximal eigenvalue of its leading principal submatrix \( A_j \), \( j = 1, 2, \ldots, n \), if and only if
\[ \lambda_1^{(n)} \leq \cdots \leq \lambda_1^{(3)} \leq \lambda_1^{(2)} \leq \lambda_1^{(1)} \leq \lambda_2^{(3)} \leq \cdots \leq \lambda_n^{(n)}. \] (11)
Proof. Let \( \lambda_1^{(j)} \) and \( \lambda_j^{(j)} \), \( j = 1, 2, \ldots, n \), satisfying (11). Observe that
\[ A_1 = [a_1] = \left[ \lambda_1^{(1)} \right] \]
and \( P_1(\lambda) = \lambda - a_1 \). To show the existence of \( A_j \), \( j = 2, 3, \ldots, n \) with \( \lambda_1^{(j)} \) and \( \lambda_j^{(j)} \) as its minimal and maximal eigenvalues, respectively, is equivalent to show that the system of equations
\[
\begin{align*}
P_j \left( \lambda_1^{(j)} \right) &= \left( \lambda_1^{(j)} - a_j \right) P_{j-1} \left( \lambda_1^{(j)} \right) - b_{j-1}^2 \prod_{i=2}^{j-1} \left( \lambda_1^{(j)} - a_i \right) = 0 \\
P_j \left( \lambda_j^{(j)} \right) &= \left( \lambda_j^{(j)} - a_j \right) P_{j-1} \left( \lambda_j^{(j)} \right) - b_{j-1}^2 \prod_{i=2}^{j-1} \left( \lambda_j^{(j)} - a_i \right) = 0
\end{align*}
\]
has real solutions \( a_j \) and \( b_{j-1}, \ j = 2, 3, \ldots, n \). If the determinant
\[ h_j = P_{j-1} \left( \lambda_1^{(j)} \right) \prod_{i=2}^{j-1} \left( \lambda_1^{(j)} - a_i \right) - P_{j-1} \left( \lambda_j^{(j)} \right) \prod_{i=2}^{j-1} \left( \lambda_j^{(j)} - a_i \right) \] (13)
of the coefficients matrix of the system (12) is nonzero then the system has unique solutions \( a_j \) and \( b_{j-1}, \ j = 2, 3, \ldots, n \). In this case, from Lemma 3 we have \( \tilde{h}_j > 0 \). By solving the system (12) we obtain
\[ a_j = \frac{\lambda_j^{(j)} P_{j-1} \left( \lambda_1^{(j)} \right) \prod_{i=2}^{j-1} \left( \lambda_1^{(j)} - a_i \right) - \lambda_1^{(j)} P_{j-1} \left( \lambda_j^{(j)} \right) \prod_{i=2}^{j-1} \left( \lambda_j^{(j)} - a_i \right)}{h_j} \] (14)
and
\[ b_{j-1}^2 = \left( \lambda_j^{(j)} - \lambda_j^{(j-1)} \right) \frac{\prod_{i=2}^{j-1} \left( \lambda_j^{(j)} - a_i \right)}{h_j}, \] (15)

Since
\[ (-1)^{j-1} \left( \lambda_j^{(j)} - \lambda_j^{(j-1)} \right) \prod_{i=2}^{j-1} \left( \lambda_j^{(j)} - a_i \right) \geq 0, \]
then \( b_{j-1} \) is a real number and therefore, there exists \( A \) with the spectral properties required.

Now we will show that if \( h_j = 0 \), the system (12) still has a solution. We do this by induction by showing that the rank of the coefficients matrix is equal to the rank of the augmented matrix.

Let \( j = 2 \). If \( h_2 = 0 \) then
\[ \tilde{h}_2 = (-1)^1 h_2 \]
\[ = (-1)^1 \left[ P_1 \left( \lambda_1^{(2)} \right) - P_1 \left( \lambda_2^{(2)} \right) \right] = 0, \]
which, from Lemma 2, is equivalent to
\[ P_1 \left( \lambda_1^{(2)} \right) = 0 \quad \text{and} \quad P_1 \left( \lambda_2^{(2)} \right) = 0 \]
and therefore
\[ \lambda_1^{(2)} = \lambda_1^{(1)} \quad \text{and} \quad \lambda_2^{(2)} = \lambda_2^{(1)}. \] (16)

In this case the augmented matrix is
\[ \begin{bmatrix} P_1 \left( \lambda_1^{(2)} \right) & 1 & \lambda_1^{(2)} P_1 \left( \lambda_1^{(2)} \right) \\ P_1 \left( \lambda_2^{(2)} \right) & 1 & \lambda_2^{(2)} P_1 \left( \lambda_2^{(2)} \right) \end{bmatrix} \]

and the ranks of both matrices, the coefficient matrix and the augmented matrix, are equal. Hence \( A_2 \) exists and has the form
\[ A_2 = \begin{bmatrix} \lambda_1^{(1)} & 0 \\ 0 & \lambda_1^{(1)} \end{bmatrix}. \]

Now we consider \( j \geq 3 \). If \( h_j = 0 \) then
\[ \tilde{h}_j = (-1)^{j-1} h_j \]
\[ = (-1)^{j-1} \left[ P_{j-1} \left( \lambda_1^{(j)} \right) \prod_{i=2}^{j-1} \left( \lambda_j^{(j)} - a_i \right) - P_{j-1} \left( \lambda_j^{(j)} \right) \prod_{i=2}^{j-1} \left( \lambda_1^{(j)} - a_i \right) \right] = 0. \]

From Lemma 2
\[ P_{j-1} \left( \lambda_1^{(j)} \right) = 0 \quad \text{and} \quad \prod_{i=2}^{j-1} \left( \lambda_j^{(j)} - a_i \right) = 0 \]
and
\[ P_{j-1} \left( \lambda_j^{(j)} \right) = 0 \quad \text{and} \quad \prod_{i=2}^{j-1} \left( \lambda_1^{(j)} - a_i \right) = 0. \]
Then $h_j = 0$ leads us to the following cases:

(i) $\lambda_1^{(j)} = \lambda_1^{(j-1)} \land \lambda_j^{(j-1)} = \lambda_j^{(j)}$,

(ii) $\lambda_1^{(j)} = \lambda_1^{(j-1)} \land \prod_{i=2}^{j-1} (\lambda_1^{(j)} - a_i) = 0$,

(iii) $\prod_{i=2}^{j-1} (\lambda_j^{(j)} - a_i) = 0 \land \lambda_j^{(j-1)} = \lambda_j^{(j)}$,

(iv) $\prod_{i=2}^{j-1} (\lambda_j^{(j)} - a_i) = 0 \land \prod_{i=2}^{j-1} (\lambda_j^{(j)} - a_i) = 0$

and the augmented matrix is

\[
\begin{bmatrix}
P_j-1(\lambda_1^{(j)}) & \prod_{i=2}^{j-1} (\lambda_1^{(j)} - a_i) & P_j-1(\lambda_1^{(j)}) \\
P_j-1(\lambda_j^{(j)}) & \prod_{i=2}^{j-1} (\lambda_j^{(j)} - a_i) & P_j-1(\lambda_j^{(j)})
\end{bmatrix}
\]  

(17)

By replacing conditions (i)–(iii) in (17), it is clear that the coefficients matrix and the augmented matrix have the same rank. From condition (iv), the system of equations (12) becomes

\[
\begin{aligned}
P_j-1(\lambda_1^{(j)}) a_j &= \lambda_1^{(j)} P_j-1(\lambda_1^{(j)}) \\
P_j-1(\lambda_j^{(j)}) a_j &= \lambda_j^{(j)} P_j-1(\lambda_j^{(j)})
\end{aligned}
\]

If $P_j-1(\lambda_1^{(j)}) \neq 0$ and $P_j-1(\lambda_j^{(j)}) \neq 0$ then $a_j = \lambda_1^{(j)} = \lambda_j^{(j)}$ and from (11)

\[
\lambda_1^{(j)} = \lambda_1^{(j-1)} = \cdots = \lambda_1^{(1)} = \cdots = \lambda_{j-1}^{(j-1)} = \lambda_j^{(j)}.
\]

Thus, $P_j-1(\lambda_1^{(j)}) = P_j-1(\lambda_j^{(j)}) = 0$, which is a contradiction. Hence, under condition (iv) $P_j-1(\lambda_1^{(j)}) = 0$ or $P_j-1(\lambda_j^{(j)}) = 0$ and therefore, the coefficients matrix and the augmented matrix have also the same rank. By taking $h_{j-1}^2 \geq 0$, there exists a $j \times j$ matrix $A_j$ with the required spectral properties. The necessity comes from the Cauchy interlacing property. □

We have seen in the proof of Theorem 1 that if the determinant $h_j$ of the coefficients matrix of the system (12) is nonzero, then the Problem II has a unique solution except for the sign of the $b_i$ entries.

Now we solve the Problem II in the case that the $b_i$ entries are required to be positive. We need the following Lemma:

**Lemma 4.** Let $A$ be a matrix of the form (1) with $b_i \neq 0$, $i = 1, \ldots, n-1$. Let $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, respectively, be the minimal and the maximal eigenvalue of the leading principal submatrix $A_j$, $j = 1, 2, \ldots, n$, of $A$. Then

\[
\lambda_1^{(j)} < \cdots < \lambda_1^{(3)} < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \lambda_3^{(3)} < \cdots < \lambda_j^{(j)}
\]  

(18)
and
\[ \lambda^{(j)}_1 < a_i < \lambda^{(j)}_j, \quad i = 2, 3, \ldots, j \]
for each \( j = 2, 3, \ldots, n \).

**Proof.** For \( j = 2 \), we have from (4)
\[
P_2(\lambda) = (\lambda - a_2)P_1(\lambda) - b_1^2
= (\lambda - a_2)(\lambda - \lambda^{(1)}_1) - b_1^2.
\]
As \( b_1 \neq 0 \), then \( P_2(\lambda^{(1)}_1) \neq 0 \) and from (5), we have
\[ \lambda^{(2)}_1 < \lambda^{(1)}_1 \leq \lambda^{(2)}_2. \]
If \( \lambda^{(2)}_1 = a_2 \) or \( \lambda^{(2)}_2 = a_2 \) then
\[ 0 = P_2(a_2) = (a_2 - a_2)P_1(a_2) - b_1^2 = -b_1^2 \]
contradicts \( b_1 \neq 0 \) and from (6) we have
\[ \lambda^{(2)}_1 < a_2 < \lambda^{(2)}_j. \]

Let \( j = 3 \). Then from (4)
\[
P_3(\lambda^{(2)}_1) = (\lambda^{(2)}_1 - a_3)P_2(\lambda^{(2)}_1) - b_2^2(\lambda^{(2)}_1 - a_2)
= -b_2^2(\lambda^{(2)}_1 - a_2) \neq 0.
\]
In the same way \( P_3(\lambda^{(2)}_2) \neq 0 \). Hence, \( \lambda^{(2)}_1 \) and \( \lambda^{(2)}_2 \) are not zeroes of \( P_3(\lambda) \) and from (5)
\[ \lambda^{(3)}_1 < \lambda^{(2)}_1 < \lambda^{(1)}_1 < \lambda^{(2)}_2 < \lambda^{(3)}_3. \]
Now, suppose that \( \lambda^{(3)}_1 = a_3 \). Then
\[
0 = P_3(a_3) = (a_3 - a_3)P_2(a_3) - b_2^2(a_3 - a_2)
= -b_2^2(a_3 - a_2) = -b_2^2(\lambda^{(3)}_1 - a_2)
\]
contradicts the inequalities (20) and (21). Same occurs if we assume that \( \lambda^{(3)}_3 = a_3 \). Then from (6) we have
\[ \lambda^{(3)}_1 < a_i < \lambda^{(3)}_3, \quad i = 2, 3. \]
Now, suppose that (18) and (19) hold for \( 4 \leq j \leq n - 1 \) and consider
\[
P_{j+1}(\lambda) = (\lambda - a_{j+1})P_j(\lambda) - b_j^2 \prod_{i=2}^j (\lambda - a_i).
\]
Since \( b_j \neq 0 \) and \( \lambda^{(j)}_1 < a_i < \lambda^{(j)}_j, i = 2, 3, \ldots, j \), then \( \prod_{i=2}^j (\lambda^{(j)}_i - a_i) \neq 0 \) and \( \prod_{i=2}^j (\lambda^{(j)}_j - a_i) \neq 0 \). Hence \( \lambda^{(j)}_1 \) nor \( \lambda^{(j)}_j \) are zeroes of \( P_{j+1}(\lambda) \). Then from (5) we have
\[
\lambda^{(j+1)}_1 < \lambda^{(j)}_1 < \cdots < \lambda^{(2)}_1 < \lambda^{(1)}_1 < \lambda^{(2)}_j < \cdots < \lambda^{(j)}_j < \lambda^{(j+1)}_j.
\]
Finally, if \( \lambda_1^{(j+1)} = a_{j+1} \) then

\[
0 = P_{j+1}(a_{j+1}) = (a_{j+1} - a_{j+1})P_j(a_{j+1}) - b_j^2 \prod_{i=2}^{j+1} (a_{j+1} - a_i)
\]

\[
= -b_j^2 \prod_{i=2}^{j+1} (a_{j+1} - a_i) = -b_j^2 \prod_{i=2}^{j+1} \left( \lambda_1^{(j+1)} - a_i \right)
\]

contradicts (22). Then from (6)

\[
\lambda_1^{(j+1)} < a_i < \lambda_j^{(j+1)}, \quad i = 2, 3, \ldots, j + 1.
\]

□

The following corollary solves Problem II with \( b_j > 0 \).

**Corollary 1.** Let the real numbers \( \lambda_1^{(j)} \) and \( \lambda_j^{(j)} \), \( j = 1, 2, \ldots, n \), be given. Then there exists a unique \( n \times n \) matrix \( A \) of the form (1), with \( a_j \in \mathbb{R} \) and \( b_j > 0 \), such that \( \lambda_1^{(j)} \) and \( \lambda_j^{(j)} \) are, respectively, the minimal and the maximal eigenvalue of the leading principal submatrix \( A_j \), \( j = 1, \ldots, n \), of \( A \), if and only if

\[
\lambda_1^{(n)} < \cdots < \lambda_1^{(3)} < \lambda_1^{(2)} < \lambda_2^{(2)} < \lambda_3^{(3)} < \cdots < \lambda_n^{(n)}.
\]

(23)

**Proof.** The proof is quite similar to the proof of Theorem 1: Let \( \lambda_1^{(j)} \) and \( \lambda_j^{(j)} \), \( j = 2, \ldots, n \), satisfying (23). To show the existence of \( A_j \), \( j = 2, 3, \ldots, n \), with the required spectral properties, is equivalent to show that the system of equations (12) has real solutions \( a_j \) and \( b_{j-1} \), with \( b_{j-1} > 0 \), \( j = 2, 3, \ldots, n \). To do this it is enough to show that the determinant of the coefficients matrix

\[
h_j = P_{j-1} \left( \lambda_1^{(j)} \right) \prod_{i=2}^{j-1} \left( \lambda_j^{(j)} - a_i \right) - P_{j-1} \left( \lambda_1^{(j)} \right) \prod_{i=2}^{j-1} \left( \lambda_1^{(j)} - a_i \right)
\]

be nonzero.

From Lemmas 3 and 4 it follows that \( \tilde{h}_j = (-1)^{j-1} h_j > 0 \). Hence \( h_j \neq 0 \) and the system (12) has real and unique solutions:

\[
a_j = \frac{\lambda_1^{(j)} P_{j-1} \left( \lambda_1^{(j)} \right) \prod_{i=2}^{j-1} \left( \lambda_j^{(j)} - a_i \right) - \lambda_j^{(j)} P_{j-1} \left( \lambda_1^{(j)} \right) \prod_{i=2}^{j-1} \left( \lambda_1^{(j)} - a_i \right)}{h_j}
\]

(25)

and

\[
b_{j-1}^2 = \frac{\left( \lambda_j^{(j)} - \lambda_1^{(j)} \right) P_{j-1} \left( \lambda_1^{(j)} \right) P_{j-1} \left( \lambda_j^{(j)} \right)}{h_j},
\]

(26)

where

\[
(-1)^{j-1} \left( \lambda_j^{(j)} - \lambda_1^{(j)} \right) P_{j-1} \left( \lambda_1^{(j)} \right) P_{j-1} \left( \lambda_j^{(j)} \right) > 0.
\]

Then it is clear that \( b_{j-1}^2 > 0 \). Therefore, the \( b_{j-1} \) can be chosen positive and then there exists a unique matrix \( A_j \) with the required spectral properties. The necessity of the result comes from Lemma 4. □
3. Partial solution to Problem I

As it was observed in Section 1, Problem I in [4] has not been solved. In fact, the matrix $A$ in (3) shows that to apply the formulae in [4, Theorem 1] may lead us to a matrix, which does not satisfy the requirements. In this section, we give a sufficient condition to solve Problem I. Previously, we give conditions under which we may construct a matrix of the form (1) with $a_i = a \in \mathbb{R}$, $i = 1, \ldots, n$ and $b_i \neq 0$. We start with the following:

**Lemma 5.** Let $A$ be a matrix of the form

$$
\tilde{A} = \begin{pmatrix}
0 & b_1 & b_2 & \cdots & b_{n-1} \\
b_1 & 0 & 0 & \cdots & 0 \\
b_2 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
b_{n-1} & 0 & 0 & \cdots & 0
\end{pmatrix}
$$

with $b_j \neq 0$, $1 \leq j \leq n - 1$. (27)

Let $\tilde{P}_j(\lambda)$ be the characteristic polynomial of the leading principal submatrix $\tilde{A}_j$ of $\tilde{A}$, $j = 1, \ldots, n$. Then, if $j$ is even, $\tilde{P}_j(\lambda)$ is an even polynomial and if $j$ is odd, $\tilde{P}_j(\lambda)$ is a odd polynomial.

**Proof.** If $a_j = 0$, $j = 1, 2, \ldots, n$, then the recurrence relation (4) become

$$
\tilde{P}_1(\lambda) = \lambda, \\
\tilde{P}_2(\lambda) = \lambda^2 - b_1^2, \\
\tilde{P}_j(\lambda) = \lambda \tilde{P}_{j-1}(\lambda) - b_{j-1}^2(\lambda)^{j-2}, \quad j = 3, \ldots, n.
$$

Clearly, $\tilde{P}_1(\lambda)$ is a odd polynomial, while $\tilde{P}_2(\lambda)$ is an even polynomial. Now, suppose that $\tilde{P}_j(\lambda)$ is even for an even $j$ and that $\tilde{P}_j(\lambda)$ is odd for a odd $j$. Let $j + 1$ be even. Then $j$ is odd with $\tilde{P}_j(\lambda)$ odd and $j - 1$ is even with $\tilde{P}_{j-1}(\lambda)$ even. From (4), we have

$$
\tilde{P}_{j+1}(-\lambda) = -\lambda \tilde{P}_j(-\lambda) - b_j^2(-\lambda)^{j-1} \\
= \lambda \tilde{P}_j(\lambda) - b_j^2(\lambda)^{j-1} \\
= \tilde{P}_{j+1}(\lambda).
$$

Hence $\tilde{P}_{j+1}(\lambda)$ is an even polynomial. Analogously if $j + 1$ is odd, $\tilde{P}_{j+1}(-\lambda) = -\tilde{P}_{j+1}(\lambda)$. □

**Definition 1.** We say that $\Gamma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is a balanced set if $\lambda_i = -\lambda_{n-i+1}$ with $\lambda_2 = 0$ for odd $n$.

Thus, if $\lambda_1^{(1)} = 0$ and $\lambda_j^{(j)} = -\lambda_j^{(j)}$, $j = 2, 3, \ldots, n$, then the minimal and maximal eigenvalues $\{\lambda_1^{(j)}, \lambda_j^{(j)}\}$ of all leading principal submatrices $\tilde{A}_j$ of $\tilde{A}$ form a balanced set.

**Corollary 2.** Let $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, be real numbers satisfying (23). Then there exists a unique $n \times n$ matrix $A = \tilde{A} + aI$, $a \in \mathbb{R}$, where $\tilde{A}$ is of the form (27), such that $\lambda_1^{(j)}$ and
Theorem 2. Let \( \lambda_j^{(i)} \) and \( \lambda_j^{(j)} \), \( i = 1, 2, \ldots, n \), be real numbers satisfying
\[
\lambda_1^{(i)} < \cdots < \lambda_1^{(3)} < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \lambda_3^{(3)} < \cdots < \lambda_n^{(n)}.
\]
(30)

Then, there exists a unique \( n \times n \) matrix \( A \) of the form (1), with \( a_i \neq a_j \) for \( i \neq j \) (i, \( j = 1, 2, \ldots, n \)) and \( b_i > 0 \), such that \( \lambda_1^{(i)} \) and \( \lambda_1^{(j)} \) are, respectively, the minimal and the maximal eigenvalue of the leading principal submatrix \( A_j \) of \( A \), if and only if
\[
\lambda_1^{(j)} + \lambda_j^{(j)} = 2\lambda_1^{(1)}, \quad j = 2, \ldots, n.
\]
(29)

Proof. Let \( \lambda_1^{(j)} + \lambda_j^{(j)} = 2\lambda_1^{(1)}, \ j = 2, \ldots, n \). It is enough to prove the result for a balanced set, that is, for \( \lambda_1^{(1)} = 0 \). Otherwise, if \( \lambda_1^{(1)} \neq 0 \), then define \( \mu_i^{(j)} = \lambda_i^{(j)} - \lambda_1^{(1)}, j = 1, 2, \ldots, n, i = 1, j \) to obtain \( \mu_1^{(1)} = 0, \mu_1^{(j)} = -\mu_j^{(j)}, j = 2, \ldots, n \). Hence, if there exists a unique matrix \( \tilde{A} \) of the form (27) such that \( \mu_1^{(j)} \) and \( \mu_j^{(j)} \) are, respectively, the minimal and the maximal eigenvalue of the leading principal submatrix \( \tilde{A}_j, j = 1, \ldots, n \), of \( \tilde{A} \), then \( A = \tilde{A} + \lambda_1^{(1)}I \) is the unique symmetric bordered diagonal matrix with the required spectral properties.

Let \( \lambda_1^{(1)} = 0 \) and \( \lambda_1^{(j)} = -\lambda_j^{(j)}, j = 2, \ldots, n \). Since (23) holds, then from Corollary 1 there exists a unique matrix \( A \) of the form (1) with the required spectral properties. It only remains to show that \( a_j = 1, 2, \ldots, n \).

Clearly, \( a_1 = \lambda_1^{(1)} = 0 \) and \( a_1 + a_2 = a_2 = \lambda_1^{(2)} + \lambda_2^{(2)} = 0 \). Suppose that \( a_k = 0, k = 1, 2, \ldots, j; j < n \). Let \( k + 1 \) be even. Then from Lemma 5, \( P_k(\lambda) \) is odd and the numerator in (25) is
\[
\lambda_1^{(k+1)} P_k \left( \lambda_1^{(k+1)} \right) \left( \lambda_2^{(k+1)} \right)^{k-1} - \lambda_2^{(k+1)} P_k \left( \lambda_2^{(k+1)} \right) \left( \lambda_1^{(k+1)} \right)^{k-1}
\]
\[
= -\lambda_2^{(k+1)} P_k \left( -\lambda_2^{(k+1)} \right) \left( \lambda_1^{(k+1)} \right)^{k-1} - \lambda_1^{(k+1)} P_k \left( \lambda_1^{(k+1)} \right) \left( \lambda_2^{(k+1)} \right)^{k-1}
\]
\[
= \lambda_1^{(k+1)} P_k \left( \lambda_1^{(k+1)} \right) \left( \lambda_1^{(k+1)} \right)^{k-1} - \lambda_2^{(k+1)} P_k \left( \lambda_2^{(k+1)} \right) \left( \lambda_1^{(k+1)} \right)^{k-1}
\]
\[
= -\lambda_1^{(k+1)} P_k \left( \lambda_1^{(k+1)} \right) \left( \lambda_1^{(k+1)} \right)^{k-1} - \lambda_2^{(k+1)} P_k \left( \lambda_2^{(k+1)} \right) \left( \lambda_1^{(k+1)} \right)^{k-1}
\]
\[
= 0
\]
from where \( a_{k+1} = 0 \). Similarly, it can be shown that \( a_{k+1} = 0 \) when \( k + 1 \) is odd.

Now, let \( A \) be the unique \( n \times n \) matrix of the form (1) with \( a_j = a, j = 1, 2, \ldots, n, b_j \neq 0 \), such that \( \lambda_1^{(j)} \) and \( \lambda_j^{(j)} \) are, respectively, the minimal and the maximal eigenvalue of the leading principal submatrix \( A_j, j = 1, 2, \ldots, n \), of \( A \). Then \( A = \tilde{A} + aI \), with \( a = \lambda_1^{(1)} \) and \( \tilde{A} \) of the form (27) having leading principal submatrices \( \tilde{A}_j \) with characteristic polynomials \( \tilde{P}_j(\lambda), j = 1, 2, \ldots, n \). Since \( \tilde{P}_j(\lambda) \) even or odd imply \( \tilde{P}_j(\lambda - 0) = 0 \), then the eigenvalues \( \mu_1^{(j)} < \mu_2^{(j)} < \cdots < \mu_j^{(j)} < \mu_j^{(j)} \) of \( \tilde{A}_j \) satisfy the relation \( \mu_1^{(j)} + \mu_j^{(j)} = 0 \). It is clear that the minimal and maximal eigenvalues of \( \tilde{A}_j \) are, respectively, \( \lambda_1^{(1)} - \lambda_1^{(1)} \) and \( \lambda_j^{(j)} - \lambda_1^{(1)} \), \( j = 1, 2, \ldots, n \). Hence \( \lambda_1^{(1)} - \lambda_1^{(1)} + \lambda_j^{(j)} - \lambda_1^{(1)} = 0 \) and consequently, \( \lambda_1^{(1)} + \lambda_j^{(j)} = 2\lambda_1^{(1)}, j = 1, 2, \ldots, n \). The proof is completed. \( \Box \)

The following result gives a sufficient condition in order Problem I to have a solution.

Theorem 2. Let \( \lambda_1^{(j)} \) and \( \lambda_j^{(j)} \), \( j = 1, 2, \ldots, n \), be real numbers satisfying
\[
\lambda_1^{(n)} < \cdots < \lambda_1^{(3)} < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \lambda_3^{(3)} < \cdots < \lambda_n^{(n)}.
\]
(30)
\[ \lambda_1^{(2)} + \lambda_2^{(2)} \neq 2\lambda_1^{(1)} \tag{31} \]

and

\[ \frac{\lambda_{j-1}^{(j-1)} - \lambda_j^{(j)}}{\lambda_1^{(j)} - \lambda_1^{(j)}} > \frac{P_{j-1}(\lambda_1^{(j)}) \prod_{i=2}^{j-1} (\lambda_j^{(j)} - a_i)}{P_{j-1}(\lambda_j^{(j)}) \prod_{i=2}^{j-1} (\lambda_1^{(j)} - a_i)} \tag{32} \]

or

\[ \frac{\lambda_1^{(j-1)} - \lambda_j^{(j)}}{\lambda_1^{(j-1)} - \lambda_1^{(j)}} < \frac{P_{j-1}(\lambda_1^{(j)}) \prod_{i=2}^{j-1} (\lambda_j^{(j)} - a_i)}{P_{j-1}(\lambda_j^{(j)}) \prod_{i=2}^{j-1} (\lambda_1^{(j)} - a_i)} \tag{33} \]

\( j = 3, 4, \ldots, n. \)

**Proof.** From Corollary 1, condition (30) guarantees the existence of a unique matrix \( A \) of the form (1) with \( b_j > 0 \) and the required spectral properties, and from Corollary 2, condition (31) is necessary and sufficient in order that \( a_1 \neq a_2 \). Now, let

\[ u_j = P_{j-1}(\lambda_1^{(j)}) \prod_{i=2}^{j-1} (\lambda_j^{(j)} - a_i) \]

and

\[ v_j = P_{j-1}(\lambda_j^{(j)}) \prod_{i=2}^{j-1} (\lambda_1^{(j)} - a_i), \]

\( j = 3, 4, \ldots, n \) (the numerator and the denominator in the right side of (32)). Suppose that (32) holds for \( j = 3 \), that is

\[ \frac{\lambda_2^{(2)} - \lambda_3^{(3)}}{\lambda_2^{(2)} - \lambda_1^{(3)}} > \frac{(-1)^2 u_3}{(-1)^2 v_3}. \]

Then

\[ (-1)^2 \left[ \lambda_1^{(3)} u_3 - \lambda_3^{(3)} v_3 \right] > \lambda_2^{(2)} (-1)^2 [u_3 - v_3] \]

and

\[ a_3 = \frac{(-1)^2 [\lambda_1^{(3)} u_3 - \lambda_3^{(3)} v_3]}{(-1)^2 [u_3 - v_3]} > \lambda_2^{(2)}. \]

From Lemma 4, we have \( \lambda_1^{(2)} < a_1, a_2 < \lambda_2^{(2)} \). Hence, \( a_3 \neq a_2 \neq a_1 \).

Similarly, if (33) holds for \( j = 3 \), then \( a_3 < \lambda_1^{(2)} \), and therefore, \( a_3 \neq a_2 \neq a_1 \).

Now, suppose that the \( a_i \) are all different, \( i = 1, \ldots, j - 1 \) and (32) holds, that is

\[ \frac{\lambda_{j-1}^{(j-1)} - \lambda_j^{(j)}}{\lambda_1^{(j)} - \lambda_1^{(j)}} > \frac{(-1)^{j-1} u_j}{(-1)^{j-1} v_j}. \]

Then

\[ (-1)^{j-1} \left[ \lambda_1^{(j)} u_j - \lambda_j^{(j)} v_j \right] > \lambda_{j-1}^{(j-1)} (-1)^{j-1} [u_j - v_j] \]
and therefore
\[ a_j = \frac{(-1)^{j-1} \left( \lambda_1^{(j)} u_j - \lambda_j^{(j)} v_j \right)}{(-1)^{j-1}[u_j - v_j]} > \lambda_j^{(j-1)}. \]

From Lemma 4, we have \( \lambda_1^{(j-1)} < a_i < \lambda_j^{(j-1)}, \) \( i = 1, \ldots, j-1 \) and then \( a_j \neq a_{j-1} \neq \cdots \neq a_2 \neq a_1. \) Similarly if (33) holds then we obtain \( a_j < \lambda_1^{(j-1)} \) and then \( a_j \neq a_{j-1} \neq \cdots \neq a_2 \neq a_1 \) again. \( \square \)

We observe that a sufficient condition for Problem I can be obtained from (30) together with (32) or (33).

4. The nonnegative case

In this section, we look for conditions for the existence of a matrix \( A \) of the form (1) with \( a_j \geq 0, b_j \geq 0 \) and such that the given real numbers \( \lambda_1^{(j)}, \lambda_j^{(j)}, j = 1, 2, \ldots, n, \) are, respectively, the minimal and the maximal eigenvalue of the leading principal submatrix \( A_j, j = 1, 2, \ldots, n, \) of \( A. \) We start by giving a necessary and sufficient condition for the existence of such a matrix, when \( \lambda_1^{(j)}, \lambda_j^{(j)} \) are all distinct.

**Corollary 3.** Let \( \lambda_1^{(j)} \) and \( \lambda_j^{(j)} , j = 1, 2, \ldots, n, \) be real numbers satisfying
\[ \lambda_1^{(n)} < \cdots < \lambda_1^{(3)} < \lambda_1^{(2)} < \lambda_2^{(2)} < \lambda_1^{(3)} < \cdots < \lambda_n^{(n)}. \] (34)

Then, there exists a unique \( n \times n \) nonnegative matrix \( A \) of the form (1), such that \( \lambda_1^{(j)} \) and \( \lambda_j^{(j)} \) are, respectively, the minimal and the maximal eigenvalue of the leading principal submatrix \( A_j, j = 1, 2, \ldots, n, \) of \( A. \) if and only if
\[ \lambda_1^{(1)} \geq 0 \] (35)

and
\[ \frac{\lambda_1^{(j)}}{\lambda_j^{(j)}} \geq \frac{P_{j-1}(\lambda_j^{(j)}) \prod_{i=2}^{j-1} (\lambda_1^{(j)} - a_i)}{P_{j-1}(\lambda_1^{(j)}) \prod_{i=2}^{j-1} (\lambda_j^{(j)} - a_i)}, \quad j = 2, 3, \ldots, n. \] (36)

**Proof.** Corollary 1 guarantees the existence of a unique matrix \( A \) of the form (1) with \( b_i > 0, \ i = 1, \ldots, n-1. \) It remains to show that the diagonal elements \( a_i \) are nonnegative. From (35), \( a_1 = \lambda_1^{(1)} \geq 0 \) and from (36)
\[ \frac{\lambda_1^{(j)}}{\lambda_j^{(j)}} \geq \frac{(-1)^{j-1} P_{j-1}(\lambda_j^{(j)}) \prod_{i=2}^{j-1} (\lambda_1^{(j)} - a_i)}{(-1)^{j-1} P_{j-1}(\lambda_1^{(j)}) \prod_{i=2}^{j-1} (\lambda_j^{(j)} - a_i)}, \quad j = 2, 3, \ldots, n. \]

Since \( 0 \leq \lambda_1^{(1)} < \lambda_j^{(j)} \) then from Lemmas 2 and 3
\[ (-1)^{j-1} P_{j-1}(\lambda_1^{(j)}) \prod_{i=2}^{j-1} (\lambda_j^{(j)} - a_i) > 0. \]
Then
\[ \lambda_{1}^{(j)}(-1)^{j-1}P_{j-1}\left(\lambda_{1}^{(j)}\prod_{i=2}^{j-1}(\lambda_{j}^{(j)}-a_{i})\right) \geq \lambda_{j}^{(j)}(-1)^{j-1}P_{j-1}\left(\lambda_{j}^{(j)}\prod_{i=2}^{j-1}(\lambda_{1}^{(j)}-a_{i})\right) \]
or
\[ \tilde{g}_{j} = (-1)^{j-1}\left[\lambda_{1}^{(j)}P_{j-1}\left(\lambda_{1}^{(j)}\prod_{i=2}^{j-1}(\lambda_{j}^{(j)}-a_{i})\right) - \lambda_{j}^{(j)}P_{j-1}\left(\lambda_{j}^{(j)}\prod_{i=2}^{j-1}(\lambda_{1}^{(j)}-a_{i})\right) \right] \geq 0. \]

Hence, from the proof of Corollary 1, we obtain
\[ a_{j} = \frac{\tilde{g}_{j}}{h_{j}} \geq 0. \]

Now, let us assume that there exists a unique \( n \times n \) nonnegative matrix \( A \) of the form (1) with \( b_{i} > 0, \lambda_{1}^{(j)}, \lambda_{j}^{(j)}, j = 1, \ldots, n \), satisfying (34) and being the minimal and the maximal eigenvalue of each leading principal submatrix \( A_{j} \) of \( A \). From Lemma 4 the condition (30) is satisfied. Moreover, from the proof of Corollary 1, the diagonal elements of \( A \) are of the form
\[ a_{j} = \frac{\tilde{g}_{j}}{h_{j}} \geq 0. \]

with \( \tilde{h}_{j} > 0 \). Then
\[ (-1)^{j-1}\lambda_{1}^{(j)}P_{j-1}\left(\lambda_{1}^{(j)}\prod_{i=2}^{j-1}(\lambda_{j}^{(j)}-a_{i})\right) \geq (-1)^{j-1}\lambda_{j}^{(j)}P_{j-1}\left(\lambda_{j}^{(j)}\prod_{i=2}^{j-1}(\lambda_{1}^{(j)}-a_{i})\right), \]

that is
\[ \frac{\lambda_{1}^{(j)}}{\lambda_{j}^{(j)}} \geq \frac{(-1)^{j-1}P_{j-1}\left(\lambda_{1}^{(j)}\prod_{i=2}^{j-1}(\lambda_{1}^{(j)}-a_{i})\right)}{(-1)^{j-1}P_{j-1}\left(\lambda_{j}^{(j)}\prod_{i=2}^{j-1}(\lambda_{j}^{(j)}-a_{i})\right)} \]
\[ = \frac{P_{j-1}\left(\lambda_{1}^{(j)}\prod_{i=2}^{j-1}(\lambda_{1}^{(j)}-a_{i})\right)}{P_{j-1}\left(\lambda_{j}^{(j)}\prod_{i=2}^{j-1}(\lambda_{j}^{(j)}-a_{i})\right)} \]
and the proof is completed. □

Now we discuss the case in which some of the given real numbers \( \lambda_{1}^{(j)}, \lambda_{j}^{(j)}, j = 1, 2, \ldots, n \), are equal. It is clear that if \( \lambda_{1}^{(n)} = \lambda_{n}^{(n)} = \alpha \), then \( \lambda_{1}^{(j)} = \lambda_{j}^{(j)} = \alpha, j = 1, 2, \ldots, n \), and therefore, \( A = \alpha I \).
Suppose that the determinant $h_j$ of the coefficients matrix of the system (12) is nonzero, that is
\[ h_j = P_{j-1}(\lambda_{1}^{(j)}) \prod_{i=2}^{j-1} (\lambda_{j}^{(j)} - a_i) - P_{j-1}(\lambda_{1}^{(j)}) \prod_{i=2}^{j-1} (\lambda_{1}^{(j)} - a_i) \neq 0. \]
In this case the solution matrix $A$ is unique, except for the sign of $b_{j-1}$, which we may choose as nonnegative. Then we examine conditions for the nonnegativity of $\lambda(j)$ from Lemma 2 we have the following cases:

(i) $P_{j-1}(\lambda_{1}^{(j)}) \prod_{i=2}^{j-1} (\lambda_{j}^{(j)} - a_i) \neq 0$ and $P_{j-1}(\lambda_{1}^{(j)}) \prod_{i=2}^{j-1} (\lambda_{1}^{(j)} - a_i) = 0$.

Then $a_j = \lambda_{1}^{(j)}$ and $a_j \geq 0$ if $\lambda_{1}^{(j)} \geq 0$.

(ii) $P_{j-1}(\lambda_{1}^{(j)}) \prod_{i=2}^{j-1} (\lambda_{j}^{(j)} - a_i) = 0$ and $P_{j-1}(\lambda_{1}^{(j)}) \prod_{i=2}^{j-1} (\lambda_{1}^{(j)} - a_i) \neq 0$.

Then $a_j = \lambda_{j}^{(j)}$ and $a_j \geq 0$ always occurs since $\lambda_{1}^{(1)} \leq \lambda_{j}^{(j)}$ and $0 \leq \lambda_{1}^{(1)}$ is a necessary condition:

(iii) $P_{j-1}(\lambda_{1}^{(j)}) \prod_{i=2}^{j-1} (\lambda_{j}^{(j)} - a_i) \neq 0$ and $P_{j-1}(\lambda_{1}^{(j)}) \prod_{i=2}^{j-1} (\lambda_{1}^{(j)} - a_i) \neq 0$.

Then $\lambda_{1}^{(j)} < \lambda_{1}^{(j-1)}$ and $\lambda_{1}^{(j-1)} < \lambda_{1}^{(j)}$ and a necessary and sufficient condition for $a_j \geq 0$ is given by (35) and (36) of Corollary 3.

Now, suppose that $h_j = 0$. From Lemma 2 we have the following cases:

(i) $\lambda_{1}^{(j)} = \lambda_{1}^{(j-1)}$ and $\lambda_{j-1}^{(j-1)} = \lambda_{j}^{(j)}$.

From (12) $a_j$ can take any real value. Then we may choose $a_j \geq 0$. On the other hand
\[ b_{j-1}^2 = 0 \lor \left( \prod_{i=2}^{j-1} (\lambda_{1}^{(j)} - a_i) = 0 \land \prod_{i=2}^{j-1} (\lambda_{j}^{(j)} - a_i) = 0 \right). \]
Thus $b_{j-1} = 0$ or $b_{j-1}$ can be chosen as nonnegative.

(ii) $\lambda_{1}^{(j)} = \lambda_{1}^{(j-1)} \land \prod_{i=2}^{j-1} (\lambda_{1}^{(j)} - a_i) = 0$.

If $P_{j-1}(\lambda_{j}^{(j)}) = 0$, then $a_j$ can take any real value. In particular, $a_j \geq 0$ and $b_{j-1} = 0$ or $b_{j-1} \geq 0$.

If $P_{j-1}(\lambda_{j}^{(j)}) \neq 0$, then
\[ a_j = \frac{\lambda_{j}^{(j)} P_{j-1}(\lambda_{j}^{(j)}) - \prod_{i=2}^{j-1} (\lambda_{j}^{(j)} - a_i) b_{j-1}^2}{P_{j-1}(\lambda_{j}^{(j)})}. \]
From Lemma 2, \( P_{j-1}(\lambda_j^{(j)}) > 0 \), and from (6), \( \prod_{i=2}^{j-1} (\lambda_j^{(j)} - a_i) \geq 0 \). Moreover, if \( b_{j-1} \geq 0 \), then \( a_j \geq 0 \) if
\[
\lambda_j^{(j)} \geq \frac{\prod_{i=2}^{j-1} (\lambda_j^{(j)} - a_i) b_j^2}{P_{j-1}(\lambda_j^{(j)})}.
\]
(iii) \( \prod_{i=2}^{j-1} (\lambda_j^{(j)} - a_i) = 0 \) \( \wedge \lambda_{j-1}^{(j-1)} = \lambda_j^{(j)} \).

If \( P_{j-1}(\lambda_1^{(j)}) = 0 \), \( a_j \) can be taken as nonnegative. Moreover, \( b_{j-1} = 0 \) or \( b_{j-1} \geq 0 \). If \( P_{j-1}(\lambda_1^{(j)}) \neq 0 \), then
\[
a_j = \frac{\lambda_1^{(j)} P_{j-1}(\lambda_1^{(j)}) - \prod_{i=2}^{j-1} (\lambda_1^{(j)} - a_i) b_j^2}{P_{j-1}(\lambda_1^{(j)})}.
\]

From Lemma 2, \((-1)^{j-1} P_{j-1}(\lambda_1^{(j)}) > 0 \), and from (6), \((-1)^{j-1} \prod_{i=2}^{j-1} (\lambda_1^{(j)} - a_i) \geq 0 \). Moreover, if \( b_{j-1} \geq 0 \), then \( a_j \geq 0 \) if
\[
\lambda_1^{(j)} \geq \frac{\prod_{i=2}^{j-1} (\lambda_1^{(j)} - a_i) b_j^2}{P_{j-1}(\lambda_1^{(j)})}.
\]
(iv) \( \prod_{i=2}^{j-1} (\lambda_1^{(j)} - a_i) = 0 \) \( \wedge \prod_{i=2}^{j-1} (\lambda_1^{(j)} - a_i) = 0 \).

In this case the system (12) reduces to
\[
\begin{align*}
P_{j-1}(\lambda_1^{(j)}) a_j + 0b_{j-1}^2 &= \lambda_1^{(j)} P_{j-1}(\lambda_1^{(j)}) \quad \text{and} \\
P_{j-1}(\lambda_1^{(j)}) a_j + 0b_{j-1}^2 &= \lambda_1^{(j)} P_{j-1}(\lambda_1^{(j)}) \quad \text{if} \\
\end{align*}
\]

We assume that \( b_{j-1} \geq 0 \). Then, if \( P_{j-1}(\lambda_1^{(j)}) = 0 \) and \( P_{j-1}(\lambda_1^{(j)}) = 0 \), we may choose \( a_j \geq 0 \). If \( P_{j-1}(\lambda_1^{(j)}) = 0 \) and \( P_{j-1}(\lambda_1^{(j)}) \neq 0 \), then \( a_j = \lambda_1^{(j)} \geq \lambda_1^{(1)} \geq 0 \). If \( P_{j-1}(\lambda_1^{(j)}) \neq 0 \) and \( P_{j-1}(\lambda_1^{(j)}) = 0 \), then \( a_j = \lambda_1^{(j)} \geq 0 \) if \( \lambda_1^{(1)} \geq 0 \). Finally, the case \( P_{j-1}(\lambda_1^{(j)}) \neq 0 \) and \( P_{j-1}(\lambda_1^{(j)}) \neq 0 \) cannot occur.

5. Examples

Example 1. The following numbers:

\[
\begin{align*}
\lambda_1^{(5)} &= -11.2369 \quad \lambda_1^{(4)} &= -11.1921 \quad \lambda_1^{(3)} &= -10.9106 \quad \lambda_1^{(2)} &= -8.7760 \quad \lambda_1^{(1)} &= -6.0043 \quad \lambda_2^{(2)} = -2.6295 \\
\lambda_2^{(3)} &= \lambda_1^{(4)} \quad \lambda_3^{(5)} &= \lambda_5^{(5)} \\
1.8532 &= 8.4266 \quad 10.4020 \\
\end{align*}
\]

satisfy the sufficient conditions (30)–(32) of the Theorem 2. Then the bordered diagonal matrix with \( b_i > 0 \) and \( a_i \neq a_j, i \neq j \) is
Example 2. We modify the previous example, in order that some given eigenvalues be equal:

\[
\begin{pmatrix}
\lambda_1^{(5)} & \lambda_1^{(4)} & \lambda_1^{(3)} & \lambda_1^{(2)} & \lambda_1^{(1)} & \lambda_2^{(2)} \\
1.8532 & 8.4266 & 10.4020 \\
\end{pmatrix}
\]

These numbers satisfy (11). One solution of Problem II is the matrix

\[
\begin{pmatrix}
-6.0043 & 0 & 6.2090 & 0 & 3.2977 \\
0 & -8.7760 & 0 \\
3.2977 & 8.4266 & 9.5989 \\
\end{pmatrix}
\]

Example 3. The numbers

\[
\begin{pmatrix}
\lambda_1^{(5)} & \lambda_1^{(4)} & \lambda_1^{(3)} & \lambda_1^{(2)} & \lambda_1^{(1)} & \lambda_2^{(2)} \\
-3.8467 & -3.4048 & -3.3900 & -1.5635 & 0.2233 & 6.0818 \\
1.8532 & 8.4266 & 10.4020 \\
\end{pmatrix}
\]

satisfy relations (30)–(32), and relations (35) and (36). Then we obtain the nonnegative bordered diagonal matrix

\[
\begin{pmatrix}
0.2233 & 3.2354 & 4.6803 & 0.5594 & 3.3490 \\
3.2354 & 4.2950 \\
0.5594 & 11.6505 \\
3.3490 & 14.4225 \\
\end{pmatrix}
\]

with the required spectral properties.

References