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Maximum and minimum toughness of graphs of small genus

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Abstract

A new lower bound on the toughness $t(G)$ of a graph G in terms of its connectivity $\kappa(G)$ and genus $\gamma(G)$ is obtained. For $\gamma > 0$, the bound is sharp via an infinite class of extremal graphs all of girth 4. For planar graphs, the bound is $t(G) > \kappa(G)/2 - 1$. For $\kappa = 1$ this bound is not sharp, but for each $\kappa = 3, 4, 5$ and any $\varepsilon > 0$, infinite families of graphs $\{G(\kappa, \varepsilon)\}$ are provided with $\kappa(G(\kappa, \varepsilon)) = \kappa$, but $t(G(\kappa, \varepsilon)) < \kappa/2 - 1 + \varepsilon$.

Analogous investigations on the torus are carried out, and finally the question of *upper* bounds is discussed. Several unanswered questions are posed.

1. Introduction

The concept of the toughness of a graph has received considerable attention since its introduction in 1973 by Chvátal [4] in his investigation of hamiltonicity. Aspects of the general theory of toughness were developed in [4, 12, 7]. Relations linking toughness to various forms of hamiltonicity and to the existence of k-factors in graphs have been widely investigated and will not be discussed here. Toughness may also be regarded as a measure of the vulnerability of graphs to disruption caused by the removal of vertices, and results linking it to other such measures have been established in [1, 7]. The problem of determining whether, for any fixed number k, the toughness is at least k has been shown in [2] to be NP-hard. The complexity of toughness in planar graphs is unresolved.

In this paper we consider graphs embedded on surfaces. Recently, Dillencourt [6] showed that in a 4-connected planar graph the removal of any two vertices leaves a

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graph that is 1-tough. His result and approach is generalized in Theorem 2 below using a result of Schmeichel and Bloom [15]. Harant [9] showed that there is a nonhamiltonian planar graph with toughness $\frac{3}{2}$. This is easily seen to be the maximum it could be as a nonhamiltonian planar graph has connectivity at most 3 (by Tutte's theorem). Dillencourt [5] also showed that certain planar graphs called *(nondegenerate) Delauney triangulations* are 1-tough. We shall establish sharp lower and upper bounds on the toughness of planar graphs and toroidal graphs and briefly consider the extension to higher genera.

2. Definitions and preliminary results

All graphs considered are finite, undirected, loopless and without multiple edges. The terminology and nomenclature of [3] will be used. Throughout the paper G will denote a graph with vertex set $V(G)$, edge set $E(G)$, order $p(G)$, size $q(G)$. Further the genus will be denoted $\gamma(G)$, the minimum degree $\delta(G)$, the maximum degree $\Delta(G)$, connectivity $\kappa(G)$, the independence number $\beta(G)$ and the number of components $k(G)$. If no ambiguity is possible, the symbols will be used without reference to G. A cut-set of G is a proper subset S of $V(G)$ such that $k(G-S)>1$.

If G is not complete, the toughness of G, $t(G)$, is defined by

$$t(G)=\min\{|S|\,/\,k(G-S)\colon S \text{ is a cut-set of } G\},$$

and any cut-set S for which the minimum is attained is called a *tough set* of G. For a real number c, G is said to be c-tough if $t(G)\geqslant c$. Chvátal observed that the condition that a graph be 1-tough is an old necessary condition for hamiltonicity. He conjectured that a sufficiently large value of toughness was a sufficient condition for hamiltonicity; this remains unresolved even for claw-free graphs.

We next list some known results on toughness.

Proposition 1 (Chvátal [4]). (a) *If H is a spanning subgraph of G, then $t(H)\leqslant t(G)$.*
(b) $t(G)\leqslant\kappa(G)/2$.

Proposition 2 (Pippert [12]). *If G is any noncomplete graph, $t(G-v)\geqslant t(G)-\frac{1}{2}$.*

Proposition 3 (Goddard and Swart [7]). *If G is a nonempty graph and m is the largest integer such that $K(1,m)$ is an induced subgraph of G, then $t(G)\geqslant\kappa(G)/m$.*

Corollary 1. (a) *If G is noncomplete and claw-free than $t(G)=\kappa(G)/2$* [10].
(b) *If G is a nontrivial tree then $t(G)=1/\Delta(G)$.*
(c) *If G is r-regular and r-connected then $t(G)\geqslant 1$.*

The following well-known results on genus will be used.

Proposition 4. *If G is a connected graph of genus γ, connectivity κ, girth g, having p vertices, q edges and r regions, then*

(a) $q \leqslant g(p+2\gamma-2)/(g-2)$, $\kappa \leqslant 2g(1+2\gamma/p-2/p)/(g-2)$;

(b) $\gamma(K_{m,n}) = \lceil (m-2)(n-2)/4 \rceil$ [13];

(c) $\gamma(K_p) = \lceil (p-3)(p-4)/12 \rceil$, $p \geqslant 3$ [14].

3. Lower bounds

In this section we establish lower bounds on the toughness of a graph in terms of its connectivity and genus.

We begin by presenting a theorem due to Schmeichel and Bloom [15]. We present a different proof, however, which is shorter than theirs.

Theorem 1. *Let G be a graph with genus γ. If G has connectivity κ, with $\kappa \geqslant 3$, then*

$$k(G-X) \leqslant \frac{2}{\kappa-2}(|X|-2+2\gamma) \quad \textit{for all } X \subseteq V(G) \textit{ with } |X| \geqslant \kappa.$$

Proof. If $X = V(G)$ then $k(G-X) = 0$ and the result is clear since the right-hand side of the inequality is at least 2. If X is not a cutset, but $X \neq V(G)$, then $k(G-X) = 1$ and again, since the right-hand side is at least 2, the result is clear.

So suppose X is a cutset in G. Let $x = |X|$ and $k(G-X) = k$. Now since G is κ-connected, each component of $G-X$ has at least κ vertices of attachment in X.

Let H be the graph obtained from G by deleting all edges with both ends in X, contracting each set of vertices constituting a component of $G-X$ to a single vertex, and removing multiple edges (if any). Then H is a bipartite graph with partite sets X and $V(H)-X$, and has genus at most γ.

Since for each vertex $w \in V(H)-X$ it holds that $\deg_H w \geqslant \kappa$, it follows that $q(H) \geqslant k\kappa$. Since H is bipartite, $q(H) \leqslant 2(p(H)+2\gamma(H)-2)$, by Proposition 4(a). Hence,

$$k\kappa \leqslant 2(x+k+2\gamma-2).$$

But then solving for k, we have the desired result. $\square$

It is now an easy matter to derive the bounds on the toughness that we seek.

Theorem 2. *If G is a connected graph of genus γ and connectivity κ, then*

(a) $t(G) > \kappa/2-1$, *if* $\gamma = 0$, *and*

(b) $t(G) \geqslant \kappa(\kappa-2)/2(\kappa-2+2\gamma)$, *if* $\gamma \geqslant 1$.

Proof. First, note that the inequalities hold trivially if $\kappa = 1$ or 2. So suppose $\kappa \geqslant 3$.

First, suppose that $\gamma = 0$. Let S be a tough set. Then since $|S| \geqslant \kappa$, by Theorem 1 we have

$$k(G-S) = k \leqslant \frac{2}{\kappa - 2}(|S| - 2 + 2\gamma).$$

So $2|S| \geqslant k(\kappa - 2) + 4$ and, hence,

$$t(G) = \frac{|S|}{k} \geqslant \frac{\kappa - 2}{2} + \frac{2}{k} > \frac{\kappa}{2} - 1,$$

and part (a) is proved.

So suppose $\gamma \geqslant 1$. Again, let S be a tough set in G. Then

$$k(G-S) = k \leqslant \frac{2}{\kappa - 2}(|S| - 2 + 2\gamma),$$

and thus

$$\kappa - 2 \leqslant \frac{2|S|}{k} - \frac{4 - 4\gamma}{k}.$$

Hence,

$$t(G) = \frac{|S|}{k} \geqslant \frac{1}{2}\left(\kappa - 2 + \frac{4 - 4\gamma}{k}\right) = \frac{\kappa - 2}{2} - \frac{2}{k}(\gamma - 1).$$

Now $|S| \geqslant \kappa$, so $t = |S|/k \geqslant \kappa/k$ and hence $k \geqslant \kappa/t$. So

$$t \geqslant \frac{\kappa - 2}{2} - \frac{2}{k}(\gamma - 1) \geqslant \frac{\kappa - 2}{2} - \frac{2t}{\kappa}(\gamma - 1).$$

But then, solving this inequality for t, the desired result is obtained. □

As a direct consequence of the above we obtain a simpler bound of

$$t \geqslant \kappa/2 - \gamma \quad \text{for } \gamma \geqslant 1,$$

but this is poorer for $\gamma \geqslant 2$.

The sharpness of the above bounds is illustrated by a subset of the complete bipartite graphs. Let $\kappa \geqslant 3$ and γ be integers such that 4γ is a multiple of $\kappa - 2$ and $4\gamma \geqslant (\kappa - 2)^2$. Then $K_{\kappa, 2+4\gamma/(\kappa-2)}$ has connectivity κ, genus γ and toughness $\kappa(\kappa-2)/(2(\kappa-2)+4\gamma)$ (see [4,13]). So the bound in Theorem 2(b) is attained by an infinite class of graphs, all of girth 4.

3.1. Planar graphs achieving the lower bound

We next investigate the sharpness of the bounds provided above if G is a planar or toroidal graph. To this end we require the definition of a *Kleetope*, $\tau(G)$, of an embedding G of a graph. If G is a graph embedded with regions $R_1, R_2, \ldots, R_r$, then $\tau(G)$ is the graph obtained from G by, for $1 \leqslant i \leqslant r$, inserting a vertex v_i into the interior of R_i and joining v_i to each vertex on the boundary of R_i. Note that the embedding of G extends naturally to an embedding of $\tau(G)$. In particular, if G is a plane graph then so is $\tau(G)$. Kleetopes are sometimes used as examples of graphs with maximum independence number for given genus and connectivity (see [8]).

The bound in Theorem 2(a) is not sharp for $\kappa = 1$ and $\gamma = 0$. But the following examples show that the bound is sharp for $\gamma = 0$ and all possible values of $2 \leqslant \kappa \leqslant 5$. Furthermore, such examples can be obtained with the maximum girth allowed for such connectivity. Note that by Proposition 4(a), if g is the girth,

$$\kappa < 2g/(g-2) \text{ for } \gamma = 0 \quad \text{and} \quad \kappa \leqslant 2g/(g-2) \text{ for } \gamma = 1.$$

Indeed, we can always obtain any girth from 3 up to the maximum allowed. This is often done by taking the example with maximum girth and adding an edge incident with a vertex in the tough set to create the desired short cycle.

Example 3.1. (a) For $\kappa = 2$ the girth can be arbitrarily large. For $n \geqslant 3$ consider the graph G_n obtained by taking n disjoint copies of the path P_n on n vertices and identifying the corresponding ends into two vertices. This is a planar graph with toughness $t(G_n) = 2/n \to 0^+$ as $n \to +\infty$ and girth $n \to +\infty$.

(b) For $\kappa = 3$ the girth is at most 5. A generalized Herschel graph H_n ($n \geqslant 1$) is defined as follows. Form a cyclic chain of 4-cycles by taking n disjoint 4-cycles $a_i b_i c_i d_i a_i$, $1 \leqslant i \leqslant n$, and identifying c_i and a_{i+1} (including c_n and a_1). Then introduce vertices b and d and make b adjacent to each b_i and make d adjacent to each d_i. The result is a 3-connected planar graph of girth 4.

Now, let G_n be obtained by replacing each of the b_i and d_i by a dodecahedron as follows. To make notation simpler we explain how to replace a generic node x of degree 3 with a dodecahedron D. Suppose the outer cycle of D is $v_1 v_2 v_3 v_4 v_5 v_1$ in clockwise order and the neighbors of x are y_1, y_2 and y_3 in clockwise order. Then replace x and its incident edges by D and the edges $v_1 y_1$, $v_2 y_2$ and $v_4 y_3$. The resulting graph G_n is 3-connected (recall that the dodecahedron is 3-connected), planar, and has girth 5, see Fig. 1.

Furthermore, for $S = \{b, d, a_1 = c_n, a_2 = c_1, \ldots, a_n = c_{n-1}\}$,

$$t(G_n) \leqslant \frac{|S|}{k(G_n - S)} = \frac{n+2}{2n} \xrightarrow[\infty]{n} \frac{1}{2}^+,$$

while $\kappa(G_n)/2 - 1 = \frac{1}{2}$.

(c) If $\kappa = 4$ then $g = 3$. Let $H_n = C_4 \times P_n$ ($n \geqslant 2$) and $G_n = \tau(H_n)$, the Kleetope of H_n. It follows that G_n is 4-connected. For $S = V(H_n)$, if $r(H_n)$ denotes the number of faces of H_n,

$$t(G_n) \leqslant \frac{|S|}{k(G_n - S)} = \frac{p(H_n)}{r(H_n)} = \frac{4n}{4n-2} \xrightarrow[\infty]{n} 1^+,$$

whereas $\kappa(G_n)/2 - 1 = 1$.

(d) If $\kappa = 5$ then $g = 3$. For positive integer n the graph D_n is defined inductively as follows: D_1 is the dodecahedron; and for $n \geqslant 2$, D_n is obtained from D_{n-1} by inserting

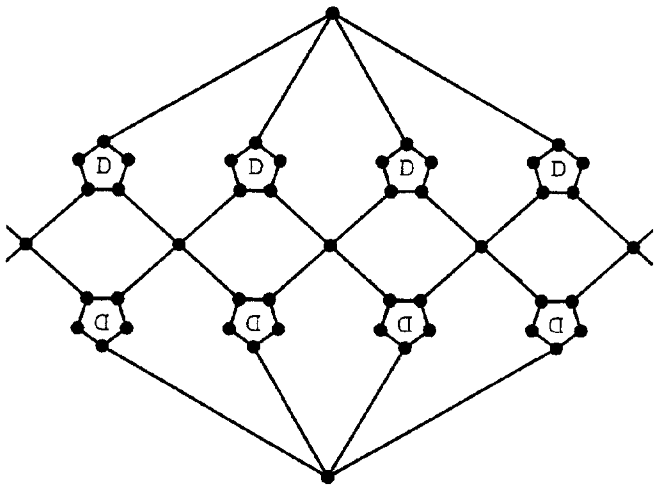

Fig. 1. Part of the 3-connected planar graph G_n with minimum toughness and maximum girth.

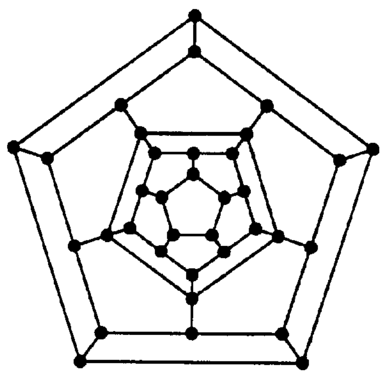

Fig. 2. The pentagonalization D_2.

into the central region of D_{n-1} a dodecahedron and identifying its exterior boundary with the boundary of the central region. The graph D_2 is shown in Fig. 2.

Let $G_n = \tau(D_n)$. Then it can be shown that G_n is 5-connected. For $S = V(D_n)$ we obtain

$$t(G_n) \leqslant \frac{|S|}{k(G_n - S)} = \frac{p(D_n)}{r(D_n)} = \frac{15n+5}{10n+2} \xrightarrow[\infty]{n} \frac{3^+}{2},$$

whereas $\kappa(G_n)/2 - 1 = 3/2$.

3.2. Toroidal graphs

We next consider toroidal graphs in more depth. For $3 \leqslant \kappa \leqslant 6$ we provide graphs with $t = \kappa/2 - 1$ and maximum girth.

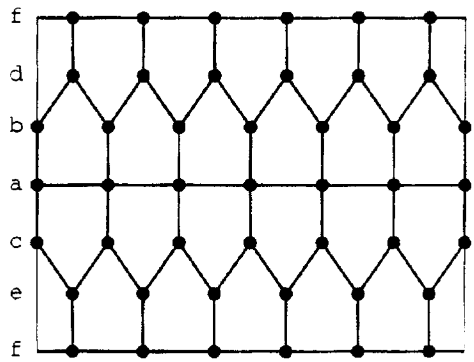

Fig. 3. The graph J_6.

Example 3.2. (a) For $\gamma = 1$ and $\kappa = 2$ the family of graphs described in Example 3.1(a) for planar graphs shows that 2-connected graphs can have toughness arbitrarily close to 0. (Examples specifically with genus 1 can be obtained by adding two edges to G_n, for $n \geqslant 4$.)

(b) If $\kappa = 4$ then $g \leqslant 4$. The graph $H_n = C_4 \times C_n$ for n an even integer has genus 1, connectivity 4, toughness 1 (since, for example, its bipartite and hamiltonian) and girth 4.

(c) If $\kappa = 5$ then $g = 3$. Consider the following graph J_n where every region is a pentagon: Let $V(J_n) = \{ a_i, b_i, c_i, d_i, e_i, f_i \colon i = 0, 1, \ldots, n-1 \}$ and $E(J_n) = \{ a_i a_{i+1}, a_i b_i, a_i c_i, b_i d_i, d_i b_{i+1}, c_i e_i, e_i c_{i+1}, d_i f_i, e_i f_i, f_i f_{i+1} \colon i = 0, 1, \ldots, n-1 \}$ where addition is taken modulo n. The graph J_6 is shown in Fig. 3.

We note that J_n is toroidal with a pentagonal embedding. Let $H_n = \tau(J_n)$. Then $\gamma(H_n) = 1$, $\kappa(H_n) = 5$ and $t(H_n) = p(J_n)/r(J_n) = \frac{3}{2} = \kappa(H_n)/2 - 1$.

(d) If $\kappa = 6$ then $g = 3$. Consider the cubic bipartite 'honeycomb' graph J_n on $12n$ vertices where every region is a hexagon. Then $H_n = \tau(J_n)$ satisfies $\gamma(H_n) = 1$, $\kappa(H_n) = 6$ and $t(H_n) = p(J_n)/r(J_n) = 2 = \kappa(H_n)/2 - 1$.

(e) If $\kappa = 3$ then $g \leqslant 6$. Consider any (3-connected) bipartite graph H which has partite sets A and B where every vertex in A has degree 3 and every vertex in B has degree 6 and is embedded in the torus with every region a quadrilateral. For example, $K_{3,6}$. Such an H_n can also be obtained by modifying the honeycomb graph J_n depicted in Fig. 4 as follows: If the bipartite sets for J_n are A and B, then add in each region a new vertex and join it to the three vertices of A on the boundary of the region; the new vertices are added to B.

Now form G_n by taking H_n and replacing every vertex of degree 3 by a dodecahedron as described in Example 3.1(b). The resulting graph H_n satisfies $\gamma(H_n) = 1$, $\kappa(H_n) = 3$ and $t(H_n) = \frac{1}{2} = \kappa(H_n)/2 - 1$.

The graph G_n constructed in Example 3.2(e) has girth $g = 5$. The lower bound given in Theorem 2(b) cannot be obtained if $\gamma = 1$, $\kappa = 3$ and $g = 6$ as is shown next.

Lemma 1. *If G is a graph with $\gamma(G) = 1$, $\kappa(G) = 3$ and $g(G) = 6$, then $t(G) \geqslant 1$.*

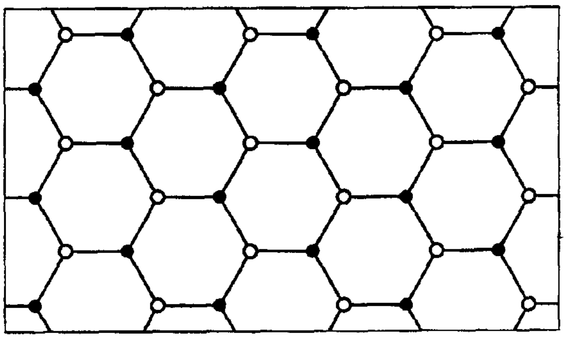

Fig. 4. A toroidal honeycomb graph.

Proof. Let G be a toroidal graph satisfying the hypothesis of the lemma. Then Euler's formula (or Proposition 4(a)) shows that the graph is 3-regular. So by Corollary 1(c) the toughness is at least 1. □

The honeycomb graph J_n is bipartite and hence has toughness exactly 1. However, we do not know the answer to the following question: Do all graphs which satisfy the hypotheses of the lemma have toughness exactly 1?

4. Upper bounds

In this section we investigate upper bounds on the toughness of a graph embedded in the plane and other surfaces. The simple upper bound of $\kappa/2$ is the best we know. Matthews and Sumner [10] showed that equality is attained for claw-free graphs (see also Proposition 3).

Let us begin with $\kappa \leqslant 2$. Any graph with $\kappa = 0$ has toughness 0. For $\kappa = 1$ consider the path and for $\kappa = 2$ consider the cycle. These have toughness $\frac{1}{2}$ and 1, respectively. One can also find graphs with specified girth and genus which have these properties.

We consider next 3-connected graphs with $t(G) = \kappa(G)/2$. To this end we require the following definition. The *inflation* $I(H)$ of a graph H with vertex set $V(H) = \{v_1, \ldots, v_p\}$ is the graph obtained by replacing each vertex v_i by a clique G_i of order $\deg_H v_i$, and replacing each edge v_iv_j of H by an edge joining a vertex of G_i to a vertex of G_j so that each vertex x of each G_i has $\deg_{I(H)} x = \deg_H v_i$. Note that, if H is cubic, then $\gamma(I(H)) = \gamma(H)$. The inflation of H may also be thought of as the line graph of the subdivision graph of H. Thus, $I(H)$ is claw-free.

Example 4.1. Let $A_1 = K_4$ and $A_{i+1} = I(A_i)$ for $i \geqslant 1$. For $i \geqslant 2$ A_i is a 3-connected planar graph with $t(A_i) = \frac{3}{2}$.

(Harant [9] showed that if one starts with a 3-connected cubic nonhamiltonian planar graph and repeatedly inflates, one obtains an infinite family of cubic nonhamiltonian planar graphs with toughness $\frac{3}{2}$.)

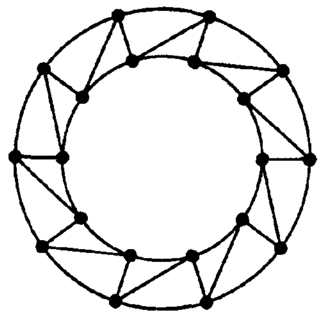

Fig. 5. A 4-connected planar graph with maximum toughness.

This and other examples suggest the question: Do there exist 3-connected planar graphs G with $g(G) \geqslant 4$ that are $\frac{3}{2}$-tough? A similar question can be asked for toroidal graphs.

Example 4.2. For $\kappa = 4$. For $n \geqslant 3$ let G_n be the graph formed from $C_n \times K_2$ with vertices (v_i, w_j), $i = 0, \ldots, n-1$ and $j = 1, 2$, by joining (v_i, w_1) to (v_{i+1}, w_2) for $i = 0, 1, \ldots, n-1$ (addition modulo n). Then G_n is planar and 4-connected and claw-free; so $t(G_n) = \kappa(G)/2$. The graph G_{10} is depicted in Fig. 5.

(This example also suffices to handle the case of $\kappa < 4$ by the removal of the appropriate number of members of a minimum cut-set.)

For $\kappa = 5$ one planar graph that is 5-connected with toughness equal to $\frac{5}{2}$ is the icosahedron. Unfortunately, the icosahedron is the only claw-free 5-connected planar graph. We have constructed several examples of planar graphs with toughness $\frac{5}{2}$ but have not been able to obtain an infinite family. One such graph is depicted in Fig. 6. The proof of toughness is straightforward if tedious. One approach is that an examination of the proof of Proposition 3 shows that if a graph has toughness less than $\kappa/2$ then there must exist a vertex v in the tough set S which is adjacent to vertices in at least three components of $G - S$. There is only one choice for v up to automorphism in the graph G and this forces two of its neighbors into S while the other three must be in separate components of $G - S$. By inspection one can then argue that to create 3 components a total of 8 vertices must be removed, and to create 4 components at least 10 vertices removed; hence, the toughness is as claimed.

This leaves the following question.

Question 1. *What, asymptotically, is the maximum value of toughness in the plane?*

On the torus the maximum value of connectivity is 6 (and this is obtained if and only if the graph is 6-regular (cf. [11])). There is an infinite number of toroidal graphs with toughness 3. One such family consists of the powers C_p^3 for $p \geqslant 7$. These are claw-free. But there are also examples which have toughness 3 and contain claws: one

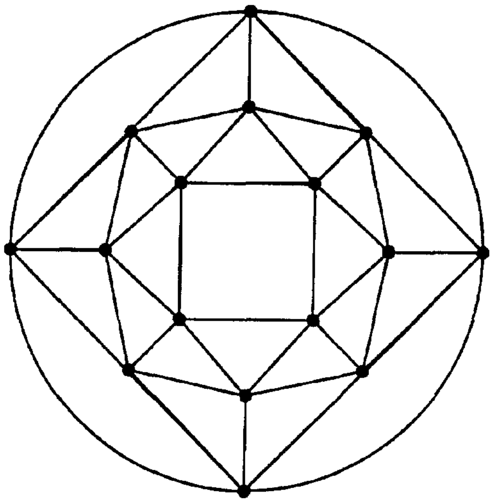

Fig. 6. A 5-connected planar graph with maximum toughness.

example is the circulant $C_{13}(1,3,4)$. That these graphs embed on the torus was shown by Negami [11]; indeed Negami produced a construction which can be used to derive all 6-regular toroidal graphs.

For completeness let us construct the embedding of the circulant $C_p(1,k,k+1)$ on the torus (the case $k=2$ is the third power of the cycle). Start with a rectangular sheet of paper and on the left and right sides place $p+1$ evenly spaced vertices (using the corners). Add all horizontal, vertical and slope-1 lines through the vertices. Then identify the top and bottom sides of the rectangle to form an annulus (with a 4-regular graph on $2p$ vertices embedded on it in which vertex i on the left is adjacent to vertices $i-1$ and $i+1$ on the left and to vertices i and $i+1$ on the right). Finally, stick the sides of the annulus together with a twist so that vertex i on the left is identified with vertex $i-k$ on the right for all i (arithmetic modulo p).

An infinite family of 5-connected toroidal graphs which are $\frac{5}{2}$-tough is obtained by deleting one vertex from the powers C_p^3 for $p \geqslant 7$ (by Proposition 2).

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